MATH EXPLORATIONS
PART 2

5th Edition

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Preface and Introduction

Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

First, learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. **A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together.** In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself. Some basic rules for discussion within a group include

1. **Encourage everyone to participate**, and value each person's opinions. Listening carefully to what someone else says can help clarify a question. The process helps the explainer often as much as the questioner.
2. If one person has a question, remember that the chances are good that someone else will have the same question. Be sure everyone understands new ideas completely, and **never be afraid to ask questions.**
3. **Don't be afraid to make a mistake.** In the words of Albert Einstein, "A person who never made a mistake never discovered anything new." Group discussion is a time of exploration without criticism. In fact, many times mistakes help to discover difficulties in solving a problem. So rather than considering a mistake a problem, think of a mistake as an opportunity to learn more about the process of problem-solving.

4. **Finally, always share your ideas** with one another, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don't understand an idea, be sure to ask "why" it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn't clear, there are several things to try.

a. First, look for simpler cases. Looking deeply at simple cases can help you see a general pattern.

b. Second, if an idea is unclear, ask your peers and teacher for help. Go beyond "Is this the right answer?"

c. Third, understand the question being asked. Understanding the question leads to mathematical progress.

d. Fourth, focus on the process of obtaining an answer, not just the answer itself; in short become problem centered, not answer centered. One of the major goals of this book is to develop an understanding of ideas that can be used to solve more difficult problems as well.

e. Fifth, after getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

Some hints to help in responding to oral questions in group and class discussion: As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times try exploring first by yourself and then discuss your ideas with others.

When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.
2. In class, be recognized if you want to contribute or ask a question.
3. Be polite and listen when others in your group or class are talking. This is one of the best ways to learn.
4. Finally, don't be shy. If you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

Last, some general advice about taking notes in math:

1. Reading math is a specific skill. Unlike other types of reading, when you read math, you need to read each word carefully. The first step is to know the mathematical meaning of all words.
2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you relate what you are learning to real world situations.
3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.
4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that "a picture is worth a thousand words." Visual cues help you understand and remember definitions of new terms.

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Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes.

It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. To this end, they have made every effort to create explanations that are as transparent as possible. Parents are encouraged to read both the book and the accompanying materials.

Possibly the most unique aspect of this book is the breadth and span of its appeal. The authors wrote this text for both a willing 5th or 6th grader and any 7^{th} grader. Students may particularly enjoy the ingenuity and investigation problems at the end of each set of exercises which are designed to lead students to explore new concepts more deeply.

The text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 modeled after the Ross program at Ohio State, teaching students to "think deeply of simple things" (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with over 115 students being named semi-finalists, regional finalists, and national finalists in the prestigious Siemens Competition in Math, Science, and Engineering. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also received significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Advisory Board.

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Preface and Introduction

In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 4–8. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts the JSMC targeted gifted students; in other districts the program was delivered to mixed group of students. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

A concern with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The Math Explorations texts that we have written have taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills) for grades 6–8 while weaving in algebra throughout. The third volume, for 8^{th} graders, allows all students to complete Algebra I. This is an integrated approach to algebra developed especially for younger students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U.S. students on international tests.

An accompanying Teacher Edition (TE) has been written to make the textbook and its mathematical content as clear and intuitive for teachers as possible. The guide is in a three-ring binder so teachers can add or rearrange whatever they need. Every left-hand page is filled with suggestions and hints for augmenting the student text. Answers to the exercises and additional activities are available in the TE and companion CD.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation, and Kodosky Foundation. A special thanks to our Advisory Board, especially Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other was Jeff Kodosky, who immediately realized the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The primary basis for the book was our Junior Summer Math Camp curriculum, coauthored with my wife Hiroko, and friend, colleague, and coauthor Terry McCabe. The three of us discuss every part of the book, no matter how small or insignificant it might seem. Each of us has his or her own ideas, which together I hope have made for an interesting book that will excite all young students with the joy of mathematical exploration and discovery. Over the summers of 2005–2012, we have been assisted by an outstanding group of former Honors Summer Math Camp students, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge the help and support from this past summer.

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Terry McCabe led the team in developing this book, Math Explorations (ME) Part 2, assisted by Hiroko Warshauer and Max Warshauer. Alex White from Texas State provided valuable suggestions for each level of the curriculum, and took over the leadership of the effort for the third volume, Algebra 1.

We made numerous refinements to the curriculum in summer 2012 incorporating the 2012 revised TEKS, additional exercises, new warm-ups, and an accompanying collection of workbook handouts that provides a guide for how to teach each section. Many of these changes were inspired by and suggested by our pilot site teachers. Amy Warshauer and Alexandra Eusebi from Kealing Middle School in Austin, who piloted Part 2 of the curriculum with their 6th grade students, helped design and put together a fabulous collection of workbook handouts that accompany the revised text. Melissa Freese from Midland provided valuable assistance on both the workbooks and in making new warm-up problems. Additional edits and proofreading were done by Michael Kellerman, who did a wonderful job making sure that the language of each section was at the appropriate grade level. Robert Perez from Brownsville developed special resources for English Language Learners, including a translation of key vocabulary into Spanish.

ME Part 1 should work for any 6^{th} grade student, while ME Part 2 is suitable for either an advanced 6th grade student or any 7th grade student. Finally, Math Explorations Algebra I completes algebra for all 8^{th} grade students. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6–8, while also covering Algebra I.

The production team was led by Claudia Hernández, an undergraduate student in mathematics at Texas State. Claudia did an outstanding job of laying out the book, editing, correcting problems, and in general making the book more user friendly for students and parents. She has a great feel for what will excite young students in mathematics, and worked tirelessly to ensure that each part of the project was done as well as we possibly could. As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsibility for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help the project would not have reached its present state.

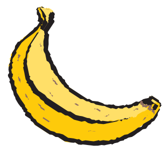
Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with dedicated teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Andrew Hsiau, Patty Amende, and Michelle Pruett have provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesn't, and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer
Director of Texas State Mathworks

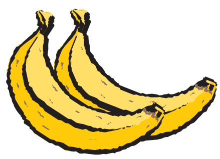
EXPLORING INTEGERS ON THE NUMBER LINE 1

SECTION 1.1
BUILDING NUMBER LINES

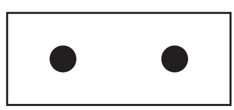
Let's begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number one, we have in mind a picture:



Similarly, the number two describes a different quantity:



We could use a picture with dots to describe the number two. For instance, we could draw:



We call this way of thinking of numbers the "**set model**." There are, however, other ways of representing numbers.

Another way to represent numbers is to describe locations with the **number line model**, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the **origin**. Label the origin with the number 0. We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.

Next, mark off some distance to the right of the origin and label the second point with the number 1.

Continue marking off points the same distance apart as above and label these points with the numbers 2, 3, 4, and so on.

The points you have constructed on your number line lead us to our first definition.

DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)

The **counting numbers** are the numbers in the following never-ending sequence:

$$1, 2, 3, 4, 5, 6, 7, \ldots$$

We can also write this as

$$^{+}1, {}^{+}2, {}^{+}3, {}^{+}4, {}^{+}5, {}^{+}6, {}^{+}7, \ldots$$

These numbers are also called the **positive integers** or **natural numbers**.

One interesting property of the natural numbers is that there are "infinitely many" of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

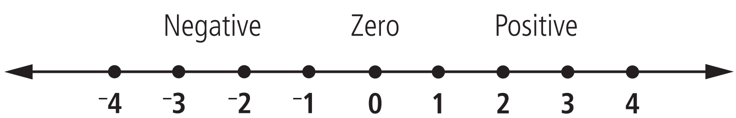
When we include the number 0, we have a different collection of numbers that we call the **whole numbers**.

DEFINITION 1.2: WHOLE NUMBERS (NON-NEGATIVE INTEGERS)
The **whole numbers** are the numbers in the following never-ending sequence: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, … These numbers are also called the **non-negative integers.**

In order to label points to the left of the origin, we use **negative integers**:

$$^{-}1,\ ^{-}2,\ ^{-}3,\ ^{-}4,\ \ldots$$

The sign in front of the number tells us on which side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. *Zero is not considered to be either positive or negative.*

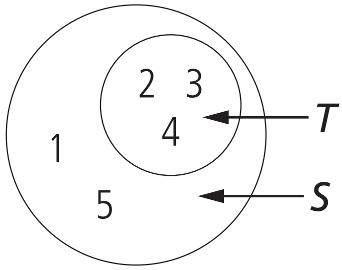


So we have seen that numbers can be used in different ways. They can help us to describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location also can tell us the distance the number is from the origin if we ignore the sign.

DEFINITION 1.3: INTEGERS
The collection of **integers** is composed of the negative integers, zero, and the positive integers: …, $^{-}4$, $^{-}3$, $^{-}2$, $^{-}1$, 0, 1, 2, 3, 4, …

Definition 1.3 leads to the **trichotomy** property, which states that there are exactly three possibilities for an integer: positive, negative, or zero.

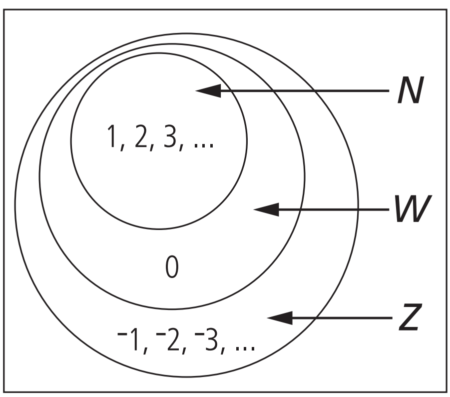
Such collections of numbers are often called sets of numbers. For example, assume ***S*** is the set {1, 2, 3, 4, 5} and ***T*** is the set {2, 3, 4}. Notice that every number in ***T*** is also an element of the set ***S***. That means that ***T*** is a subset of ***S***. This relationship is represented by the following diagram:



EXAMPLE 1

Call the set of positive integers *N*, the set of whole numbers *W*, and the set of integers *Z*. Draw a diagram of the relationship of these three sets.

SOLUTION



N= {1, 2, 3, ...}

W= {0, 1, 2, ...}

Z= {..., -3, -2, -1, 0, 1, 2, ...}

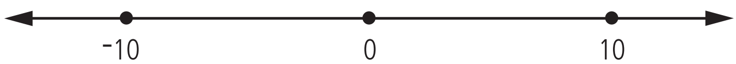
EXPLORATION: CONSTRUCTING A NUMBER LINE

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate the numbers 1, 2, 3, …, 10, and ⁻1, ⁻2, ⁻3, …, ⁻10.
4. Where would 20, 30, 50 be located? 100? 1000?
5. Find the negative numbers corresponding to the numbers in question 4.

EXERCISES

1. The post office is located at the origin of Main Street. We label its address as 0. The laboratory has address 6 and the zoo has address 9. Going in the other direction from the origin, we find a candy shop with address $^{-}4$ and a space observatory with address $^{-}7$. Draw a number line representing Main Street. Label each of the above locations on the number line. Watch your spacing.

2. a. Copy the line below to mark off and label the integers from 0 to 10 and from 0 to $^{-}10$. Use a pencil to experiment because you might need to erase. Watch your spacing.

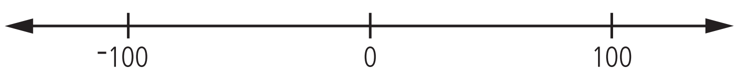


   b. Make a number line from $^{-}20$ to 20.

3. Draw a section of the number line containing the number 77. Mark the number 77 on your line. Now label your number line with the first few integers to the right of 77 and the next few to the left of 77, at least three each way.

4. Do the same thing you did in the previous exercise, but this time start with the number $^{-}77$ instead of 77.

5. Use a line like the one below to mark off the numbers with equal distances by tens from 0 to 100 and from 0 to $^{-}100$. Use a pencil to experiment.



   a. What is the distance from 0 to 50 and from 50 to 100? Are they the same?

   b. What are the distances from 10 to 20, 30 to 40, and 70 to 80? Are they the same?

   c. Explain whether you need to rework your markings on the number line.

   d. *Estimate* the locations of the following numbers and label each on your number line:

   15, 25, 55, $^{-}15$, $^{-}25$, $^{-}75$, 34, 31, $^{-}34$, $^{-}31$, 87

6. Draw a number line so that the number ⁻1000 is at the left end and 1000 is at the right end. *Estimate* the locations of the following integers:

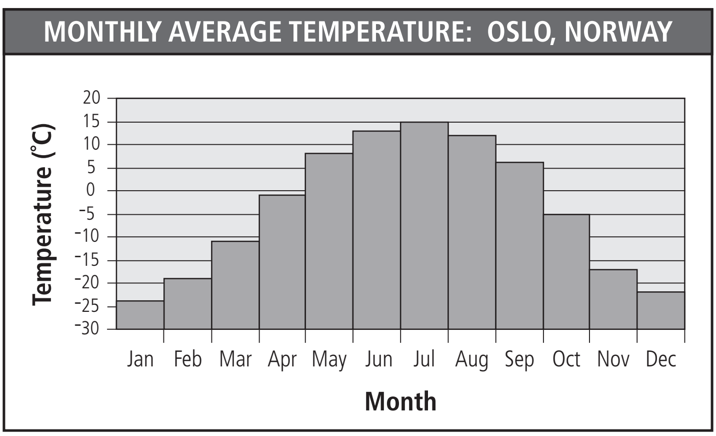
   500, ⁻500, 250, ⁻100, ⁻800, 10, ⁻990, 342, ⁻781, 203, ⁻407

7. Draw a number line. Find all the integers on your number line that are greater than 15 and less than 20. Color each of these locations blue.

8. Given two numbers on the number line, how do you decide which number is greater?

Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.

9. The chart below shows the monthly average temperatures for the city of Oslo, Norway. Based on the data, put the twelve months in order from coldest to warmest.



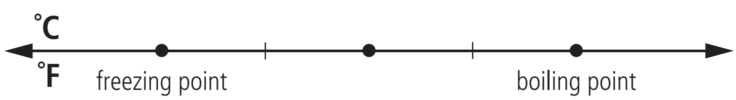
10. Chris visits Edmonton, Canada, where it is ⁻7 °C. Carmen visits Winnipeg, Canada, where it is 9 °C. Which temperature is closer to the freezing point? Explain. Remember, when we measure temperature in degrees Celsius (°C), 0 °C is the freezing point of water.

11. The temperature in Toronto, Canada, one cold day, is ⁻10 °C. The next day the temperature is 4 °C. Which temperature is closer to the freezing point?

12. The temperature in Fargo, North Dakota, is ⁻15 °F while it is ⁻20 °F in St. Paul, Minnesota. Which temperature is colder? How much colder?

13. **Ingenuity:**
On a cold winter day in Iowa, the temperature at 6:00 p.m. is 10 °F. If the temperature decreases an average of 4 °F for each of the next five hours, what will the temperature be at 11:00 p.m.?

14. **Investigation:**
Use the number line below as a thermometer with the Celsius scale above the line and the Fahrenheit scale below the line to discover how the two scales are related. Write the temperatures above and below the number line.



a. At what temperature does water boil on each scale?

b. At what temperature does water freeze on each scale?

c. What is the Fahrenheit reading for 50 °C?

d. Is the Celsius reading for 25 °F a positive or negative number?

e. A nice day is 77 °F. What is this temperature in Celsius?

f. A hot day is 100 °F. Estimate this temperature in Celsius.