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TEACHER VERSION

Math Explorations

Part 1

2013 Teacher Edition

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preface and introduction

Math Explorations follows several fundamental principles. First, learning math is not a spectator sport. The activities that fill the text and accompanying workbooks allow students to develop the major concepts through exploration and investigation rather than being given rules to follow. Second, students should learn algebraic thinking and the precise use of mathematical language to model problems and communicate their ideas. This is not done as an afterthought, but is woven in throughout the text. Third, students should be continually challenged to see patterns, and through this process develop the ability to solve progressively more difficult problems.

Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

First, learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together. In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself. Some basic rules for discussion within a group include

1. Encourage everyone to participate, and value each person's opinions. Listening carefully to what someone else says can help clarify a question. The process helps the explainer often as much as the questioner.

2. If one person has a question, remember that the chances are someone else will have the same question. Be sure everyone understands new ideas completely, and never be afraid to ask questions.

3. Don't be afraid to make a mistake. In the words of Albert Einstein, "A person who never made a mistake never discovered anything new." Group discussion is a time of exploration without criticism. In fact, many times mistakes help to discover difficulties in solving a problem. So rather than considering a mistake a problem, think of a mistake as an opportunity to learn more about the process of problem-solving.

4. Finally, always share your ideas with one another, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don't understand an idea, be sure to ask "why" it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn't clear, there are several things to try.

a. First, look for simpler cases. Looking deeply at simple cases can help you see a general pattern.

b. Second, if an idea is unclear, ask your peers and teacher for help. Go beyonk "Is this the right answer?"

c. Third, understand the question being asked. Understanding the question leads to mathematical progress.

d. Focus on the process of obtaining an answer, not just the answer itself; in short become problem centered, not answer centered. One of the major goals of this book is to develop an understanding of ideas that can solve more difficult problems as well.

e. After getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

MATH EXPLORATIONS

preface and introduction

Some hints to help in responding to oral questions in group and class discussion:

As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times try exploring first by yourself and then discuss your ideas with others. When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.

2. In class, be recognized if you want to contribute or ask a question.

3. Be polite and listen when others in your group or class are talking. This is one of the best ways to learn.

4. Finally, don't be shy. If you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

Last, some general advice about reading and taking notes in math.

1. Reading math is a specific skill. Unlike other types of reading, when you read math, you need to read each word carefully. The first step is to know the mathematical meaning of all words.

2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you relate what you are learning to real world situations.

3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.

4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that "a picture is worth a thousand words." Visual cues can help you understand and remember definitions of new terms.

Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes.

It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. To this end, they have made every effort to create explanations that are as transparent as possible. Parents are encouraged to read both the book and the accompanying materials.

Possibly the most unique aspect of this book is the breadth and span of its appeal. The authors wrote this text for both a willing 4^{th} or 5^{th} grader and any 6^{th} grader. Students may particularly enjoy the ingenuity and investigation problems at the end of each set of exercises which are designed to lead students to explore new concepts more deeply.

The text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 modeled after the Ross program at Ohio State, teaching students to "think deeply of simple things" (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with over 115 students being named semi-finalists, regional finalists, and national finalists in the prestigious Siemens Competition in Math, Science, and Engineering. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also received significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Advisory Board.

In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 4–8. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts the JSMC targeted gifted students; in other districts the program was delivered to mixed group of students. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

A concern with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The Math Explorations texts that we have written have taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills), for grades 6-8 while weaving in algebra throughout. The third volume for 8th graders allows all students to complete Algebra I. This is an integrated approach to algebra developed especially for younger students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U. S. students on international tests.

An accompanying Teacher Edition (TE) has been written to make the textbook and its mathematical content as clear and intuitive for teachers as possible. The guide is in a three-ring binder so teachers can add or rearrange whatever they need. Every left-hand page is filled with suggestions and hints for augmenting the student text. Answers to the exercises and additional activities are available in the TE and companion CD.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation, and Kodosky Foundation. A special thanks to our Advisory Board, especially Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other was Jeff Kodosky, who immediately realized the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The primary basis for the book was our junior summer math camp curriculum, coauthored with my wife Hiroko, and friend, colleague and coauthor Terry McCabe. The three of us discuss every part of the book, no matter how small or insignificant it might seem. Each of us has his or

her own ideas, which together I hope have made for an interesting book that will excite all young students with the joy of mathematical exploration and discovery. Over the summers of 2005-2012, we have been assisted by an outstanding group of former Honors Summer Math Camp students, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge the help and support from this past summer.

Hiroko Warshauer led the team in developing this book, Math Explorations (ME) Part I, assisted by Terry McCabe and Max Warshauer. Terry McCabe led the team in developing Math Explorations (ME) Part II. Alex White from Texas State provided valuable suggestions for each level of the curriculum, and took over the leadership of the effort for the third volume, Algebra 1.

We made numerous refinements to the curriculum in summer 2012 incorporating the 2012 revised TEKS, additional exercises, new warm-ups, and an accompanying collection of workbook handouts that provides a guide for how to teach each section. Many of these changes were inspired by and suggested by our pilot site teachers. Denise Girardeau from Austin's Kealing Middle School did a fabulous job working with our team on edits and new exercises. Additional edits and proofreading was done by Michael Kellerman, who did a wonderful job making sure that the language of each section was at the appropriate grade level. Robert Perez from Brownsville developed special resources for English Language Learners, including a translation of key vocabulary into Spanish.

The production team was led by Namakshi P. Kant, a graduate student in mathematics education at Texas State. She did an outstanding job of laying out the book, editing, correcting problems, and in general making the book more user friendly for students and parents. Nama has a great feel for what will excite young students in mathematics, and worked tirelessly to ensure that each part of the project was done as well as we possibly could. As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsiblity for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help the project would not have reached its present state.

ME Part 1 should work for any 6th grade student, while ME Part 2 is suitable for either an advanced 6th grade student or any 7th grade student. Finally, Math Explorations Algebra I completes algebra for all 8th grade students. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8, while also covering Algebra I.

ME Part I was piloted in 2010 - 2012 by teachers and students in San Marcos, New Braunfels, and Midland. The results of these pilots have been extremely encouraging. We are seeing young 6th and 7th grade students reach (on average) 8th grade level as measured by the Orleans-Hanna test by the end of 7th grade.

Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with dedicated teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Andrew Hsiau and Patty Amende have provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesn't and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer

Texas Mathworks,
Director

[redacted]

MATH EXPLORATIONS

Exploring Integers

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Section 1.1 introduces the number line model as a visual way to locate integers. Important elements of scaling and ordering numbers are incorporated into a linear model while discussing different sets of numbers that can be identified. In section 1.2, the location of the integers is used to compare whether one integer is greater or less than another integer using the visual representations. As an application of a number line, section 1.3 has the students construct historical time lines that integrate mathematics and history. In 1.4 students explore distance on the number line then use this context to add and subtract integers in sections 1.5 and 1.6. In order to weave algebra early into the curriculum, we introduce algebraic terminology in section 1.7 and use the number line to illustrate relationships between numbers, operations, and extend to corresponding algebraic relationships. Finally, in section 1.8, we look at the Cartesian coordinate system by using two perpendicular number lines that help to locate points in the plane. This chapter

Section 1.1 - Building Number Lines

Big Idea

Constructing the number line and modeling integers

Key Objectives

- Use the linear model to order numbers: number lines.
- Realize importance of order and spacing when constructing number lines.
- Identify types of numbers (counting, natural, whole and integers).
- Compare and order integers within the context of temperature.

Materials

Thermometer, Objects to count (bananas might be fun, but counters are more useful), Tape, Index cards, Number line, Graph paper, Rulers

Pedagogical/Orchestration

- Several of the homework questions require students to think about and draw a thermometer. You might want to have one handy to show or pass around. The big ones that you can find for outside patios and gardens would be easy to read.
- Within launch, include background of Roman Numbers being adopted from letters—does not have a zero, starts with counting numbers.

Activity

"United We Stand"

Exercises

Ingenuity foreshadows ideas of distance with integers.

#12 Investigation is a good problem for a discussion on locating numbers left and right of each other.

Note that blank number lines are available in the Teacher's Edition at the end of this section.

Vocabulary

set model, number line model, origin, counting numbers, positive integers, natural numbers, whole numbers, non-negative integers, integers, negative integers, opposites

TEKS

6.1(A,C); 6.12(A)

New: 6.1 A, D; 6.2 A, C

Launch for Section 1.1

Lead the class through the beginning of the section by modeling the idea of the set model using bananas or other counters. Ask the class, "Why is this called the 'set model?'" A possible response might be that numbers are thought of as sets of objects (e.g. bananas). Another possibility is that a number can describe how many objects are in a set, or a collection, of objects. Ask students, "In the set model, what does adding objects to a collection do to the total set?" Students will respond that addition makes the total set larger. "What about subtraction?" The response should be that it makes the total set smaller. The set model mostly describes "How many?" as in, "How many bananas do I have?" Tell the class that there are other models for numbers and we are going to learn one that answers "How far and where?" At this point, lead the class into the "United We Stand" activity on the CD. At the end of the "United We Stand" activity, ask the students if they can think of examples in the real world where people use negative numbers. There are some examples below:

- For a sports example, use yards gained or lost in attempting a first down.
- The old but good standby of very cold temperatures and warm temperatures.

Tell students they will be expanding their knowledge of number lines in today's lesson.

Alternative to Exploration: Constructing a Number Line (on page 11(4))

For this activity have the students turn their paper horizontally. As the students do parts 1-3 be looking to see who makes $^{-}10$, and 10 all the way to the edge of their paper. As you get to part 4 they will observe that they needed to have more room on either side and they might need to redraw their number line again to correct it. Let them make this mistake because learning to scale a number line (and later coordinate graphs) is very important. We ultimately want them to be able to judge and create appropriate scales for themselves. This will be formally discussed in Section 1.3. For the bigger numbers it might be best not to make the students redraw the number line, but instead to estimate where it might be. For example: in the next classroom, outside the building, in the next town? This is a great opportunity to practice estimation.

Objective: The students will work collaboratively to physically position themselves on a number line.

Materials:
1) Index Cards, numbered with random integers. You will need one card per student.
(Be sure to include positives, negatives, and zero when you are numbering your cards.)
2) Colored Tape

Activity Instructions:
Before the students get to class, use the colored tape to make an unmarked number line along the floor in the front of your room. Do not mark intervals on the number line.

After shuffling the cards to make sure they are in random order, pass one card to each student.

Once all of the cards have been handed out, the students will come up one at a time and position themselves on the number line that you made on the floor. Allow the important issue of spacing to arise naturally as the students arrange themselves.

When all students are standing in the correct order at the front of the room, take this opportunity to ask them some extension questions to check for understanding of the lesson. Some examples of questions you could ask are:

-Who is standing in the middle of the line, and why?

-If we had a card with the number one million, where would it go?

-If we had a card with the number ________, it would be between which two students?

-Who represents the biggest number on our line?

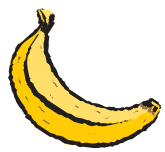
-Who represents the smallest number on our line?

-Is the spacing between integers important?

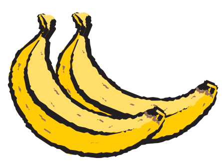
1 EXPLORING INTEGERS ON THE NUMBER LINE

SECTION 1.1 BUILDING NUMBER LINES

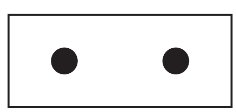
Let's begin by thinking carefully about numbers. Numbers are part of the mathematical alphabet, just like letters are used in English to form words. We use numbers for counting and representing quantities. When we think of the number *one*, we have in mind a picture:



Similarly, the number *two* describes a different quantity:



We could use a picture with dots to describe the number 2. For instance, we could draw:

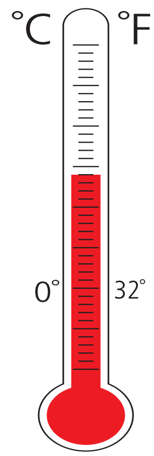


We call this way of thinking of numbers the "set model." There are, however, other ways of representing numbers.

The students will have an opportunity to include the definitions and examples from this section on their worksheets. Encourage careful wording and appropriate examples that will be useful when students look back at their work.

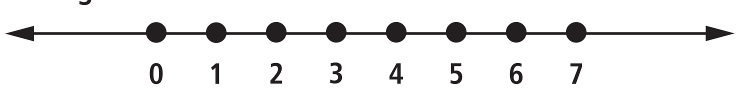
Throughout this book your students will be exposed to different forms of mathematical notation. The "..." known as an ellipsis in math, means that the list of numbers continues in the same way it starts. For instance, in Definition 1.1, students can assume that the number after "7" is "8" in the first sequence and that the number after "+7" is "+8" in the second sequence. Another inference with the ellipsis is that the list continues without end, the idea of going on forever as an idea of infinitely long.

Another way to represent numbers is to describe locations with the **number line model**, which is visually similar to a thermometer. To construct a number line, begin by drawing a straight line and picking some point on the line. We call this point the **origin**. Label the origin with the number 0. We can think of 0 as the address of a certain location on the number line. Notice that the line continues in both directions without ending. We show this with arrows at the ends of the line.



Next, mark off some distance to the right of the origin, and label the second point with the number 1.

Continue marking off points the same distance apart as above, and label these points with the numbers 2, 3, 4, and so on. Deciding how we label our marks is called **scaling**.



Make sure your students can tell you the difference between the set of whole numbers and the natural or counting numbers.

Emphasize regularly that left is negative and right is positive on the number line. Right now this directionality only involves identifying integers, but it is also essential in computation with integers.

What is the relationship between the set of integers and the sets of whole numbers and natural or counting numbers?

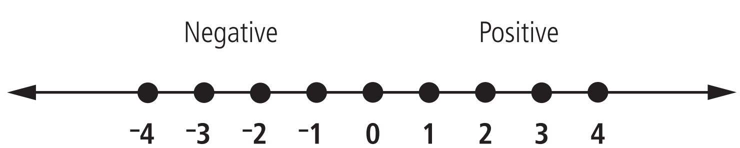
DEFINITION 1.1: COUNTING NUMBERS (POSITIVE INTEGERS)
The **counting numbers** are the numbers in the following never-ending sequence: $1, 2, 3, 4, 5, 6, 7, \ldots$ We can also write this as $^{+}1, ^{+}2, ^{+}3, ^{+}4, ^{+}5, ^{+}6, ^{+}7, \ldots$ These numbers are also called the **positive integers,** or **natural numbers**.

One interesting property of the natural numbers is that there are "infinitely many" of them; that is, if we write down a list of natural numbers, there is always some natural number that is not on the list.

When we include the number 0, we have a different collection of numbers that we call the **whole numbers**.

DEFINITION 1.2: WHOLE NUMBERS (NON-NEGATIVE INTEGERS)
The **whole numbers** are the numbers in the following never-ending sequence: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$ These numbers are also called the **non-negative integers.**

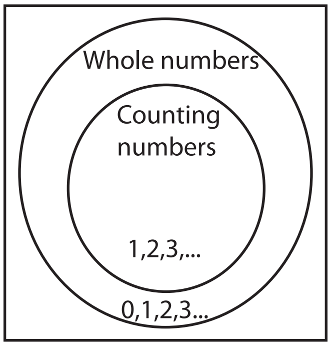
In order to label points to the left of the origin, we use **negative integers**: $^{-}1, ^{-}2, ^{-}3, ^{-}4, \ldots$ The sign in front of the number tells us on what side of zero the number is located. Positive numbers are to the right of zero; negative numbers are to the left of zero. Zero is neither positive nor negative.



We have seen that numbers can be used in different ways. They can help us describe the quantity of objects using the set model or to denote a location using the number line model. Notice that the number representing a location also can tell us the distance the number is from the origin if we ignore the sign.

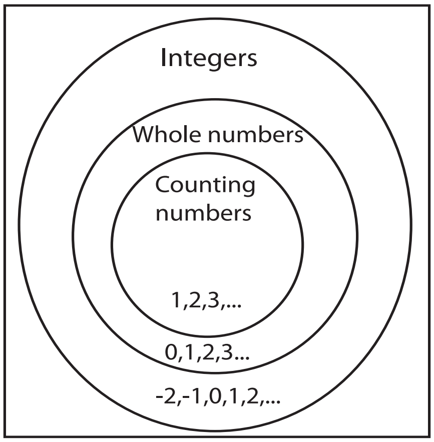
DEFINITION 1.3: INTEGERS
The collection of **integers** is composed of the negative integers, zero, and the positive integers: $\ldots, {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, 4, \ldots$

Notice that every counting number is a whole number, so the set of counting numbers is a subset of the set of counting numbers. One way to represent the relationship between these sets is called a Venn Diagram:



Notice that all of the whole numbers except 0 are inside the counting numbers.

Similarly, every whole number is an integer. This is the Venn Diagram of the three sets:

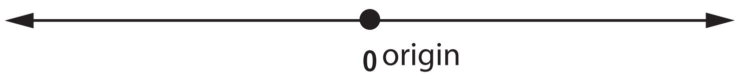


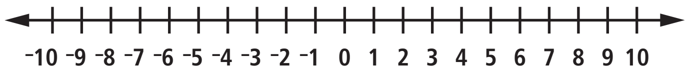
Exploration: Constructing a Number Line

In this class activity, you might want the students to draw their line on the board, or use tape to draw the line on the floor. Some important issues to discuss include: Are the numbers evenly spaced? How big is big? i.e. where are the large numbers located? How do you decide which numbers are greater and which numbers are less? Does the origin have to be placed in the center of the line? For an alternative to this exploration, see the end of this section You may wish to use graph paper so that students are aware of the importance of equal spacing.

Include horizontal and vertical number line templates - one page of each.

1. 

2. 

3. 

4. 20 is located 20 units to the right of zero

   Other numbers will follow this model.

*Note: Students may also describe these in terms relative to other numbers, i.e. 20 is 10 units to the right of 10, etc.

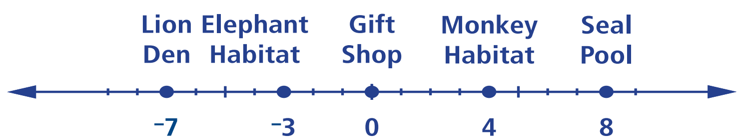
5. -20 is located 20 units to the left of zero

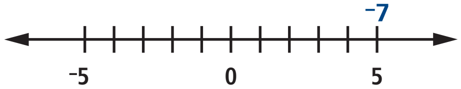
   Other numbers will follow this model.

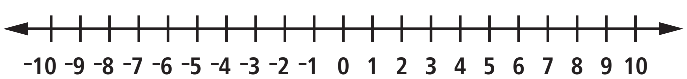
Teacher Tip: Have students create a number line from about ⁻20 to 20 on a half sentence strip, punch holes in them so they can keep it in their binder. Students can have a horizontal number line and a vertical number line. They can also write "clue" words that identify with negative direction and positive direction. You may suggest using a "folding" technique to ensure that the number line is evenly spaced.

EXERCISES

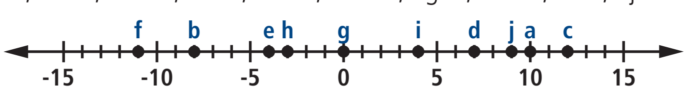
Use graphing paper for solutions of each exercise. Pre-designed number lines will be provided at the end of the section in the Teacher's Edition for exercises 5 and 6.

1. 

2. a. 

   b. 

3. a. 10, b.-8, c. 12, d. 7, e. -4, f. -11, g. 0, h. -3, i. 4, j. 9

   b. 

The integers that are not whole numbers are called the negative integers, so the integers are divided into three parts:

The positive integers.

Zero, which is a whole number but not a positive integer.

The negative integers.

The fact that every integer is in exactly one of these three sets is called Trichotomy, since it cuts the integers into these three parts. The prefix "tri" means three.

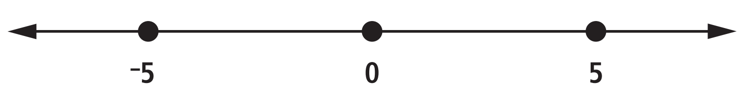
EXPLORATION: CONSTRUCTING A NUMBER LINE

1. Draw a straight line.
2. Pick a point on the line and call this point the origin. Label the origin with the number 0.
3. Locate the numbers 1, 2, 3, …, 10, and ⁻1, ⁻2, ⁻3, …, ⁻10.
4. Where would 20, 30, 50 be located? 100? 1000?
5. Find the negative numbers corresponding to the numbers in question 4.

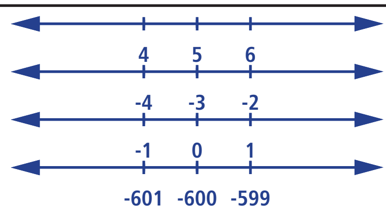
EXERCISES

1. At the zoo, the gift shop is located at the origin of Oak Street. Its address will be labeled as 0. Going right from the origin, the seal pool is located at address 8, and the monkey habitat is at address 4. Going in the other direction from the origin, the elephant habitat is located at address -3, and the lion den is at address -7. Draw a number line representing Oak Street. Label each of the locations on the number line. Watch your spacing. **See TE.**

2. a. Copy the line below to mark off and label the integers from 0 to 5 and from 0 to ⁻5. Use a pencil to experiment with the spacing because you might need to erase.



4. a.
b.
c.
d.

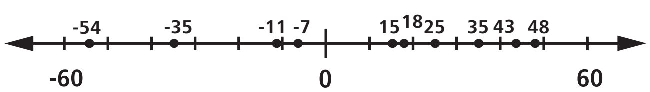


5. a. Students measure with a ruler or other instrument. Answers may vary but the aim is to get 30,30, yes.

b. Students measure with a ruler or other instrument. Answers may vary but the aim is to get 10, 10, 10, yes

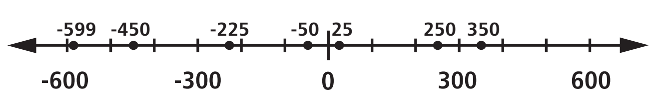
c. If students' measurements are not equal in part a and in part b, they need to rework their markings so that the numbers on the line are equal distances apart.

d.



6. a. Count by 100s

b.



b. Make a new number line from –10 to 10. Use what you learned about spacing to make it accurate.

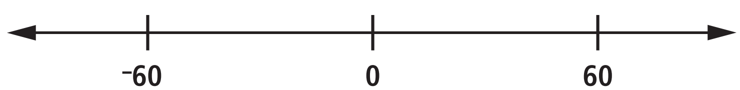
3. Write an integer for each situation below. Find the point on the number line that corresponds to the integer. (create a number line from -15 to 15, counting by fives but leaving a mark for each integer.)

a. Score 10 points | b. 8° below zero
c. A deposit of $12 | d. A gain of 7 pounds
e. 4 ft. below sea level | f. A debt of $11
g. Neither positive or negative | h. A withdrawal of $3
i. The opposite of -4 | j. Put in 9 gal

4. For each of the following integers, create a section of a number line to illustrate which integer is immediately before and which integer is immediately after the given integer. Be sure to label which is which.

a. 5 | b. -3
c. 0 | d. -600

5. Draw a number line like the one below to mark off the numbers with equal distances by tens from 0 to 60 and from 0 to – 60. Use a pencil to experiment.



a. Measure the distances from 0 to 30 and from 30 to 60. Are they the same? **50, 50. Yes**.

b. Measure the distances from 10 to 20, 30 to 40, and 40 to 50. Are they the same? **10, 10, 10. Yes.**

c. Explain whether you need to rework your markings on the number line.

d. Estimate the location of the following numbers and label each on your number line: **See TE.**

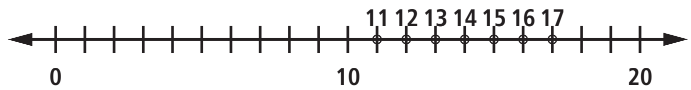
25, 15, -35, -7, -54, 43, 18, 35, -11, 48

6. Draw a number line so that the number -600 is at the left end, and 600 is on the right end.

a. What is the best amount to count by to ensure proper scaling?

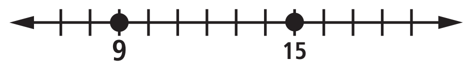
b. Sketch the locations of the following integers:

7.



8. Answers may vary.

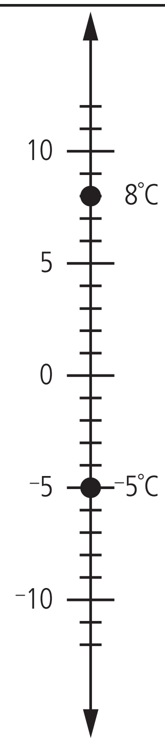
Ex. Since 15 is to the right of 9, 15 is greater than 9.



9. In order from coldest to warmest:

Jan, Dec, Feb, Nov, Mar, Oct, Apr, Sept, May, Aug, June, July

10.



-5 degree C is closer to the freezing point

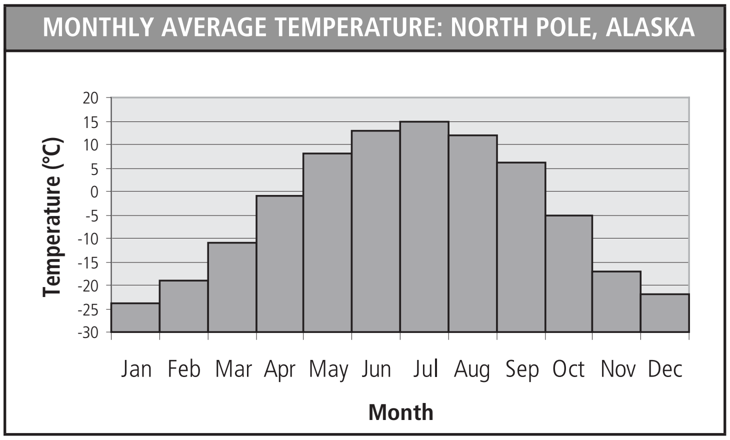
0, 350, ⁻225, 250, ⁻450, 25, ⁻599, ⁻50

7. Draw a number line. Find all the integers on your number line that are greater than 10 and less than 18. Circle the numbers you found.

8. How do you decide which number on a number line is greater? Draw 2 examples to explain your answer. **The number to the right is greater.**

Notice that we can move the number line from the horizontal position to a vertical position. We would then have a number line that looks like a thermometer. Draw a thermometer (vertical number line) on the side of your paper to help you answer questions 9 through 12.

9. The chart below shows the monthly average temperatures for the city of North Pole, Alaska (not the actual North Pole, which is farther north). Based on the data, put the twelve months in order from coldest to warmest. **See TE.**



10. Nicholas visited his cousin, Marissa, in Anchorage, Alaska where the temperature was ⁻5 °C. Michael visited his friend in Portland, Oregon where the temperature was 8 °C. Which temperature is closer to the freezing point? Draw a thermometer to prove your answer. Remember, when we measure temperature in degrees Celsius (°C), 0 °C is the freezing point of water.

11. The temperature in Toronto, Canada, one cold day is ⁻7 °C. The next day, the temperature is 5 °C. Which temperature is closer to the freezing point? Draw a thermometer to prove your answer.

12. -13˚C is colder by 4 ˚C.

Spiral Review 5.2 (C)

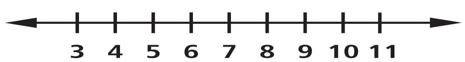
13. Answers will vary ($\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$)

Spiral Review 5.2 (B)

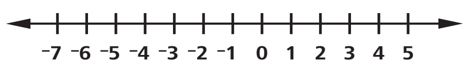
14. $\frac{11}{4}$

Ingenuity

15. a. 9 markings



b. 13 markings



c. 26 markings

d. (270-210) + 1 = 61 markings

e. (150 - -120) + 1 = 150 + 120 + 1 = 271 markings

Investigation

16. a. 12 b. 141 c. 7 d. -2 e. -7 f. 9 g. -8 h. -317

12. One cold day, the temperature in Oslo, Norway is -9 °C and the temperature in Stockholm, Sweden is -13 °C. Which temperature is colder? How much colder?

Spiral Review:

13. Name three fractions less than $\frac{2}{3}$.

14. Write $2\frac{3}{4}$ as an improper fraction.

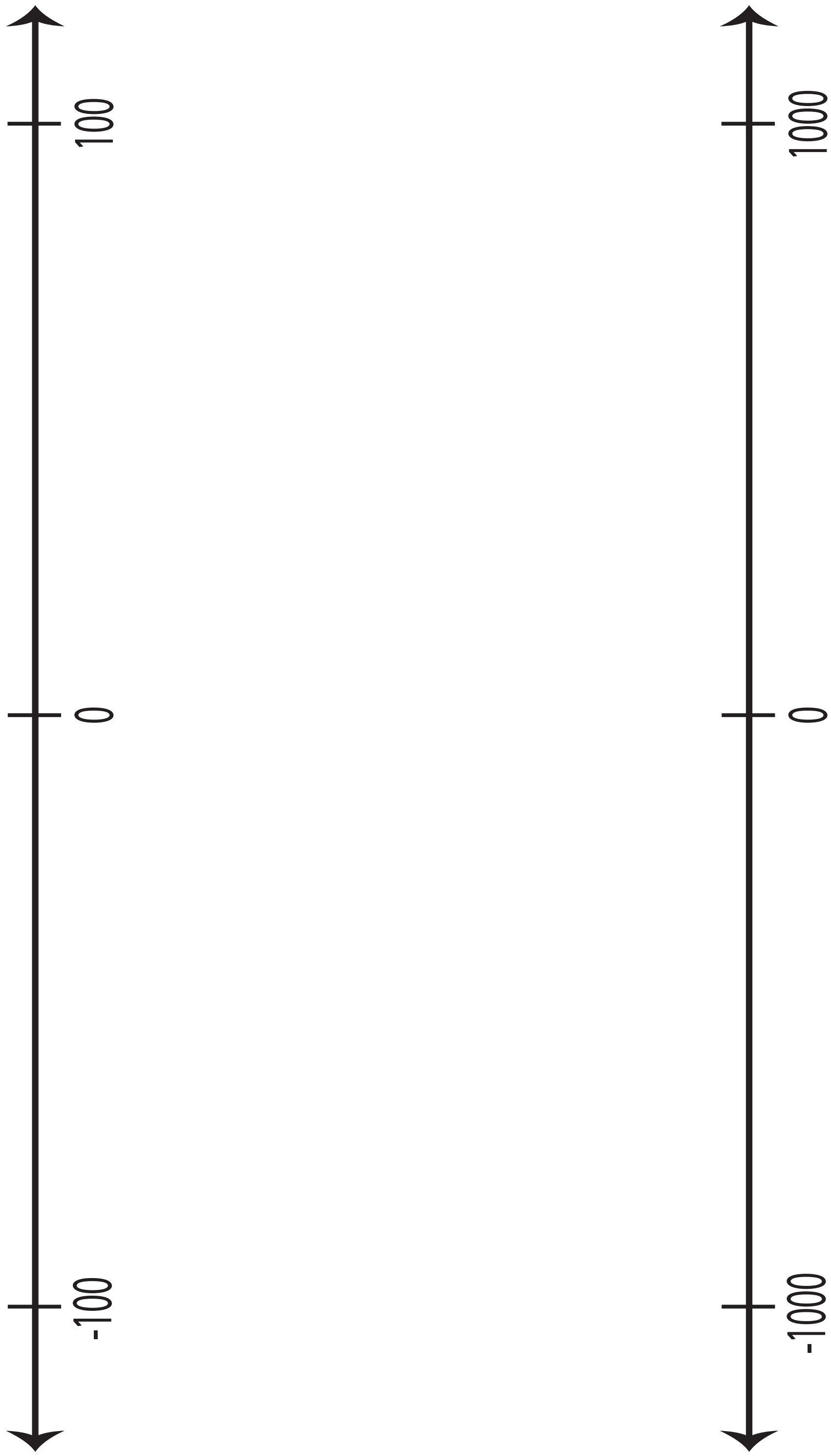
15. **Ingenuity:**
    a. Draw a number line, mark and label all of the integers from 3 to 11. How many integers have you marked?
    b. Draw a number line, mark and label all of the integers from -7 to 5. How many integers have you marked?
    c. Suppose we drew a number line, and marked all of the integers from 12 to 37. How many integers would we mark if we did this?
    d. Suppose we drew a number line, and marked all of the integers from 210 to 270. How many integers would we mark if we did this?
    e. Suppose we drew a number line, and marked all of the integers from -120 to 150. How many integers would we mark if we did this?

16. **Investigation:**

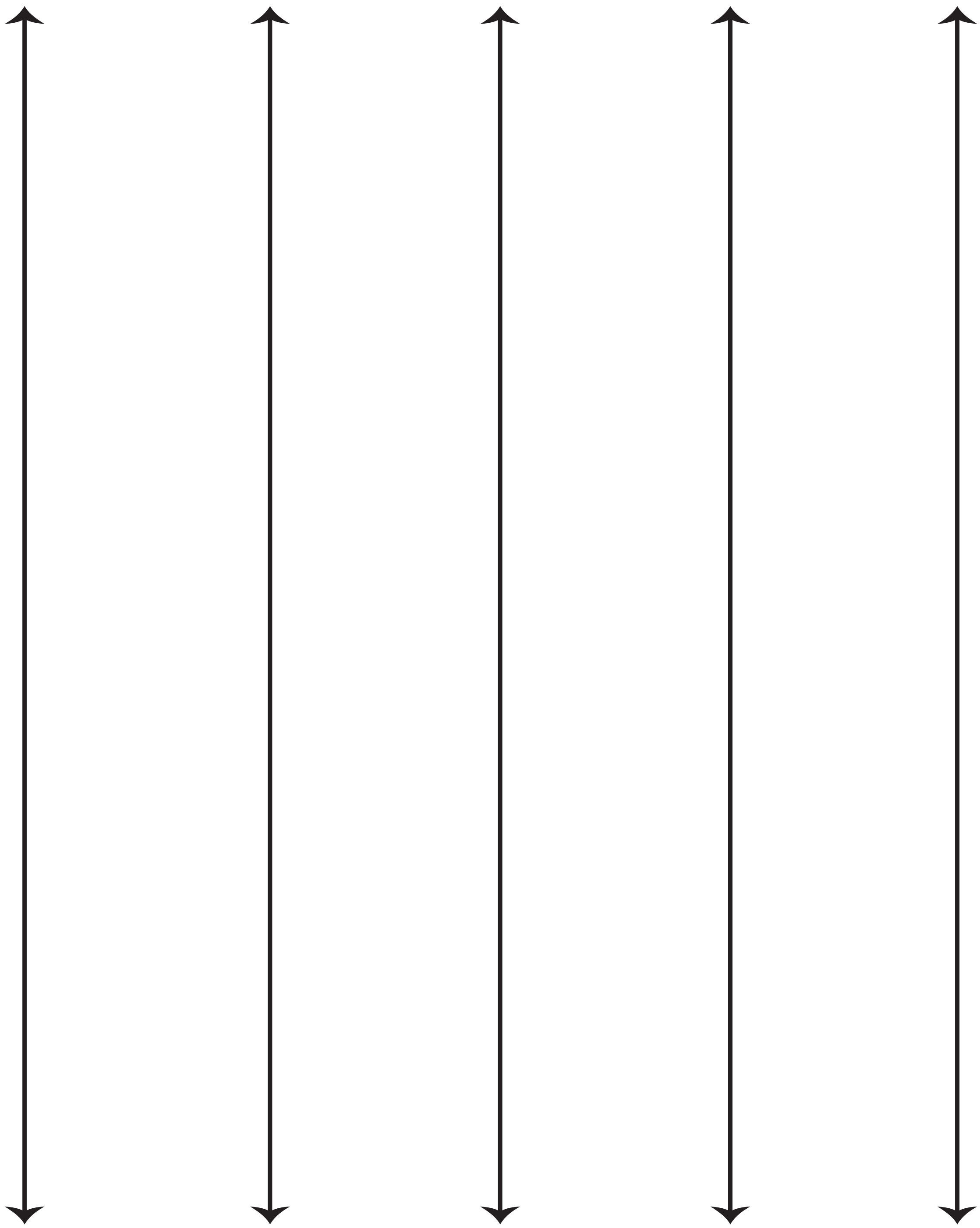
    For each of the following pairs of integers, decide which integer is further to the right on the number line.

    a. 5 and 12 **100 °C, 212 °F**
    b. 141 and 78 **0 °C, 32 °F**
    c. 7 and -1 **122 °F**
    d. -4 and -2
    e. -7 and -10
    f. 9 and -55
    g. -8 and -21
    h. -355 and -317 **Negative**.

Number lines to be used with Exercises 5 and 6



Blank Number Lines



Section 1.2 - Less Than and Greater Than

Big Idea:

Comparing and Ordering Integers

Key Objectives:

- Construct different types of number lines.
- Develop the linear model of numbers.
- Use Inequality Notation: $<$ and $>$.
- Define $a < b$: a is less than b, and a is to the left of b.
- Define $a > b$: a is greater than b, and a is to the right of b.

Materials:

Tape, Index cards, Deck of cards, Graph Paper, Rulers

Pedagogical/Orchestration:

- Prerequisites of this section are creating a number line: spacing/placing zero/positive & negative integers.
- Have students work out problems to compare which number is bigger than another. Example: Compare which is bigger. 7 or 2? 28 or 40? 13 or 25? Raise the question: "What do you mean by a number being bigger? What about 2 or -2? Both are 2's?" Have students re-evaluate the word "bigger" & change to "greater than." Help students tie "greater than or less than" to a relative position on the number line, so that a number to the left is always less than a number to the right on a number line.

Activity:

"Integer War"

Exercises:

Be sure to assign Exercises 2 , 3 and 4 so that patterns can be verbalized. Also make sure you assign Exercise 13 so that you can use this information to begin the next lesson, Section 1.3: Applications of the Number Line.

Vocabulary:

less than, greater than, numerical expression, variable, inequality

TEKS:

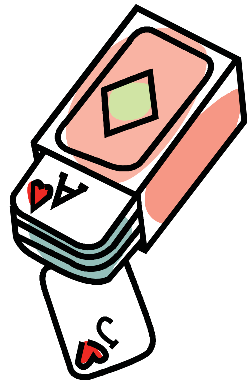
6.1(A,C); 6.8(B); 6.11(B)

New: 6.1A,D; 6.2C

Launch for Section 1.2:

To launch Section 1.2, students will be reviewing what they learned yesterday about number lines, and building on that concept. Ask students, "What are the important characteristics of a number line?" Put students' responses on the chalkboard or chart paper. Essential characteristics should include: numbers should be in order, numbers should be equally spaced, greater numbers are to the right of lesser numbers, there are arrows on each end of the line to show that the numbers extend infinitely in both directions, etc. It is important to locate the zero and identify 1 unit for scaling purposes. Assign students to groups and have each group make a number line on the floor using tape for the line and index cards for the numbers. The numbers should extend from -10 to 10 and be spaced at least one foot apart. Have groups compare the number lines from other groups to make sure they satisfy the essential criteria of the number line as outlined by the students at the beginning of class. Also check to see that the markings are symmetrical, for instance that 5 and -5 are the same distance from 0. If you see a number line that is not spaced properly, measure and cut a piece of string so that it is the length of the distance between two numbers like 3 to 5. Use the string length to compare equivalent distances on the same number line such as 7 to 9 or -3 to -1. This has a great visual impact and will encourage the students to correct their spacing. Tell students, "Today we will be using these number lines to model the concepts of greater than and less than."

Integer War



Objective: The students will work in pairs or small groups to practice comparing the values of certain integers.

Materials:
One complete deck of cards (including Jokers) for each group

Activity Instructions:
This card game is played like "War". The students will shuffle the deck and pass out all of the cards so that all players start with the same number of cards in their pile. Each student in the group will then take the top card from their pile and turn it face up. The students will decide which card has the highest value, and the winner will pick up all of the cards played and put them face down in the bottom of their pile. The game is called "Integer WAR" because if there is ever a tie, the players with the tied cards must announce "WAR". These two players will then count out 6 cards (the first 5 will be face down, the 6th and last will be turned face up). As they count out these six cards, they will recite the words "I Declare War On You". (There are six syllables in this sentence, and they lay down one card per syllable). The player with the highest value on the card that is face up wins the whole pile. If they are still tied, the students will declare war again and follow the same procedure until there is a winner for this round. The game continues until either 1 player has collected all of the cards, or until you designate when they will stop playing. If you stop the game before 1 player has collected all of the cards, the player with the most cards wins.

The values of the cards are as follows:
All red cards are negative.
All black cards are positive.
All aces are 1.
All jacks are 10.
All queens are 50.
All kings are 100.
All jokers are 0.

Directing Inequalities

Objective: The students will understand the relationship among numbers in terms of magnitude.

Materials:
Cards labeled from integers -10 to 10 and a "<" and a "=" card.

Activity Instructions:
Distribute an integer card to each student. If there are more than 21 students then create more cards with other numbers or the same number. Have two students go up with their cards. Select a student to then take either a < or = card and place it correctly between the two numbers. Ask the "middle" student to justify his/her decision. Confirm the choice with the whole class.

Have another pair of students put their cards and select another student to select the inequality/equality card and place it correctly.

Have one student stand at 5 and another at 2. Ask the class which is bigger (further along to the right). As they answer write "5 is greater than 2" and "$5 > 2$" on the board. Discuss this notation if necessary. Ask how else we might represent the relationship between 2 and 5. "2 is less than 5" and "$2 < 5$." Write these on the board and again go over the notation. Notice that the inequalities identify relationships between two numbers. In section 1.4, the students will be introduced to operations of two numbers. The students often have difficulty distinguishing the two. For example, 2 is less than 5 is a relation $2 < 5$ and is different from 2 less than 5 which is an operation $5 - 2$.

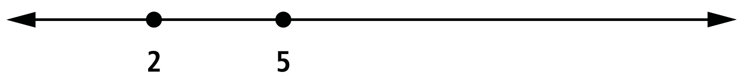
Next introduce the definition in box 1.4 and take as much time as needed to ensure the students understand x and y, less than and greater than, and the notation.

This may be the first time the students are seeing variables used. We should note that the variables used here are to talk in general about the relationship of number. So the x and the y are for a given number x and a given number y but that these numbers can be any integer for which the relationship makes sense. By that, $x < y$ works for say $x = 2$ and $y = 5$ but not if $x = 5$ and $y = 2$. We are not ready to generalize the notion of variables as we shall see with conditional equations when we solve equations.

If your students are having trouble figuring out which symbol to use, explain to them that the $<$ and $>$ symbols match the symbols at the ends of the number line. If they are asked to compare 2 and 5, they can just place these numbers on the number line and choose the symbol that is closest to 2 because 2 came first in the comparison, proving that $2 < 5$. Similarly, if they are asked to compare 5 and 2, they would choose the symbol that is closest to 5. Because 5 came first in the comparison, 5 must be greater than 2. Sometimes it helps the students understand the concept of these symbols if they have something visual to help it make sense.

SECTION 1.2 LESS THAN AND GREATER THAN

We say that 2 is less than 5 because 2 is to the left of 5 on the number line. "Less than" means "to the left of" when comparing numbers on the number line. We use the symbol "$<$" to mean "less than." We write "2 is less than 5" as "$2 < 5$." Some people like the "less than" symbol because it keeps the numbers in the same order as they appear on the number line.



We also say that 5 is greater than 2 because 5 is to the right of 2 on the number line. "Greater than" means "to the right of" when comparing numbers on the number line. We use the symbol "$>$" to mean "greater than", so we write "5 is greater than 2" as "$5 > 2$." Just as "$=$" is the symbol for equality, "$<$" and "$>$" are symbols for **inequalities**.

DEFINITION 1.4: LESS THAN AND GREATER THAN
Suppose x and y are integers. We say that x is **less than** y, $x < y$, if x is to the left of y on the number line. We say that x is **greater than** y, $x > y$, if x is to the right of y on the number line.

Here x and y are called **variables**, which we will formally introduce in the next section. Variables give us a simple way to describe math objects and concepts. In this case, x and y represent two integers, and the way that we tell which is greater is to compare their positions on the number line. The two number lines below demonstrate the cases $x < y$ and $x > y$. Can you tell which is which?

EXAMPLE 1

When ready to begin Example 1, ask students to represent (a) 3 and 7 and ask the class which is greater and which is less. Have students write these symbolically on the board and in their notes. Discuss how they know. How does the number line help? Continue with parts (b), (c), and (d).

a. $3 < 7$ c. $^-1 > {}^-5$
b. $^-2 < 9$ d. $4 > {}^-4$

Teachers, what do your students know about the relationship between any positive number and any negative number? Do they recognize that positive numbers are always greater than negative numbers?

What about the relationship between zero and the positives and negatives? Zero is always less than any positive number and greater than any negative number.

What about the order of negative numbers? Ask students to think about how it is counterintuitive, by that -100 <-2 though the one hundred seems “big” compared to two.

ACTIVITY
Play the **Integer War** activity at the end of this section for more practice on this important concept.

EXAMPLE 2

For Example 2, ask the class to determine the order from least to greatest of the numbers given. Give them a few minutes to think and record a response. Have students come forward to represent each number. When in position, have the student that is leftmost of zero (-5) identify him/herself as least, and continue to the student at position 7. Ask students how they know -5 is least and 7 is greatest of those two numbers. Emphasize (if needed) that when ordered along a number line, you can easily tell which integers are least, greatest, and everything in between.

Also, if $a < b$ and $b < c$, then $a < c$, and if $a > b$ and $b > c$, then $a > c$.

For the number line in the solution for Example 2, notice that we didn’t label all of the integers on the number line this time. It is common practice not to write all of the numbers on the diagram if you don’t need them and if doing so would make the picture look cluttered.

Talk to students about determining how to label the numbers on a number line. Talk about the multiples of five or ten as possibilities.

EXAMPLE 1

For each pair of integers below, locate them on a number line to determine which one is greater and which one is smaller. Express your answer as an inequality of the form $x < y$, where x and y are the given integers.

a. 3 ☐ 7
b. -2 ☐ 9
c. -1 ☐ -5
d. 4 ☐ -4

SOLUTION

a. Begin by drawing a number line from -10 to 10. Using this number line, we see that 3 is to the left of 7, so $3 < 7$.

b. We observe that -2 is to the left of 9 on the number line, so $-2 < 9$. We can also see this in a different way: We know that -2 is to the left of 0 because -2 is negative, and 0 is to the left of 9 because 9 is positive. Thus, -2 must be to the left of 9, and we have $-2 < 9$.

c. We notice that -5 is to the left of -1, so $-5 < -1$ or $-1 > -5$.

d. Because -4 is to the left of 0, and 0 is to the left of 4, we have $-4 < 4$ or alternatively, $4 > -4$.

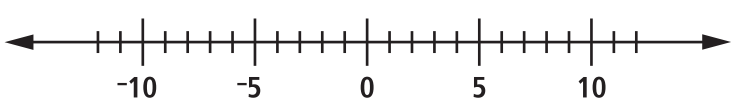
EXAMPLE 2

Use the number line to put the following integers in order from least to greatest:

2, -2, 7, -1, -4, -5, 4, 6, 3

SOLUTION

Again, we can use the number line to help us put the integers in order:



We can locate our nine given integers on the number line. You might try doing this by copying the number line above, and labeling the given numbers on your number line. After comparing the nine numbers given, we get the following order:

-5, -4, -2, -1, 2, 3, 4, 6, 7

Summary:

Before they begin the exercises ask the students to explain again how they know if numbers are less than, greater, or equal to one another.

A final activity might be to ask each student in order around the room to volunteer an integer, you might have students write their integer on their own personal white board or an index card so they can order them more easily. Record these as you go. Then have the students order these from least to greatest. Encourage students to volunteer negative numbers if they are not doing so. If no one offers 0, then you can provide it as your example integer. If students volunteer really big numbers or small numbers (1,000 or -1,000) ask them to estimate where those might be if the class number line were to continue past the classroom.

PROBLEM: -11, -6, -3, -1, 0, 2, 5

EXERCISES

1. a. $8 > 5$
   b. $3 < 9$
   c. $-4 > -7$
   d. $^-3 < 2$
   e. $6 > 0$
   f. $^-5 < {}^-3$

2. a. $5 > 3$; $3 < 5$
   b. $0 < 2$; $2 > 0$
   c. $^-5 < 0$; $0 > {}^-5$
   d. $^-4 > {}^-5$; $^-5 < {}^-4$
   e. $^-6 < 7$; $7 > {}^-6$
   f. $3 > {}^-2$; $^-2 < 3$

3. a. $7 > 3$; $^-7 < {}^-3$
   b. $^-8 < {}^-4$; $8 > 4$
   c. $5 > 0$; $^-5 < 0$
   d. $^-4 < 0$; $4 > 0$
   e. $^-2 < 3$; $2 > {}^-3$
   f. $6 > {}^-6$; $^-6 < 6$

4. In Exercise 2, the students should observe that when the order in which two number are written changes, then the inequality changes from greater than to less than or visa versa. The relationship between the two numbers, however, has not changed; that is "3 is less than 4" is merely a restatement of "4 is greater than 3" and 3 is still to the left of 4.

   The pattern in Exercise 3 is to observe that when one examines the relationship between the inequality between a pair of numbers a and b to its negatives, ^-a and ^-b that the inequality changes signs. For example, in the first relationship between 7 and 3, we have 7 is greater than 3. However when we look at $^-7$ and $^-3$, we see that $^-7$ is less than $^-3$. The students should observe this on the number line and the relative position of the paired numbers. We do not expect the students to conclude that when one multiplies an inequality, $a < b$ by a negative number such as $^-1$, that $^-a > {}^-b$. This exercise foreshadows this property but we do not expect to formalize this relationship here.

   You might ask your students to remember that "negative" also means "opposite." We will be careful when the operation subtraction is considered with the use of the word "negative".

PROBLEM

Put the following integers in order from least to greatest:

-3, -1, 0, -11, 2, -6, 5

EXERCISES

1. Rewrite each of the following as a statement using < or >. Compare your statements to the relative locations of the two numbers on the number line. Example: -3 is less than 8. $-3 < 8$.

   a. 8 is greater than 5.
   b. 3 is less than 9.
   c. -4 is greater than –7.
   d. -3 is less than 2.
   e. 6 is greater than 0.
   f. -5 is less than -3

2. Compare the numbers below and decide which symbol, < or >, to use between the numbers. In each, show the relationship of these numbers on a number line.

   a. 5 $\square$ 3
      3 $\square$ 5
   b. 0 $\square$ 2
      2 $\square$ 0
   c. -5 $\square$ 0
      0 $\square$ -5
   d. -4 $\square$ -5
      -5 $\square$ -4
   e. -6 $\square$ 7
      7 $\square$ -6
   f. 3 $\square$ -2
      -2 $\square$ 3

3. Compare the numbers below and decide which symbol, < or >, to use. In each, show the relationship of these numbers on a number line.

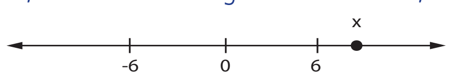
   a. 7 $\square$ 3
      -7 $\square$ -3
   b. -8 $\square$ -4
      8 $\square$ 4
   c. 5 $\square$ 0
      -5 $\square$ 0
   d. -4 $\square$ 0
      4 $\square$ 0
   e. -2 $\square$ 3
      2 $\square$ -3
   f. 6 $\square$ -6
      -6 $\square$ 6

4. Describe any patterns you see in Exercises 2 and 3. For example, I noticed that larger positive numbers are to the right of smaller positive numbers on a number line.

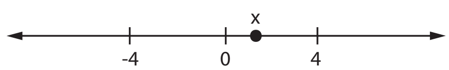
5. a. $6 > 5$
   $5 < 6$
   b. $3 > -1$
   $-3 < 1$
   c. $-7 < 3$
   $-3 < 7$
   d. $0 < 4$
   $-4 < 0$
   e. $2 < 8$
   $-2 > -8$
   $-2 < 8$
   $2 > -8$

6. 5, 6, 7

7. a. If an integer is greater than 6, then it is to the right of 6. Therefore, it is certainly to the right of 6.
   True



   b. Suppose the integer is 1. We know that 1 is certainly less than 4 (it is positioned to the left of 4), but is to the right of -4, 1 > -4.
   False

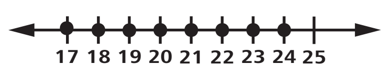


   Of course, if one had considered -5, then it is true that -5 is both less than 4 and less than -4 and it would satisfy the statement, "If an integer is less than 4, then it is less than -4." However, in mathematics, a statement is considered true if it is true all the time and otherwise, we say the statement is false. It is interesting for students to be able to note that there are conditions when the statement is true, that is if the chosen number is less than -4 but is not true if the chosen number is less than 4 but greater than or equal to -4.

8. The temperature fell 4 °C

9. Mr. Canales, at 34 steps above the ground, is farther from the ground.
   Mr. Garza is only 15 steps from the ground.
   In fact, Mr. Canales is 34 - 15 = 19 steps farther from the ground than Mr. Garza.

10.



5. Compare the numbers below and decide which symbol, < or >, to use. Use your rules from Exercise 4 to help you.

a.	6	$\square$	5	d.	0	$\square$	4
	5	$\square$	6		$^{-}4$	$\square$	0
b.	3	$\square$	$^{-}1$	e.	2	$\square$	8
	$^{-}3$	$\square$	1		$^{-}2$	$\square$	$^{-}8$
c.	$^{-}7$	$\square$	3	f.	$^{-}2$	$\square$	8
	$^{-}3$	$\square$	7		2	$\square$	$^{-}8$

6. What are the possible values for an integer that is greater than 4 and less than 8? Circle these values on the number line.

7. Determine whether each of the following statements is true or false. Explain your answers.
   a. If an integer is greater than 6, then it is greater than $^{-}6$.
   b. If an integer is less than 4, then it is less than $^{-}4$.

8. In Green Bay, Wisconsin, the morning temperature is $^{-}5$ °C. In the evening, the temperature reads $^{-}9$ °C. Did the temperature rise or fall? Draw a thermometer to show how much the temperature rose or fell.

9. Mr. Canales is on a flight of stairs 34 steps above the ground. Mr. Garza has gone into the basement of the same building and is 15 steps down from ground level (let's call it the $^{-}15^{th}$ step). Who is farther from ground level? How do you know?

10. Luis had some jellybeans. He has at least 17 jellybeans and fewer than 25. How many jellybeans could he have? Give all possible answers

Sprial Review 5.3(B)

11. Jim worked for 3 days totaling 24 hours and Sally worked 4 days, totaling 20 hours.
    So, Jim worked more hours last week.

Sprial Review 5.10(A)

12. Jose walked 132 yards. $\frac{396 \text{ ft}}{1} \cdot \frac{1 \text{ yd}}{3 \text{ ft}}$

Ingenuity

13.

3	>	2	>	1	<	4
∧		∧		∧		∨
4	>	3	>	2	>	1
∨		∧		∧		∧
1	<	4	>	3	>	2
∧		∨		∧		∧
2	>	1	<	4	>	3

Spiral Review:

11. Jim worked 8 hours each day on Monday, Tuesday, and Wednesday last week. Sally worked 5 hours each day on Monday, Tuesday, Wednesday, and Thursday. Who worked more hours last week?
12. Jose walked from his house to school. It was a distance of 396 feet. How many yards did Jose walk from his house?
13. **Ingenuity:**

    Fill in the boxes in the grid below with the numbers 1 through 4 so that each row contains all of the numbers from 1 to 4, each column contains all of the numbers from 1 to 4, and the numbers obey the inequality signs in the grid.

	>		>		<	
$\wedge$		$\wedge$		$\wedge$		$\vee$
	>		>		>	
$\vee$		$\wedge$		$\wedge$		$\wedge$
	<		>		>	
$\wedge$		$\vee$		$\wedge$		$\wedge$
	>		<		>	

INVESTIGATION

14. Teachers, this problem is necessary for the **Exploration: Constructing a Number Line** in Section 1.3.

Challenge Problem

15. Careful, the question does not address how many different combinations are possible, only the possible values of each number. So, *a* could be 1-7, *b* could be 2-8, and *c* could be 3-9, making sure to keep the order as defined by the problem.

14. **Investigation:**
    a. Ask your family for some important events and the years they occurred in your family's history, such as the year someone was born or married. Go back as far as you can, for instance the year one of your grandparents was born. Find at least 10 events and their dates.
    b. Make a list of important historical events, people or discoveries and try to find the year they occurred, lived or appeared in history over the last three thousand years. You could also pick a time in history that you would like to visit. Find at least 10 events and their dates.

15. **Challenge Problem:**

    If a, b, and c are positive integers such that $a < b < c < 10$, how many possible values are there for the three numbers?

Section 1.3 - Applications of the Number Line

Big Idea:

Applying number lines to timelines and other applications

Key Objectives:

- Construct number lines with bigger numbers.
- Understand timelines as a form of number lines.
- Use appropriate scales.
- Compare integers.
- Determine distances between numbers on number lines.
- Apply number lines to altitude, temperature and maps.

Materials:

Colored tape, Index cards, Sentence strips, Grid paper, Rulers

Pedagogical/Orchestration:

- Have a class discussion on real life situations such as altitude, temperature, sea level, etc.
- Discuss the relationship between timelines and number lines. Students should be guided to the understanding that timelines are special types of number lines. Historic time lines use B.C.E and A.D. in place of the negative and positive signs. Smaller range time lines may have a specific date set up as the origin and use a symbol like B (for before) to indicate dates occurring before the origin. For instance, the year that is 3 years before the origin can be indicated as B3 or -3.

Activities:

Construct a Historical Timeline: See Class Exploration (Can be connected to a Social Studies lesson.)

"Family Time Line" - This activity can be an extension or parallel to Historical Timeline.

Exercises:

The exercises will begin to ask questions related to distance from 0. The idea of absolute value will be formally introduced in 7th grade, so hold off on mentioning it just yet. If needed you might suggest they count along a number line to determine distance between locations.

Exercises 10, 12, 13 involve foreshadowing of absolute value and subtracting integers.

Vocabulary:

timeline, scaling

TEKS:

6.1(C); 6.11(A); 6.12(A,B); 6.13(A,B) New: 6.1(A, C, E); 6.2(C, E)

Launch for Section 1.3:

Ask the students to get out their important family dates they gathered from the Investigation in Section 1.2 as well as the dates of major historical events. Ask different students to share a few dates they learned (weddings, births, moving to a new town, started middle school, etc.) Have some important dates from your family already posted somewhere: chart paper, side of the chalkboard, etc. Make sure that some are spread apart in time and some are more recent. (An example of a historical date that amuses students is for the teacher to ask them what important historical event happened in _____. After listening to student responses, the teacher can say, "The important historical event that happened that year is I was born!") Once the dates are discussed ask students, "What would be a way to organize these dates?" Accept most answers and listen to see if anyone mentions a number line. Tell students, "Today we will be organizing dates like this on a special number line called a timeline."

Note: Elementary students in Europe use graph paper for all their math work. They claim it keeps numbers in the correct columns and is very helpful for number lines. Grid paper is useful for maintaining place value in computation and for constructing number lines. At this time you may make grid paper available for students.

Family Time Line

Objective: The purpose of this activity is to have kids use their own experiences, and experiences of their family, to more deeply understand the concept of the number line.

Materials:
Plain white paper, cut in strips and taped together (the size will depend on the needs of individual students)
Markers, colored pencils, crayons
Ruler, or any other straight edge

Activity Instructions:

1. Explain to your class that they are to start interviewing their family members about significant family events that occurred before they were born and after they were born. Each student should to try to get a minimum of 10 events for each. The students will need to know the exact date that these particular events occurred.
2. After compiling this information, the students will organize this information on a number line. The center of the number line, which is usually zero, will represent the day the student was born. All events occurring before their birth will be placed on the number line to the left of the center, and all events occurring after their birth will be placed on the number line to the right of the center.
3. Be sure that your students understand that their time line, like all number lines, must be equally spaced, numerically accurate, legible, and neat. It would probably be a good idea if all students made a rough draft of their ideas before actually creating the time line on their white paper strips.
4. After the time line is created and has been checked for accuracy, this is when they can begin adding color and interest to their final draft.
5. I highly suggest that you, the teacher, create a family time line too. If you do yours in advance, you can use it as a model for your class. If you do yours at the same time as the students, then don't forget to share your final product with them at the end of the activity.
6. As a modification for kids that are struggling, you might want to suggest that they only represent their events in the YEAR that it occurred. Kids that are stronger in math should have no problem dividing their time line into equal intervals using both months and years. Your top students may even find it fun to be more accurate by dividing the time line into days, months and years.

Begin by asking your students to help you create a timeline or a number line of your family history. Tell them that you only want it to go back 100 years and that we can mark today or this year as 0. For now this timeline will be all negative (to the left), but leave some space to the right. Ask them what an appropriate spacing might be. Tell them that this is a scale. A suggestion of ten years might be best. Have someone come fill in the ten year marks on your timeline. That is if this year is 2008 then ten years ago was 1998, then 1988, then 1978, etc. until you get to 1908 or $^{-}100$ years. Finish by having the class help decide how to fill in your important dates with global events above and local or family events below the timeline.

When ready, tell them that like most people you have future goals so you want to include a future part of this timeline and you want it to go forward 50 years. Draw this and ask them what calendar year corresponds to 50 years from now. For instance, if this year is 2008, then 2058. Ask them to help you brainstorm some important future dates, like possible weddings, births or events related to your children, completing a masters degree, sibling events, moving to a new town, and attending their high school graduation.

Activity: Part of Exploration or Follow-up Project

Have the students now complete a timeline for their important family dates and future expectations using the important events and dates from the investigation in Section 1.2. The "Family Time Line" activity plan is at the end of this section. When completed these can be hung on a bulletin board or along the walls. Tell them they can take them home to decorate or to add family photos.

When ready ask each student on a new sheet of paper to draw a timeline that represents 3,000 years back into history and 1000 years forward. Ask, "How many years is that in all? What scale would be reasonable?" Remember that 0 is this year.

CLASS EXPLORATION: Extending our Timelines

What scale should we use?

approximately 650 years

SECTION 1.3 APPLICATIONS OF THE NUMBER LINE

When we make a number line, we place marks on it to denote different locations. We then label these marks with integers. The distance between these marks may represent more than one unit. If the numbers we want to represent on the number line are very large, we may wish to use each mark to represent ten units or perhaps even a hundred units instead of one unit. We decide the scaling by the size of the numbers we want to represent on the number line.

EXPLORATION: CONSTRUCTING A TIMELINE

In this activity, we will construct a special kind of number line called a **timeline**. Let's begin by building a timeline that goes back 100 years and forward 50 years. The first step is to draw a number line and label the origin with 0. The origin corresponds to the present year. Write the year above the line and the length of time from this year (our zero year) below the same mark. We want to label our timeline so that years that have already passed are labeled with negative numbers and years in the future are labeled with positive numbers. What scale should we use? In other words, how many years will the distance from 0 to the first mark represent? How many marks did you decide to use on your timeline? Make sure you can fit in all 150 years. Plot the special dates that you gathered from home in the Investigation from the previous section.

Next, make a timeline that goes back three thousand years and forward in time one thousand years. How long is the total span of this timeline?

CLASS EXPLORATION: EXTENDING OUR TIMELINES

Make a timeline that charts American history starting with Columbus landing in America in 1492 and continuing until the present day. Remember to think about how many years we will have to fit onto our timeline. What scale should we use?

It is important to talk about changing scales for smaller measures as well. Let's say that we wanted to display the heights of the students in our class on a number line. Would all of our heights be whole numbers? Because the answer is "No," what scale could we use to accurately display all of our heights on a number line? You might refer to a ruler and talk about the scale differences between the centimeter measures and the inch measures. After a thorough discussion about these small measurements, you can talk about something like population. What is a good scale to accurately display different populations, whose numbers are large, on a number line?

Create a model of the timeline and illustrate. Work as a class. Pick one for time considerations. Allow use of various technologies as appropriate. Consider collaborating with social studies and science teachers. Must fit 4700 years onto time line; example: 1cm = 500 yrs

Make sure all your students know that B.C.E. means "before the Common Era" and is the same as the old B.C. These years work just like negative integers.

Summary:

To summarize, ask the class to share some of their timelines, where 0 is, and how students decided on appropriate scales. Ask if anyone has a general strategy for deciding on a scale.

Chart important dates on the timeline. Here are some suggestions:

- First moon landing: 1969
- Pearl Harbor: 1941
- Henry Ford builds his first car: 1893
- The beginning of the Civil War: 1860
- The end of the Civil War: 1865
- Women given the right to vote: 1920
- Declaration of Independence: 1776
- Founding of Ysleta, Texas: 1682
- Hernando Cortes leads an expedition to Mexico: 1519
- The beginning and end of interesting eras. For instance, when was the Wild West?

Next, make another timeline that charts important events in world history. Start with the completion of the first pyramid (2690 B.C.E.) and continue until the present day. How many years will we have to fit onto this timeline? What scale should we use?

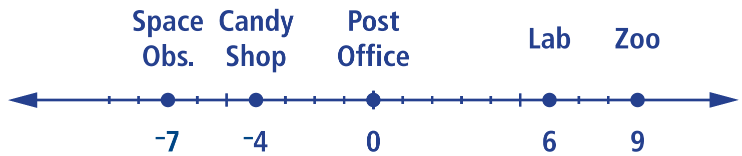
Chart important or interesting dates on this timeline. Here are some suggestions:

- Trojan War: 1250 B.C.E.
- Sacking of Rome: 410 C.E.
- Marco Polo travels to China: 1300 C.E.
- Code of Hammurabi: 1790 B.C.E.
- American Revolution: 1776 C.E.
- Signing of the Magna Carta: 1215 C.E.
- Dictatorship of Caesar in Rome: 46 C.E.

Exploration 2:

Note that the students may have counted from one location to another location. Another student may have subtracted. For example, in the problem: Find the distance between 7 and 2 the students may either count from 2 to 7 and find 5 or 7 - 2 = 5. However, in finding the distance between -7 and -2, the students will have a hard time doing the subtraction, -7 - (-2) and connecting to its absolute value and the subtraction of a negative number. Counting up or down is sufficient at this time.

1. a. **3 units**



   b. **3 units**

   Post to laboratory = 6 units
   Post to zoo = 9 units
   Post to candy shop = 4 units
   Post to observatory = 7 units

   c. **13 units**
   d. **13 units**
   e. **16 units**
   f. **10 units**

2. To find distance between two locations on a number line, you can count (this is what students generally conclude), or subtract the coordinates and take absolute value.

EXERCISES:

1. Pythagoras; because -540 < -240

2. Mission; 5 ˚F

EXPLORATION 2: DISTANCES ON MAIN STREET

Draw a number line, and label it Main Street.

a. Plot the following locations on Main Street: the laboratory at address 6, the zoo at address 9, the candy shop at address $^-4$ and the space observatory at address $^-7$.

b. What is the distance of each location from the post office, which is located at 0?

Using your number line for Main Street from the previous exercise, find the following distances:

a. The distance between the laboratory and the zoo.

b. The distance between the space observatory and the candy shop.

c. The distance between the zoo and the candy shop.

d. The distance between the space observatory and the laboratory.

e. The distance between the space observatory and the zoo.

f. The distance between the laboratory and the candy shop.

Discuss how you determine the distance between two locations on the number line.

EXERCISES

Use graph paper to draw a number line for each exercise.

1. In the year 540 B.C.E., Pythagoras, a Greek philosopher and mathematician, formulated a theorem still used today. The theorem is called the Pythagorean Theorem in his honor. Archimedes, another famous Greek mathematician, worked on a variety of problems. In 240 B.C.E., he developed formulas for the area and volume of a sphere. Which discovery occurred earlier in history? How did you decide?

2. The temperature in McAllen, Texas, on a hot summer day is 98 ˚F. The temperature in neighboring Mission, Texas, is 103 ˚F. Which city has the hotter temperature? How many degrees hotter is that city?

3. Cave Abby is 88 feet further from the surface. -413 < -325.

4. Joseph is 512 feet from the surface while Aaron is 128 feet from the surface.

5. The person exploring the cave is closer to the surface. 436 < 463

6. a. -1
   b. 1
   c. 2
   d. 7
   e. -7

7. -3, -2, -1, 0, 1, 2, 3, 4

8. 6, 6

9. -2

10. a. 5 b. 2 c. 7 d. 5

Spiral Review 5.1 (B) and 5.2 (A)

11. .05, .5, .55, 5

12. $\frac{1}{3} = \frac{3}{9}, \frac{2}{6}, \frac{4}{12}$

3. Cave Travis is located 325 feet below the surface of the ground, and Cave Abby is located 413 feet below the surface. Which cave is farther from the surface of the ground? Explain your answer.

4. Joseph is in a cave 512 feet below sea level and directly below Aaron. Aaron is on a hill 128 feet above sea level. Who is farther from sea level? You may wish to use a number line model.

5. A pilot in a helicopter hovers 463 feet above the surface. There is a person exploring in a cave 436 feet below the surface. Who is closer to the surface? Explain your choice.

6. Suppose time is measured in days, and 0 stands for today. What number would represent

   a. yesterday?
   b. tomorrow?
   c. the day after tomorrow?
   d. a week from today?
   e. a week ago?

7. On a cold day in Boston, Massachusetts, the temperature reaches a low of $^{-}4$ °F. The high temperature that day was 5 °F. What are some other temperatures that were reached in Boston that day?

8. What is the distance from 8 to 2? What is the distance from 2 to 8?

9. Which of the following integers is farthest from 12: 7, 9, $^{-}2$, or 23? Explain the reason for your decision by drawing a number line.

10. Find the distances between the following pairs of numbers on a number line.

    a. 7 and 2
    b. $^{-}4$ and $^{-}2$
    c. 6 and $^{-}1$
    d. $^{-}3$ and 2

Spiral Review:

11. Order the following decimals from least to greatest:

    0.5, 0.05, 5, 0.55

12. Darius ate $\frac{1}{3}$ of the pie his mother baked. Write 3 fractions equivalent to $\frac{1}{3}$.

INGENUITY:

13. a. 0
    b. A, D, B, C
    c. 40

INVESTIGATION:

14. a. -3, 3
    b. -8, 8
    c. -67, 67
    d. 4, 10
    e. 1, 9
    f. 1916, 204
    g. 64

CHALLENGE PROBLEM:

15. Saturday

13. **Ingenuity:**

Suppose that A, B, C, and D are points on a number line. The distances among the points A, B, C, and D are given in the following table:

	A	B	C	D
A	X	42	55	15
B	42	X	13	27
C	55	13	X	?
D	15	27	?	X

a. If we wanted to put numbers in place of the X's in the table, what numbers should we put?

b. Assuming that A is to the left of B, list the four points in order from left to right.

c. What is the distance between C and D?

14. **Investigation:**

Draw a number line , and mark the integers from -10 to 10. Use your number line to answer the following questions:

a. Name two integers that are 3 units away from 0.

b. Name two integers that are 8 units away from 0.

c. Without using a number line, name two integers which are 67 units away from 0.

d. Name two integers that are 3 units away from 7.

e. Name two integers that are 4 units away from 5.

f. Without using a number line, name two integers which are 44 units away from 1960

g. The integers 45 and 83 are both the same distance away from a certain integer. What is this integer?

15. **Challenge Problem:**

What day of the week was January 1, 2000?

Section 1.4 - Distance Between Points

Big Idea:
Understanding and using absolute value

Key Objectives:
- Define absolute value as distance from origin (magnitude).
- Find numeric distances between integers by counting from one to the other.
- Compare absolute values.

Materials:
- Rulers
- Number Line Handouts at the end of the section or the CD
- White boards for students (optional)

Pedagogical/Orchestration:
- Absolute Value prerequisite is the importance of spacing from Section 1.1.
- Be aware that some students confuse absolute value with opposite of a number.
- Before the students start the exercises, it might be a good idea to go through a few examples together. It is really important to ask the class "What is the absolute value of zero?" and make sure they know the correct answer and more importantly know the reason why their answer is correct.

Internet Resources:
Battleship Game: Integers Review- http://www.quia.com/ba/164425.html
Jeopardy Game: Integers Review- http://www.quia.com/cb/64603.html
Rags to Riches: Integers Review- http://www.quia.com/rr/41496.html

Activity:
"Absolute Value Bingo" Activity at the end of the section and on the CD

Exercises:
Exercises 7-9 foreshadow subtraction.

Exercises 10-12 foreshadow addition, and would be nice problems to refer to when introducing Section 1.5 on addition of integers.

Vocabulary:
absolute value, magnitude

TEKS:
6.13(B); 7.1(A); 7.13(A,C); 7.14(A); 7.15(A) New: 6.2(B, C); 6.1(C, D)

WARM-UPS for Section 1.4

1. Which of the following fractions is between $\frac{2}{3}$ and $\frac{3}{4}$? Explain why.

   a. $\frac{3}{5}$ b. $\frac{4}{5}$ c. $\frac{7}{10}$ d. $\frac{1}{2}$

2. Polly Nomial's science class wants to make a thermometer, but they have learned from experience that it is better to make a sketch before they begin construction. The class wants to include the following temperatures on their thermometer: -23ºF, 7ºF, 38ºF, and -9ºF (it's a thermometer for a cold place). Sketch a design of a thermometer that is practical for these temperatures. Do not label every degree. **Ans: (Possible answer) Range from -60ºF to 60ºF and label every 5ºF.**

3. Look at the table below and answer the questions:

Element Name	Approximate melting point in degrees Celcius
Hydrogen	-259
Fluorine	-220
Radon	-71
Mercury	-39
Bromine	-7
Francium	27
Rubidium	40
Sodium	98
Tin	232
Neon	248
Silver	961
Tungsten	3422

   a. How much hotter is the melting point of Francium than Bromine?
   b. How much hotter is the melting point of Rubidium than Mercury?
   c. How much hotter is the melting point of Sodium than Radon?
   d. How much hotter is the melting point of Tin than Fluorine?
   e. How much hotter is the melting point of Silver than Neon?
   f. How much hotter is the melting point of Tungsten than Hydrogen?

Launch for Section 1.4

Ask the students to describe how they know whether 7 or -7 is greater. They should respond that 7 is greater because it is further to the right of zero. Tell students, "Today we are going to be talking about the size or magnitude of a number rather than its direction from zero." At this point, lead the students through the measuring activity using the Number Line Handouts.

Pass out the Number Line Handout. Assign students pairs of numbers such as 1,-1; 2,-2; 3,-3; etc. Have students use rulers to measure the distance from 0 to each of their numbers. For example, 5 is 5 cm from 0 and –5 is also 5 cm from 0. Ask them to share their results and generalize. They should discover that a number and its negative are the same distance from 0. Also, this distance is non-negative as are all distances. Point out that yesterday the class was concerned with relative position (left/right of 0). Today we are working on distance from 0 and less concerned with left/right. After the students have tried out some number pairs and discovered that both a number and its negative (i.e. –11 and 11) are the same distance from 0, then tell them that this distance is called the absolute value. Also show them the notation.

Ask the students to share an integer (try to encourage a mix of negatives and positives and zero). Record their integers. Choose a few and have the class determine the absolute value of that integer. Reinforce the notation by writing it.

Teacher Tip: Use personal white boards so that students can work through the process of the notation, maybe show a number line to prove the distance.

Next choose a few pairs and have the class determine which absolute value is greater. For example, if you have $|-14|$ and $|7|$, the absolute value of –14 is 14 and the absolute value of 7 is 7. Therefore $|-14|>|7|$. This might be difficult for some students to grasp because $-14 < 7$. Point out again if needed that absolute value measures the distance from 0. If you draw –14 and 7 on a number line then it is visually apparent that –14 is further from 0 than 7 is and so –14 has the greater absolute value.

Finally, still using the student examples of integers, choose a few pairs to determine the distance between them on the number line. For example, if we look at 8 and –9, and we plot them on a number line, how many units are they apart from one another? Although there are rules you can generalize that will make solving this type of problem more efficient, try not to lead the conversation towards these generalizations just yet. Some students may need more experience just counting "tick" marks before they are ready to learn a rule. In fact, Exercise #14 will ask students to generalize this process so be sure to assign it. This can be a great way to start tomorrow's lesson (see review problems section).

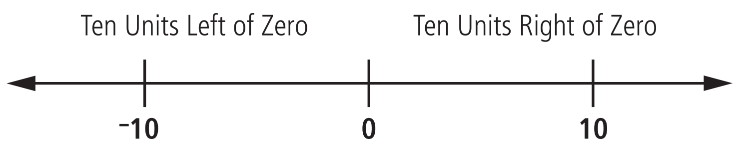
Students seem to understand the rope analogy for the absolute value more readily than other explanations

Many students have difficulty with the concept of absolute value. Thinking of a length of rope tethered at the origin helps many of these students. If you stretch the rope 5 units to the left of the origin, it will touch -5. If you stretch the same rope 5 units to the right of the origin, it will touch 5. The rope's length represents the distance from 0 or the absolute value of a number. In that case, $|-5| = |5|$.

Note: The absolute value symbol, | |, should not be confused with parentheses, ( ).
Some calculators, e.g. TI - 82, use the "abs" notation.

SECTION 1.4 DISTANCE BETWEEN POINTS

We locate the numbers 10 and -10 on the number line.



Notice that 10 and -10 are each 10 units from 0. We have a special name for the distance of a number from 0: the **absolute value** of the number.

In mathematics, we have a special symbol to represent absolute value. For example, we write |10| and read it as "absolute value of 10." We write |-10| and read it as "absolute value of -10." Because 10 and -10 are both 10 units from 0 we have the following:

The absolute value of 10 equals 10 or $|10| = 10$.

The absolute value of -10 equals 10 or $|-10| = 10$.

The absolute value of a number not only tells us its distance from the origin, it also measures the size of the number that we call its **magnitude**. The positive or negative sign tells us the direction of the number relative to 0. Because 10 and -10 are the same distance from 0, they have the same absolute value. In other words, $-10 < 10$ but $|-10| = |10|$.

EXPLORATION

Using the number line that you have constructed, find the distance between each pair of numbers:

a. 0 and 5	d. -1 and -5	g. 3 and 9
b. -0 and -5	e. 1 and -5	h. -3 and -9
c. 1 and 5	f. -1 and 5	i. -3 and 9

In addition to the previous examples, you may also see -|5| which is read as "the negative absolute value of 5" or -|-5| which is read as "the negative absolute value of -5." Since $|5| = 5$ and $|-5| = 5$, then we have

$$-|5| = -5 \text{ and } -|-5| = -5.$$

Summary

Before concluding class or allowing the students to work on the exercises, ask them to summarize what the absolute value of a number means. **What is it?** *The distance from* 0*, therefore always positive.* **What is it not?** *The relative position to the left or right of zero.*

Use Absolute Value Bingo at the end of this section as a review.

Although the absolute value concept is often dealt with at a later time, conceptually, it is essential in the understanding of the four basic operations with integers.

1. a. 0 d. 13 g. 37
   b. 4 e. 13 h. 58
   c. 4 f. 42 i. 26

2. a. 0 b. 12 c. 17 d. -17
   e. -17 f. 6

3. The absolute value of $^{-}34$ is 34. The absolute value of 34 is 34. The answer, therefore, is 34.

4. a. 10 b. 10 c. 0

5. a. $|{}^{-}3| < |5|$
   b. $|3| < |5|$
   c. $|-3| = 3$
   d. $|-5| > |0|$
   e. $|2| > {}^{-}4$
   f. $3 < |{}^{-}4|$
   g. $|{}^{-}7| < |{}^{-}8|$
   h. $-7 > -8$

6. The difference between 3 and 8 is so automatic and students have known it so long that thinking about it in two new ways might be hard. Direct them to find the two numbers on the number line and to think of their distances from the origin.

7. This exercise foreshadows using subtraction and absolute value to find the distance between any two numbers on the number line. Students may either draw a number line and use counting or they can use what they know about the number line to find the distance.

a.		b.		c.	
	2		5		9
	8		13		9
	8		5		33
	2		13		

EXERCISES

1. Find the absolute values of the following numbers.

a. 0	d. 13	g. 37
b. 4	e. -13	h. -58
c. -4	f. 42	i. -26

2. Calculate the following:

   a. |0| b. |12| c. |-17| d. -|17| e. -|-17|
   f. |- 6|

3. What is the absolute value of the absolute value of -34?

4. Find the distance between each number and zero:

   a. 10 10 b. -10 10 c. 0 0

5. For each pair of numbers below, place the correct symbol < , >, or =.

a. |-3| ☐ |5|	e. |2| ☐ -4
b. |3| ☐ |5|	f. 3 ☐ |-4|
c. |-3| ☐ 3	g. |-7| ☐ |-8|
d. |-5| ☐ 0	h. -7 ☐ -8

6. Find the distance between 3 and 8. Did you use the number line? Can you use absolute values? 3. See TE.

7. For each pair of integers given below, find the distance between the two integers on the number line. See TE.

a.	b.	c.
3 and 5	4 and 9	12 and 21
3 and -5	-4 and 9	21 and 12
-3 and 5	-4 and -9	-12 and 21
-3 and -5	4 and -9	

8. 7 and -7 because the absolute value of both gives distance seven. You add 11 and -7 and you get back to 4 or you add 4 to four and go to 11. (Answer for now before told what to put)

9. a. 3 and 7 b. -2 and 0 c. 0 and 8 d. 8 and 22 e. -5 and 11

10. Prior: 2012, 2013, 2014 Past: 2016, 2017, 2018

Ingenuity 11. 23

Investigation

12. The students may generalize from their investigation that if they subtract the smaller of the two numbers from the larger, l – s, whether the numbers are positive, negative, or zero, the difference will always be positive and give the distance. The students may, however, look at cases of positives and negatives also, but still reach the same conclusion. Some students may decide to take the difference without regard to order and then take the absolute value of the difference to find the distance. These results will become more apparent in Section 2.4 when we discuss subtraction of integers.

8. A number is distance 4 from 11. Use the number line to find this number. Is there exactly one possible number? Is there more than one? Explain using a number line.

9. Find numbers that are a distance:

   a. 2 from 5 b. 1 from -1 c. 4 from 4 d. 7 from 15

   e. 8 from 3

10. Certain school records must be kept for 3 years prior and past. If the year is 2015, identify the years of school records that must be kept.

11. **Ingenuity:**

    The distance between two cities on a highway is 118 miles. If all the exits between these two cities are at least 5 miles apart, what is the largest possible number of exits between these two cities?

12. **Investigation:**

    Write a process for finding the distance between two numbers. Remember to address all possible cases: two positive numbers, two negative numbers, one of each, and at least one number equal to zero.

Absolute Value Bingo

Objective: To help students practice their absolute value skills.

Materials:
Copy of blank "Absolute Value Bingo" grids (one per person)
Two color counter chips (or any small item to be placed on the bingo cards)
Overhead transparency of bingo cards, cut them out to make 27 individual cards
"Absolute Value Bingo Answers" transparency

Activity Instructions:
Pass out one blank "Absolute Value Bingo" grid to each student. Lay the "Absolute Value Bingo Answers" transparency on the overhead and have students copy 15 of the answers, in any random order, onto their grids. There are 16 spaces on the grids, the students can write FREE SPACE in the 16th box. (You might want to have them fill in their grids in pen or marker to avoid any temptation of cheating, and so the cards can be reused later). Pass out 10-15 chips to each student, and now the game is ready.

One at a time, place a bingo card on the overhead and have the students figure out the answer. To keep track of the correct answers, you may want to make a mark on your answer transparency of each problem you have presented to the class. The students will continue to figure out the absolute value cards until someone in the class has shouted "Bingo." You can decide whether to play a row, column, or diagonal at a time, or if you want to play blackout, around the world, or any other method.

After each round, it might be a good idea to have the kids exchange cards before you play again. Another good idea is to give the students more than one grid each.

ABSOLUTE VALUE BINGO

Students: Select 16 numbers from the Absolute Value Bingo Answers sheet. There should be ONE number per box

BINGO CARDS

FIND THE DISTANCE BETWEEN: -10 AND 10	FIND THE DISTANCE BETWEEN: 0 AND -12	WHICH IS SMALLER? 4 OR |-3|
WHICH IS SMALLER? |-2| OR -2	WHICH IS SMALLER? 15 OR |-10|	WHICH IS SMALLER? 50 OR |-100|
>, <, OR =? |15| □ |-15|	>, <, OR =? 20 □ |25|	>, <, OR =? -1 □ -|-5|

BINGO CARDS

$\|5-2\|$	$\|-2\|$	$\|7\|$
$\|14\|$	$\|0\|$	$\|-41\|$
$\|22\|$	$-\|-5\|$	$-\|-12\|$

BINGO CARDS

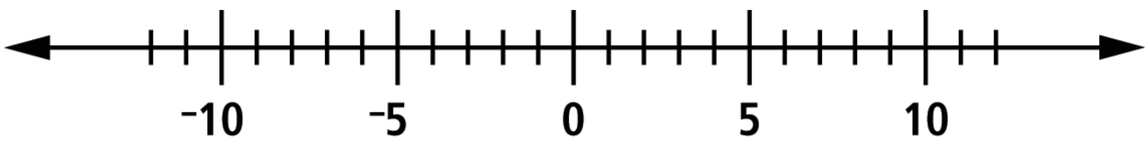
WHICH IS GREATER? -|10| or |-10|	WHICH IS GREATER? |-8| OR |4|	WHICH IS GREATER? |7| OR |-13|
WHICH IS GREATER? |-28| OR 25	WHICH IS GREATER? |13| OR 17	FIND THE DISTANCE BETWEEN -10 AND 0
FIND THE DISTANCE BETWEEN 4 AND 9	FIND THE DISTANCE BETWEEN 3 AND -1	FIND THE DISTANCE BETWEEN -25 AND 5

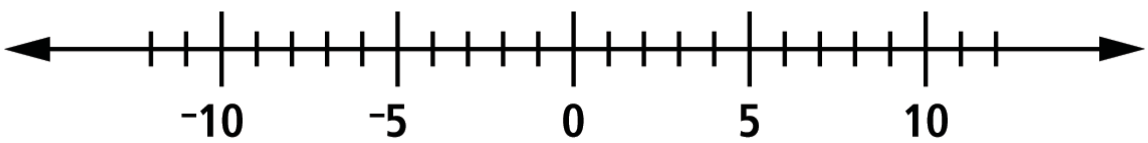
ABSOLUTE VALUE BINGO ANSWERS

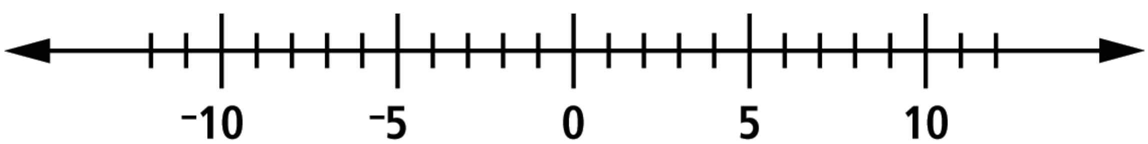
(SELECT ANY 16 ANSWERS TO PLACE ON YOUR BINGO CARD)

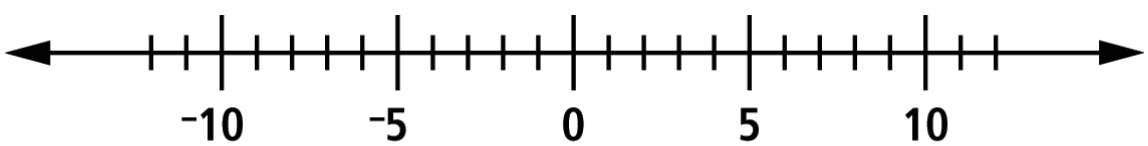
14	3	2
7	0	41
22	-5	-12
-10	$\lvert -8 \rvert$	$\lvert -13 \rvert$
$\lvert -28 \rvert$	17	5
4	30	20
$\lvert -3 \rvert$	-2	$\lvert -10 \rvert$
50	$=$	$<$
$>$	12	10

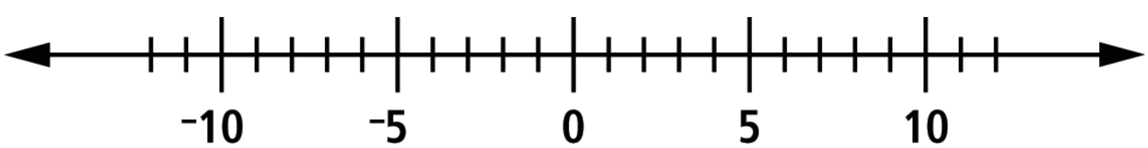
Number Lines for 1.4 Launch

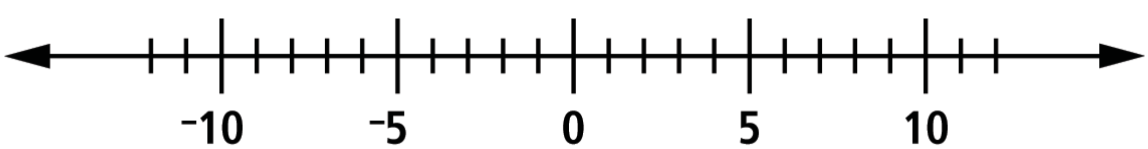












Section 1.5 - Addition of Integers

Big Idea:
Adding integers using a number line

Addition is going forward on the number line.

Key Objective:
Observe that the direction to face depends on the sign of the integer.

Pedagogical/Orchestration:
Using the activities, begin to stress key words for positive and negative, such as forward, backwards, ascend, descend, deposit, withdrawal, up, down, rise, drop, below sea level, etc. Maybe as an exercise, have kids use their own number lines to add the key words to the number lines created in this section. Teachers could keep a running list of words that mean positive and negative.

Internet Resource:
Matching Game: Adding Integers- http://www.quia.com/cm/112736.html

Materials:

- Counters, Cars or cutouts of cars from the back of the section or the CD
- Sentence strips for making number lines
- Adding Machine Tape
- Number line from -15 to 15
- Deck of playing cards

Activity:
Car Activity at the end of the section and on the CD
Double Sided Chip Addition Activity at the end of the section and on the CD

Exercises:
Students work on number lines with their cars to solve the addition problems and start the reflection process of noticing patterns.

Vocabulary:
Net yardage, (For the teacher) nested parentheses, transfer, withdraw, deposit

TEKS:
6.13(A); 7.2(C); 7.2(E)(F); 7.9(A); 7.14(A); 7.15(A) 8.1(A); 8.2(B); 8.15(A); 8.16(A);

New: 6.1(A, B, C); 6.3(D, F)

WARM-UPS for Section 1.5

1. Kayla McNutt has a family of squirrels living in a tree in her yard. She observes them for a few days and records her findings.

Day of the Week	Pecans Collected	Pecans Eaten (by the squirrels, of course)
Monday	30	12
Tuesday	23	9
Wednesday	25	11

If the squirrel family has 321 nuts before Monday, how many do they have Thursday morning?

a. 370 b. 367 c. 365 d. 362

Answer: (b) because 321 + 30 – 12 + 23 – 9 + 25 – 11 = 367 pecans

2. (Extra) Draw a model, using a number line to illustrate the following. What do you notice?
   a. 4 + 6
   b. 6 + 4
   c. 0 + 14
   d. 14 + 0

Launch for Section 1.5:

Demonstrate how to think of the set model of addition. Have a handful of counters in each hand and demonstrate putting them together and totaling their sum. This is a model most students are familiar with from previous grades. Tell your students that today they will learn a new model for adding integers. Have the students create a number line that extends from -15 to 15. This is a good size so that the little cars can be used later on for "driving" on the number line. Make sure that students are putting some thought into equally spacing the numbers; however, allow them to come up with their own strategies: paper folding, use of ruler, etc. Discuss the different strategies that students use. Tape the number lines to the students' desks, or hole punch them so they can be put in their binders and used as needed. Tell students, "Today we will use these number lines to develop the concept of adding integers."

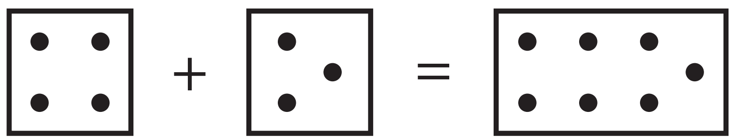
Start a brainstorm list on the board about patterns that the students notice throughout Sections 1 and 2, including exercises. This will continue through Section 1.6.

Walk your students through the CLASS EXPLORATION so they know how to use their cars. Model this at the board with the two board cars you created from the pdfs. Be sure that they understand the meaning of pointing the car right (positive) or pointing the car left (negative). Because we are adding, the car moves forward in the direction it is pointing.

Remind your students that the work and check, or more often called the guess and check, method is one of the strongest in learning mathematics.

SECTION 1.5 ADDITION OF INTEGERS

Addition is a mathematical operation for combining integers. Visually, using the "set model," when we add two integers we are combining the sets. To add 4 and 3 we draw the picture below:



We can also use our number line model to describe addition.

CLASS EXPLORATION: DRIVING ON THE NUMBER LINE WITH ADDITION

We can visualize adding two numbers using a car driving on the number line. The final location gives the sum. Let us practice how this works on a small scale. Use a number line from $^{-}15$ to 15 as your highway. You will also need a model car or something that can represent this model car.

Step 1: Place your car at the origin, 0, on the number line.

Step 2: If the first of the two numbers that you wish to add is positive, the car faces right, the positive direction. If the first of the two numbers is negative, the car faces left, the negative direction. Drive to the location given by the first number. Park the car.

Step 3: Next examine the second of the two numbers. If this number is positive, point the car to the right, the positive direction. If the second number is negative, point the car to the left, the negative direction.

Step 4: Because you are adding, move the car forward, the way that it is facing, the distance equal to the absolute value of the second number.

Use your car and the four-step process to compute each of the following examples. Attempt the process on your own first, and then compare your answer with the provided solution.

Go through the three examples together. Notice the differences.

- The first is a positive + a positive.
- The second is a negative + a positive.
- The third is a negative + a negative.

We will discuss the commutative property, which will help students see that adding a negative and a positive is just like adding a positive and a negative.

After completing Example 1, have students work with a partner to demonstrate additional problems using the car model. This should be done after each example for this section.

EXAMPLE 1

Find the sum $3 + 4$, and describe how you obtain your answer using the number line model.

SOLUTION

The two numbers we are adding are 3 and 4 (which we also know as $^{+}3$ and $^{+}4$).

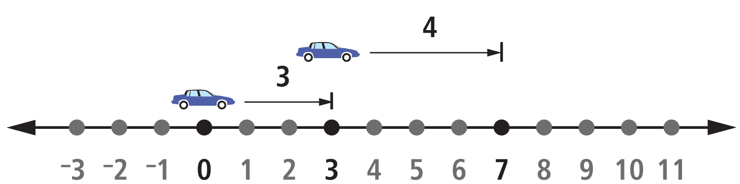
Step 1: Begin with your car at 0.

Step 2: Because the first number is positive, the car faces to the right. Drive to the location $^{+}3$. Park the car.

Step 3: Point the car to the right because the second number, $^{+}4$, is positive.

Step 4: Move the car 4 units to the right. Park the car.

You are now at location 7.



EXAMPLE 2

Find the sum $^{-}3 + 4$. How do we start the process? In which direction does your car move first and how far? Explain how you reached your solution using a car on your number line.

SOLUTION

Step 1: Begin at 0.

Step 2: Because $^{-}3$ is a negative number, point the car to the left and drive 3 units. Park the car.

Step 3: Because 4, the number added to $^{-}3$, is positive, turn the car to face right

Don't worry about discussing the rules of addition now. That is, do not feel you have to tell your students that two positives results in a positive; two negatives results in a negative; or when adding two different signs, subtract and take the sign of the larger absolute value. The first problem in the exercises is designed to get students thinking about this pattern. Let them discover it.

Summary

Pair students and distribute one playing card to each student. Red cards represent negative numbers and black cards represent positive numbers. Ignore the face cards if you want. Jokers equal 0 and Aces are 1 or -1.

Have each student pair create an illustration of a number line and the car process that models their two cards. The first illustration should be Student A's card + Student B's card. The next illustration should be Student B's card + Student A's card.

Possible dialog: "If in your group Student A's card + Student B's card was the same as Student B's card + Student A's, let me know. Why do you think this happened?"

It is because of the commutative property, covered later in Section 2.4.

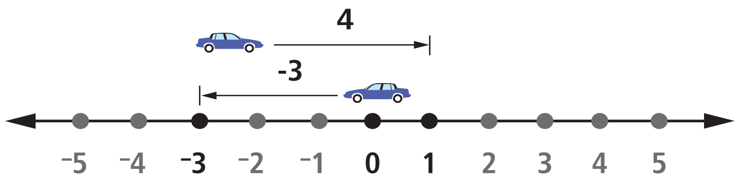
When ready, ask the class to summarize this process of using the number line for addition.

After the exercises, make a list of the four example problems you did together as a class and the four correct solutions. See if the class can come up with any theories about adding integers. Why are some of the answers positive and some negative? Why are some of the answers the sum of the two numbers, while others are the difference? Please don't give the students the "rule" for adding integers, but see if you can lead them into figuring it out for themselves.

Step 4: Move 4 units to the right, ending up at location 1. Park the car.8.

A number is distance 4 from 11. Use the number line to find this number. Is there exactly one possible number? Is there more than one? Explain using a number line.

The result $^-3 + 4 = 1$ is demonstrated below:

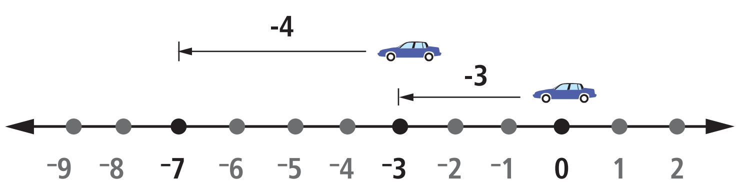


EXAMPLE 3

Find the sum $^-3 + (^-4)$, or simply $^-3 + {}^-4$, using the same process.

SOLUTION

Point the car to the left and move forward 3 units. Leave the car pointing to the left because the next number is negative. Move the car forward 4 units to the location $^-7$. We have: $^-3 + (^-4) = {}^-7$.

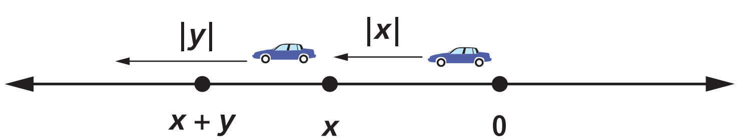


Remember, **when you are adding, the car always moves forward, in the direction that it is facing.** The signs of the integers tell us whether we face right, if positive, or left, if negative, before moving.

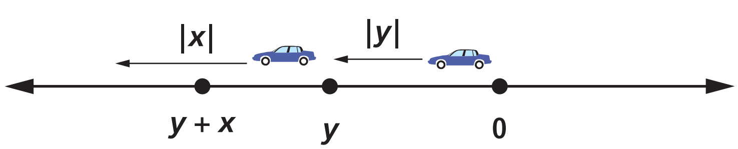
You have observed and noticed patterns in addition of integers. There are additional patterns that we can write as rules that work for any integer that we can represent by the variable x

Ask students what the addition would look like on the number line if x and y are negative numbers

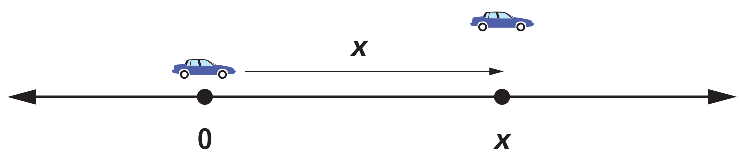
x + y



and y + x



Think about what happens when you add 0 to a number *x*. You first drive |*x*| units in the direction of the sign of *x*, and then you drive 0 units, remaining exactly where you were before:



In other words, adding 0 to any number does not change its value. Because 0 has this property, we call 0 the **additive identity**.

PROPERTY 2.1: ADDITIVE IDENTITY
For any number *x*, $x + 0 = 0 + x = x$

EXAMPLE 4

For each of the numbers below, find an equivalent expression using the additive identity 0.

a. 4 b. $^-2$ c. 0 d. 9 e. X

SOLUTION

a. $4 + 0 = 4$, so 4+0 and 4 are equivalent expressions

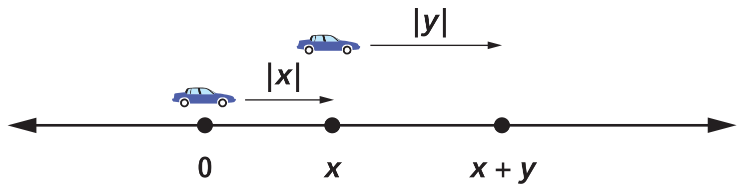
b. $^-2 + 0 = {}^-2$, so -2+0 and 2 are equivalent expressions

c. $0 + 0 = 0$, so 0+0 and 0 are equivalent expressions

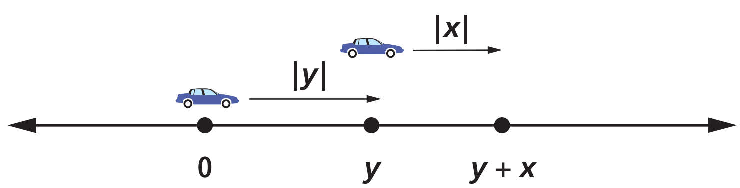
d. $0 + 9 = 9$, so 9+0 and 9 are equivalent expressions

e. $X + 0 = X$ so X+0 and X are equivalent expressions

Suppose that *x* and *y* represent integers. Remember, to find the sum $x + y$, we started at the point 0 on the number line, moved |*x*| units in the direction of *x*, and then moved |*y*| units in the direction of *y*. If *x* and *y* are positive we can model the addition as,



To find the sum $y + x$, we started at 0 and performed these two steps in the reverse order.



Reversing the order of these steps does not change the final outcome. In either case, we end up in the same place. We call this the **commutative property of addition**.

PROPERTY 2.4: COMMUTATIVE PROPERTY OF ADDITION
For any numbers x and y, $$x + y = y + x$$

PROBLEM 2

Generate equivalent expressions for each side of the equalities below to show that addition is commutative.

a. 4 + 3 = 3 + 4 b. 4 + -6 = -6 + 4 c. -2 + -3 = -3 + -2
d. 13 + -9 = -9 + 13

SOLUTION

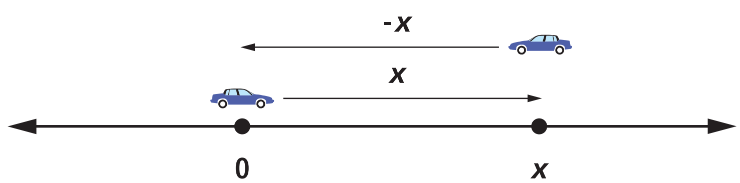
a. 4+3=7 3+4 =7

b. -4+-6=-2 -6+4 =-2

c. -2+-3 =-5 -3+-2=-5

How can we find a pair of numbers on the number line that are the same distance from zero? To do this, we need to go the same distance from 0 but in opposite directions. For example, the numbers 1 and $^{-}1$ are both 1 unit from 0 and $|1| = |-1|$. Similarly, 2 and $^{-}2$ are the same distance from 0 and $|2| = |-2|$. We call pairs of numbers like 2 and $^{-}2$ "opposites" or **additive inverses**.

What happens when you add a number to its additive inverse? Beginning at the origin, you first move a certain distance in one direction, and then move exactly the same distance in the opposite direction:



Your final position is back at the origin, 0. So, the sum of any number and its opposite is 0. We call this the **additive inverse property**:

PROPERTY 2.2: ADDITIVE INVERSE PROPERTY
For any number x, there exists a number ^{-}x, called the **additive inverse** of x, such that $$x + {}^{-}x = 0$$

EXAMPLE 5

a. $2 + {}^{-}2 = 0$ b. $5 + {}^{-}5 = 0$ c. $-9 + 9 = 0$

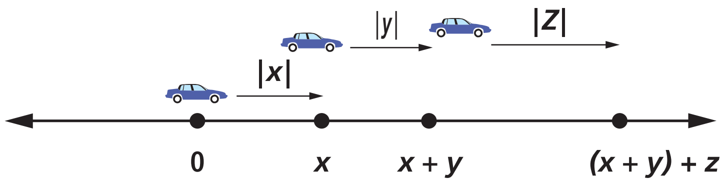
d. $4 + -4 = 0$

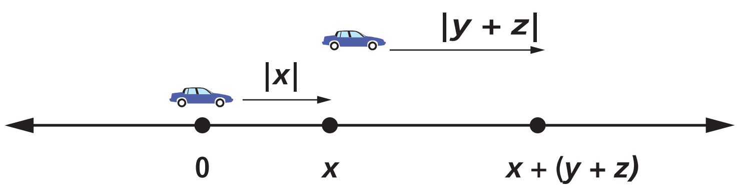
What number is $-(-2)$? In general, what number is $^{-}(^{-}x)$?

Because $^{-}(^{-}x)$ is the opposite of ^{-}x, $^{-}x + {}^{-}(^{-}x) = 0$.

On the other hand, because ^{-}x is the opposite of x, $x + (^{-}x) = 0$. We can write this as $^{-}x + x = 0$ by the commutative property.

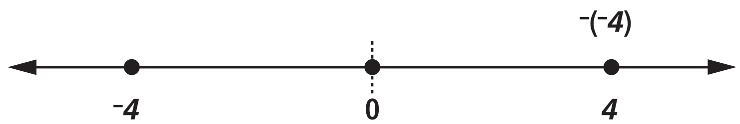
Comparing these two equations shows us that $^{-}(^{-}x)$ must equal x. In short, the opposite of the opposite of x, $^{-}(^{-}x)$, is the number x itself.


|y|
|Z|
|x|
0
x
x + y
(x + y) + z

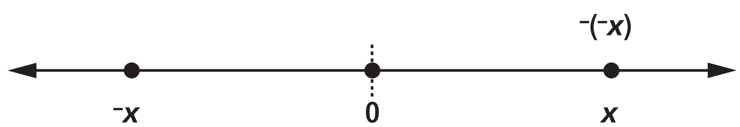

|y + z|
|x|
0
x
x + (y + z)

THEOREM 2.1: DOUBLE OPPOSITE THEOREM
For any number x, $$^{-}(^{-}x) = x$$

We can see this more easily on a picture. For example, if x is 4, the picture shows us that the opposite of the number 4 is -4 and the opposite of -4 written –(-4) is the same as 4.



If x is -7, the opposite of x is 7 and the opposite of the opposite of x is -7.
So, in general we have the following picture that shows us $^{-}(^{-}x) = x$



EXAMPLE 6

Use the number line to show that the following statements are true.

a. –(-10) = 10 b. –(-5) = 5 c. –(-25) = 25 d. –(-7) = 7

Now, we look at the expressions $(x + y) + z$ and $x + (y + z)$, where x, y, and z are integers. To calculate the first expression, we first add x and y and then add z; to calculate the second expression, we first add y and z, and then add x. Draw a picture using the car model for each of the expressions. Just as before, the order does not matter in determining the final value. This is called the **associative property of addition**.

PROPERTY 2.5: ASSOCIATIVE PROPERTY OF ADDITION
For any numbers x, y, and z, $$(x + y) + z = x + (y + z).$$

EXERCISES

1. a. -14 b. 14 c. 0 d. -x e. z

2. a. commutative property of addition b. double opposite theorem c. additive inverse
   d. additive identity e. associative property of addition

3. a. -5 + -4 b. 7 + (-9 + 6)

PROBLEM 3

Generate equivalent expressions to each of the following using the associative property. Check each expression by evaluating the expression, first computing what is inside the parentheses.

a. (4 + 7) + 2 b. -3 + (5 + -8)

SOLUTION

a. (4 + 7) + 2 = 4 + (7 + 2).

Check: (4+7)+2=11+2 = 13

4+(7+2)=4+9=13

b. (-3 + 5) + -8 = -3 + (5 + -8)

Check: -3+(5+-8)=-3+-3=-6

(-3+5)+-8=2+-8=-6

EXERCISES

1. Find the additive inverse of each number below. Generate equivalent expressions for 0 using your additive inverse and the number given. Remember, that that additive inverse of t is –t for any number t.

a. 14 b. -14 c. 0 d. x e. –z

2. Name the property associated with each of the following:

a. 16 + -5 = -5 + 16 b. –(-1) = 1

c. 11 + -11 = 0 d. -11 + 0 = -11

e. (-2 + 4) +1 = -2 + (4 + 1)

3 + (6 +-5) (-4 + 0) + -4

e. Write a rule to describe any patterns you see in problems a-d.

4. a. 4
   b. 4
   c. -6
   d. -7
5. a. 0
   b. 0
   c. 0
   d. 0

   Foreshadowing additive inverse.

6. a. 7
   b. -8
   c. -6
   d. 4
   e. 4
   f. 5
   g. -2
   h. -2
   i. -4
   j. 7
   k. - 8
   l. -6
   m. 4
   n. 4
   o. 5
   p. -2
   q. -2
   r. -4

7. Your students should observe that adding two positives yields the positive sum of the numbers. Adding two negative yields the negative sum. Adding a positive and a negative yields the difference between the absolute values of the two numbers and the sign of the number with the largest absolute value.

8.
a. 9, 9
b. 4, 4
c. 8, 8
d. -8, -8
e. Students should notice that the placement of parenthesis in the addition problems does not change

3. Use the indicated property and write an equivalent number expression.

   a. Use the commutative property of addition to write an equivalent number expression to -4 + -5

   b. Use the associative property of addition to write an equivalent number expression to (7 + -9) + 6

For exercises 4–6, find each sum. You may use your car and number line. The sum you generate is an equivalent expression to the original.

4.	a. 4 + 0	c. 0 + -6
	b. 0 + 4	d. -7 + 0
5.	a. -3 + 3	c. 4 + -4
	b. 0 + -0	d. 6 + -6
6.	a. 2 + 5	j. 5 + 2
	b. -3 + -5	k. -5 + -3
	c. -2 + -4	l. - 4 + -2
	d. -1 + 5	m. 5 + -1
	e. -2 + 6	n. 6 + -2
	f. -3 + 8	o. 8 + -3
	g. 6+ - 8	p. -8 + 6
	h. 2+ - 4	q. -4 + 2
	i. 4+ -8	r. -8 + 4

7. Write rules to describe any patterns you see in problems 4–6. See TE.
   - Do you see a pattern when adding two positives?
   - adding two negatives?
   - adding a positive and a negative?
   - adding a negative and a positive?

   Explain how your rules work using a number line.

8. For this exercise, let's pay careful attention to the order in which we add. We use parentheses to specify order. For example, (1 + 2) + 3 means first add 1 to 2, and then add the result and 3; 4 + (5 + 6) means first add 5 and 6, and then add 4 to the result. For each sum below, generate an equivalent sum using the associative property, and then calculate the result of each equivalent sum:

9. a. 13
   b. –10
   c. –11
   d. –10
   e. –3
   f. 6
   g. 3
   h. 0

Teacher Tip for Exercise 10: You might demonstrate how to do this with vertical lines and arrows so students can see that the number line does not have to be horizontal. Later you might remind them about the y-coordinate.

Teacher Tip: Throughout these exercises, we keep emphasizing working through the algorithm physically. We feel this is very important for students learning a new skill like adding integers.

10. (c) and (d) are good activities for a class floor number line. You might find that you have some students who only respond if they can act out the addition of integers themselves.
    a. 5 + 6 = 11 °C b. -2 + -7 = -9 °C d. 2 steps forward, 7 + -5 = 2
    e. 6 steps backwards, -10 + 4 = -6 f. 4th floor, 3 + 3 + -2 = 4

a. (2+ 3) + 4
   2 + (3+4)

b. (3+ 6) + -5
   3 + (6 +-5)

c. 9 + (-8 + 7)
   (9 + -8) + 7

d. -4 + (0+ -4)
   (-4 + 0) + -4

e. Write a rule to describe any patterns you see in problems a-d.

9. Predict the sign of the answer. Then find the sums. Use the number line if you need to.

a. 7 + 6

b. -6 + -4

c. -8 + -3

d. -6 + -6

e. 3 + -6

f. -3 + 9

g. 8 + -5

h. -7 + 7

For exercises 10–16, write each problem as an addition problem and use positive and negative numbers where appropriate. Show your work on a number line.

10. a. Jeff observes that the temperature is 5˚C. If it rises 6˚C in the next three hours, what will the new temperature be?

b. Alex observes that the temperature is -2˚C. During the night, it falls 7˚C. What was the low temperature that night?

c. Denise observes that the temperature is -4˚C. If it rises 6 ˚C in the next two hours, what will the new temperature be?

d. Carlos takes 7 steps forward then takes 5 steps back. How far is Carlos from where he started?

e. Marissa takes 10 steps backward and then 4 steps forward. At what location does she end?

f. Anna is looking for the hotel restaurant to meet her family for dinner. She starts at her hotel room on the 3rd floor, goes up to the 6th floor, goes down two floors and finds the restaurant. What floor is the restaurant?

11. Jim checks the temperature and it is -8˚C. If the temperature warms up 12˚C, what is the new temperature?

11. -8 + 12 = 4 ˚C

12. -6 + 7 = 1 ˚C

13. 4 + -9 = -5 ˚C

14. Ask your students if they know what overdraft protection is. If not, explain to them how they can have a negative balance with overdraft protection and not get into trouble.

24 + -30 = $-6

16. Some students may not be familiar with football. Some introduction to the basics of football may be in order. The main idea needed in this problem is for the students to be aware that a player moves the ball up or down the field, and the opposing team tries to prevent the player from progress towards his team's goal. Forward progress measured in yards is a gain, and we designate this as a positive number. We will designate loss in yards as a negative number.

17. 250 + 3(50) = $400

18. 150 + -89 = $61

19. $58 dollars transferred from checking to savings $325 +-(99 + 259 + 25) = $-58

12. It was $^-6$ °C in the morning. The temperature rose 7 °C. What is the temperature now?

13. Carlos checks the temperature and it is 4 °C at 5 PM. By 10 PM, the temperature has dropped 9 °C. What is the new temperature?

14. Chris has $24 in his bank account. If he withdraws $30, what will his balance be?

15. The temperature Alaska is $^-5$ °F on a cold winter day. If the temperature falls another 6 °F, what will the new temperature be?

16. If a football player loses 6 yards in one play, loses 2 yards in another play, and then gains 6 yards in the final play, what is the net gain or loss?

17. Eric has $250 in his bank account. Each week he earns $50 from his life guarding job and deposits it into his banking account. How much money will Eric have after 3 weeks of work?

18. Juan Carlos has $150 in his bank account. He wants to buy a new cell phone, so he makes an $89 withdrawal from his bank account. How much money will be left in his bank account after he withdraws the money?

19. Veronica has $325 in her checking account. She wants to write a check from her checking account to purchase some items. She bought a cell phone for $99, a new television for $259, and $25 books for her summer reading assignment. How much money needs to be transferred from her savings account to her checking account so that she can pay for all of the items?

Ingenuity

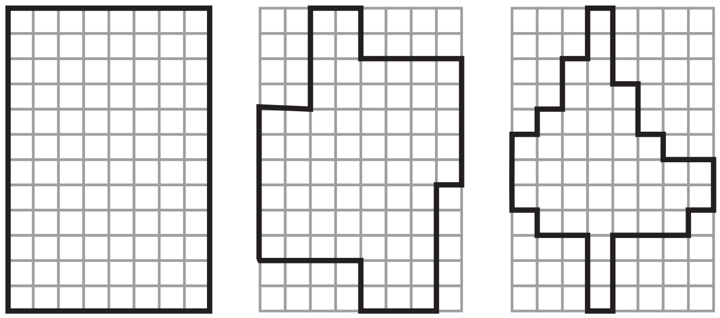
20. 40 inches, 40 inches, 40 inches, all of the perimeters are the same!

Investigation

21. At this point we are not concerned with a correct answer. We are concerned with students looking for patterns, making conjectures, and testing them. You might want to expand this into a class exploration in anticipation of the next section.

20. **Ingenuity:**

In the diagrams below, assume that each of the small squares has sides of length one inch. Find the perimeter of each of the figures below. What surprising result do you notice?



21. **Investigation:**

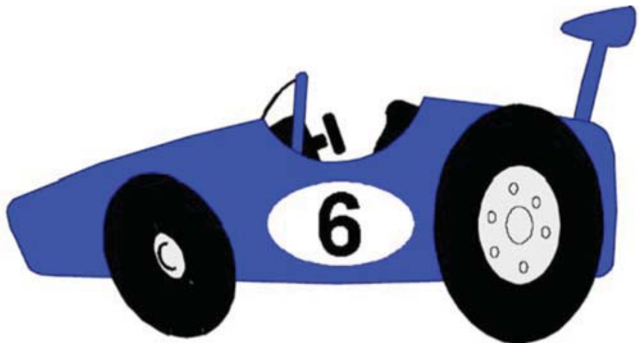
With our car model, the car moves forward when we add. What do you think we should do when we want to subtract a number from another number? Write your best guess about how to subtract two numbers. See TE.

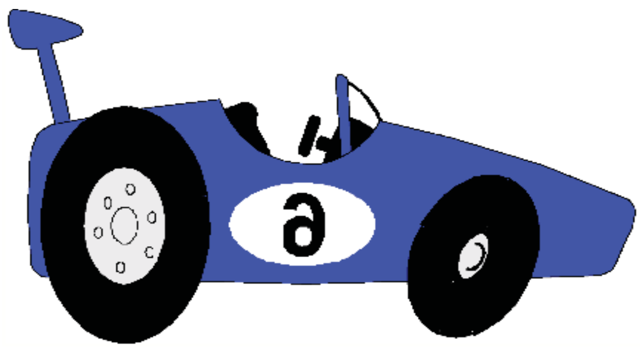
CAR ACTIVITY

Objective: This activity is an extension to the example in the math book that will help the students visualize the motion of the car as it moves on the number line.

Materials:
An evenly spaced number line on the board marked from -15 to 15
Both car cut-outs (left and right)
Magnetic Tape

Activity Instructions:
1) Make your number line on the board.
2) Copy both car cut-outs, and color them however you see fit. Attach the magnetic tape to the back of the cut-outs so the cars will stick to the board.
3) Use the number line and the cars to demonstrate the motion of several examples of integer addition and subtraction problems. Let the students come to the board and move the cars if time allows.





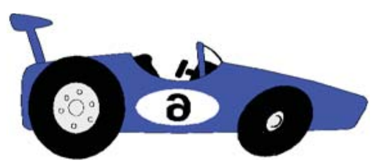
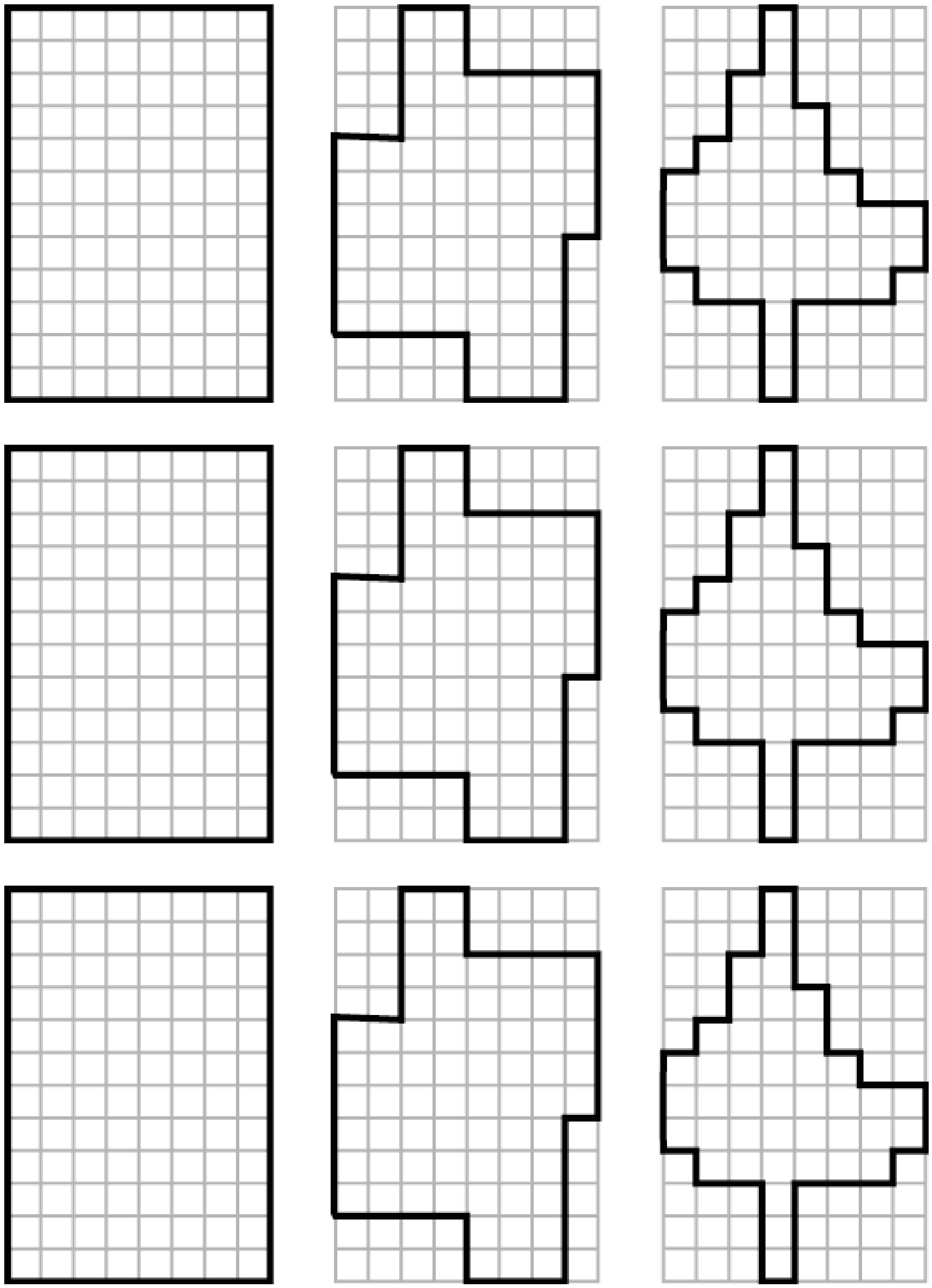


Diagram for Ingenuity 1.5



Double-Sided Chip Addition

Objective: Students will increase their ability to add integers using double-sided chips

Materials:
Double-sided chips (coins or checker pieces can be used as long as the two sides are different)

Activity Instructions:

1) We establish one side of the chip to be positive (+) and the other side to be negative (-).
2) We also consider any pair of (+) and (-) to be equivalent to zero and we'll say:
   (+) (-) = 0
3) The Chip Addition is based on a set union, $\cup$, concept. For example: 3+2 = 5 is demonstrated with (+)(+)(+) $\cup$ (+)(+) gives us (+)(+)(+)(+)(+) all together.
   3+-2= 1 is demonstrated with (+)(+)(+) $\cup$ (-)(-) giving us (+)(-)(+)(-)(+) or (+)

*Note that subtraction exercises have not been pre-written for this activity.

Section 1.6 – Subtraction of Integers

Big Idea:
Subtraction is moving backward.

Key Objectives:

- Observe that subtraction is like adding a negative.
- Observe that subtracting a negative is like adding its opposite.

Pedagogical/Orchestration:

- Students continue the reflection process of noticing patterns.
- Teacher can start a list on the board or chart paper of patterns noticed.
- Students continue using cars and number lines to work through examples and exercises.

Materials:

- Sentence strip number lines created by students
- Cars
- Number line handout from CD to help with exercises
- Playing cards

Activity:
Continue the Car Activity from Section 1.5. Use examples in book with students physically (using cars) and demonstrating (acting out) the given subtraction problems.

Lily Pad Race Activity from the end of this section and on the CD

Exercises:
After Exercise 1, have class discussion and ask students what patterns they see. They should discover the equivalence of these two computations: 3 – 4 means 3 subtract a positive 4; 3 + (-4) means 3 plus the opposite of positive 4. Make sure you use groups of related problems to help students see patterns, such as: -4 - 3; -4 + -3; -4 – -3; -4 + 3. Exercise 11 is a finance application.

Vocabulary:
backwards, subtraction, withdraw, transfer

TEKS:
6.13(A); 7.2(C); 7.13(A); 7.14(A); 7.15(A) 8.1(A); 8.2(B); 8.15(A); 8.16(A);

New: 6.1(A, B, C); 6.3(D, F)

WARM-UPS for Section 1.6 (Subtraction of Integers)

1. Anna has \$20, saved \$32, spent \$17, earned \$5, and lost \$15. How much money does she have?

   Answer: 20 + 32 – 17 + 5 – 15 = \$25

2. Ben is designing a new chip for a computer. The length of the chip needs to be between 7.45 and 7.75 centimeters. Which of the following lengths are within these specifications? Explain your answers.
   a. 7.6 cm
   b. 7.4 cm
   c. 7.38 cm
   d. 7.8 cm **Answer: a**

Launch for Section 1.6:

Have a number line with integer values spaced about a foot apart on the wall. Place a rolling chair at the zero and ask for a student volunteer. Have the student sit in the chair, and ask the class the following question: "How can we model 2 + 3 with this car model?" Follow the students' directions by pushing the student in the chair. Now ask the students, "What do you think the car should do when we want to subtract?" Allow students to answer. If no one suggests moving backwards, then move the student backwards in the chair and ask students what that would represent. Then tell students, "Today we will be making sense of subtracting numbers on a number line." Proceed with Example 1. If a rolling chair is not available, students can walk along the number line.

It might help some of your students if you remind them that going backward is like driving with the car in reverse.

It is very important for kids to write these steps in their own words so that they are clear about what they are doing. This is helpful to English Language Learners so that they are "learning" the language as they write with complete sentences.

Have students explore Examples 2 and 3 at the same time. Have class share their attempts.

EXAMPLE 1

Your students already know that 5 – 2 equals 3. This example will simply show proof that the answer is indeed 3. Before moving on to the next example, be sure that the students understand the movement procedure that produces the correct answer. Students might also use a sentence strip number line for this problem.

Have students work with a partner to demonstrate subtraction problems using the car model. This should be done after each Example in this section.

Model the three examples with the class, having your students draw and act out the movements on their number lines. Summarize the process with your students.

Proceed thoughtfully and carefully through this and perform some formative assessment to make sure each student is understanding the model. Subtraction can be tricky.

A student might reason in the following way: Because 5 is a positive number, the car faces to the right. The second number is also positive, so the car remains facing to the right. The subtraction sign just means we are putting the car in reverse 2 units. The car will end up at 3.

SECTION 1.6 SUBTRACTION OF INTEGERS

With our car model, addition involves moving the car forward in the direction indicated by the signs of the numbers we are adding. What do you think we should do when we want to subtract one number from another number? One way to model subtraction is to use our number line and move the car **backward, the opposite of forward.**

Try each example first, and then check your answer by comparing it to the solution given. Describe each of the steps you are using in words.

EXAMPLE 1

Compute the difference 5 – 2, and show how to model this with a number line. Use a model car and a number line to simulate your solution. What is your final location?

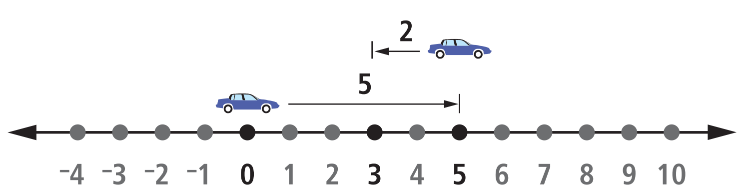
SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 5 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Instead of moving forward 2 spaces, move backward 2 spaces, ending up at location 3. Remember, we move backward because we are subtracting. We can write this movement as 5 – 2 = 3.



EXAMPLE 2

At this point, ask your students to consider **why** this problem produced a negative solution.

EXAMPLE 2

Compute the difference 2 – 5. Use a number line to show how you solved the problem.

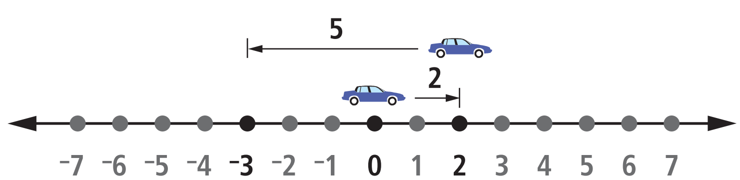
SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the right because the number being subtracted is positive.

Step 4: Since we are subtracting, move backward 5 spaces, ending up at location ⁻3. We can write this movement as 2 – 5 = ⁻3.



EXAMPLE 3

Compute the difference ⁻7 – 3. Use a number line to show how you solved the problem.

SOLUTION

Step 1: Place your car at the origin. Because the first number is negative, the car faces left.

Step 2: Move the car forward 7 units to the location given by the first number, ⁻7. Park the car.

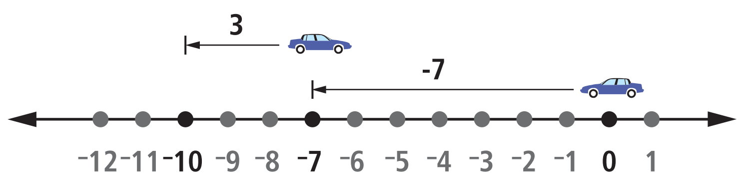
Step 3: Point your car to the right because the number being subtracted is positive.

EXAMPLE 4 Complete this example as a class.

Teacher Tip: Before moving on to the exercises, make sure your students have a really good understanding about why the car changed directions in this particular problem. You might need to do a few more examples together with the class before they are ready to work alone.

The first number is positive, so the car faces right and moves two spaces to the right. The second number is negative, so the car faces left. This time, the car needs to move opposite the direction it is facing, so we end up at 7.

Step 4: Now move backward 3 spaces, ending up at location ⁻10. Be careful! This time your car was pointing to the right, so when you back up you will move backwards to the left. We can write this movement as ⁻7 – 3 = ⁻10.



EXAMPLE 4

Compute the difference 2 – (⁻5). Use a number line to show how you solved the problem.

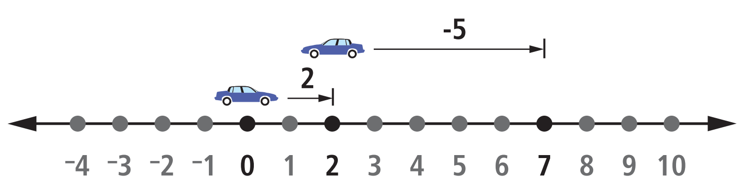
SOLUTION

Step 1: Place your car at the origin. Because the first number is positive, face the car to the right.

Step 2: Move the car forward 2 units to the location given by the first number. Park the car.

Step 3: Point your car to the left because the number being subtracted is negative.

Step 4: Now move backward 5 spaces, ending up at location 7. Be careful! This time your car was pointing to the left, so when you back up you will move to the right. We can write this movement as 2 – (⁻5) = 7.



Summary

Have a class discussion of how subtraction is different from addition. Students should understand that to subtract, you follow the same procedures as when adding, but go backwards as a final step.

Using the deck of cards from the Integer War, deal the cards giving one to each student, and pair students up to create examples of their subtraction problems. Remember, reds are negative, blacks are positive, and jokers are zero. Have each group subtract Student A's card – Student B's card followed by Student B's card – Student A's card.

When finished, have the class share its observations.

When you subtract a negative, be aware that students will ask how to write a subtraction of a negative. For instance, 4 – ⁻3 or 4 – (⁻3).

EXERCISES

1. a. 5 -2 5+ -2 TE 3,3
   b. 6-3 6 + -3 TE 3,3
   c. 2 – 5 2 +-5 TE -3,-3
   d. 4 - 8 4 + -8 TE -4,-4
   e. - 4 – 2 - 4 + -2 TE -6,-6
   f. -7 -4 - 7 + -4 TE -11, -11
   g. 0 – 4 0 + -4 TE -4, -4
   h. 0 – 8 0 + -8 TE -8,-8

Reflect on patterns noticed. Start a brainstorm list on the board or wall. Add observations to the list after each example and exercise.

2. For this exercise, use the cumulative class list of patterns to have the class write rules for adding and subtracting integers. Subtraction is the inverse, or opposite, of addition. Subtracting a positive is like adding a negative. Another possible statement of the same rule might be "subtracting is the same as adding the inverse, or x-y = x + (-y).

3. a. 7 + 2 7 - (-2) TE 9,9
   b. 2 + 5 2 – (-5) TE 7,7
   c. -3 – (-4) -3 + 4 TE 1,1
   d. - 8 + 3 -8 – (-3) TE -5,-5
   e. - 5 – (-5) - 5 + 5 TE 0,0

SUMMARY

In order to compute $x - y$, we proceed as follows:

Step 1: Place the car at 0, the origin. Then face the car in the direction of the sign of the first number x.

Step 2: Move the car $|x|$ units forward in the direction the car faces. Park the car.

Step 3: Next, face the car in the direction of the sign of the second number, y.

Step 4: Move the car $|y|$ units backward, the opposite direction from what the car faces. The car is positioned on the difference $(x - y)$.

EXERCISES

Use the car model with your number line to calculate each of the following exercises. Drive carefully!

1. a. 5 -2   5+ -2
   b. 6-3   6 + -3
   c. 2 – 5   2 +-5
   d. 4 - 8   4 + -8
   e. - 4 – 2   - 4 + -2
   f. -7 -4   - 7 + -4
   g. 0 – 4   0 + -4
   h. 0 – 8   0 + -8

2. What patterns do you see in exercise 1? In your own words, write a rule for any patterns you observe.
3. Use the car model with your number line to calculate each of the following exercises.

   a. 7 + 2   7 - (-2)
   b. 2 + 5   2 – (-5)
   c. -3 – (-4)   -3 + 4
   d. - 8 + 3   -8 – (-3)
   e. - 5 – (-5)   - 5 + 5

4. For this exercise, use the cumulative class list of patterns to have the class write rules for adding and subtracting integers. An observation students might make is "Adding is subtracting the inverse, or x + y = x – (-y).

5. a. 3 – 2 2 – 3 TE 1.-1

b. 5 – 2 2 – 5 TE 3,-3

c. - 3 – 8 8 – (-3) TE -11, 11

d. 0 – 9 9 – 0 TE -9, 9

e. 6 – (-8) - 8 – 6 TE 14, -14

f. 5 – (-2) -2 – 5 TE 7, -7

g. - 1 – (-6) - 6 – (-1) TE 5, -5

h. -4 – (-3) -3 – (-4) TE -1,1

i. For this exercise, use the cumulative class list of patterns to have the class write rules for adding and subtracting integers. Students should notice that $x - y \neq y - x$ (does not follow commutative property)

6. a. 5, -5, 11 b. 3,11,3 c. -8, -8, 2 d. 4,4,-8

7. 8 -10 = -2° C

8. 20 + 40 + 35 + 25 – 100 =$20

9. -4 -3 = -7°F

4. What patterns do you see in exercise 3? In your own words, write a rule for any patterns you observe.

5. Solve the following exercises using the car model and your rules from the previous exercises.

a.	3 – 2	2 – 3
b.	5 – 2	2 – 5
c.	- 3 – 8	8 – (-3)
d.	0 – 9	9 – 0
e.	6 – (-8)	- 8 – 6
f.	5 – (-2)	-2 – 5
g.	- 1 – (-6)	- 6 – (-1)
h.	-4 – (-3)	-3 – (-4)

   i. What patterns do you observe in a-h above? Describe any patterns that you observe.

6. Calculate the following sums and differences. Use the card model as needed.

a. 3 – 8	b. 7 – 4	c. - 3 – 5	d. -2 – (-6)
3 + - 8	7 – (-4)	- 3 + -5	-2 + 6
3 – (-8)	7 + -4	- 3 – (-5)	-2 + -6

For exercises 7- 11, write a subtraction expression and compute.

7. It was 8°C at 7 A.M. The temperature dropped 10°C over the next three hours. What was the temperature at 10:00 A.M.?

8. Benjamin opens a checking account at the bank in January. He deposits \$20 in February, \$40 in March, \$35 in April, and \$25 in May. He needs to withdraw \$100 in June to pay for a computer camp he wants to attend this summer. What is his balance after he makes his June withdrawal?

9. The temperature in Canada is -4°F on a cold winter day. If the temperature falls another 3°, what will the new temperature be?

10. -3 – 30 = -33 feet below the deck or 3 – (-30) = 33 feet below the deck

11. 440 – 395 – 25 = \$20

Ingenuity

12. Start at the beginning. Maria takes 2 steps forward and one back, ending at the first step. For every step's progress, Maria takes 3 steps. Check the number of steps Maria takes to reach step 2 on her journey. She starts on step 1, goes to step 3, then back to step 2. Maria's progress is from step 1 to step 2, but to do that, she has taken three more steps for a total of six steps to end up at step 2. So for every step's progress, she must take 3 steps. After step 28, Maria takes 2 steps forward and reaches the mailbox before she has a chance to step back. So (3)(28) + 2 = 86 steps. The best way to really understand this is to do it physically, counting steps.

10. Whitney is going scuba diving. She will jump into the water from a deck 3 feet above the water surface. She jumps in the water and descends 30 feet below the surface. What is Whitney's position relative to the deck?

11. Adam has a checking account and a savings account. If he goes below $50 in his checking account the bank charges his account a $25 penalty fee. Adam has $440 in his checking account. He withdraws $395 from his checking account for a new surfboard. What will be the new balance in Adam's checking account? $30

12. **Ingenuity:**

    Maria has an unusual morning ritual that she performs when she goes outside to get her mail. The distance from her front door to her mailbox is 30 steps. She steps outside the front door, takes two steps forward, and then takes one step back. She then takes another two steps forward and one step back. She continues doing this until she reaches her mailbox. In all, how many steps does Maria have to take before she gets to her mailbox? 86 steps

    Hint: Working this out for a 30-step trip can be quite difficult. You might want to start by seeing what happens if the mailbox is closer to the front door, perhaps 5 steps rather than 30.

Investigation: Skip Counting and Scaling

13. For (a) and (b) it would be useful to have a sheet with number lines ready for students so they do not spend too much time drawing number lines.

Use 13 (c), (d), and (e) to help your students realize that they are not dealing with another mathematical universe when they are skip counting left, using the negative numbers. Also remind them what skip counting led to in the third grade. Let them remember how skip counting is to multiplying as crawling is to walking. When you crawl, you can get to most of the places you can when you walk. It's just a longer and harder way to get there.

The purpose of this investigation is to let students discover that the product they get when they multiply a positive by a negative is logical when they consider the number line.

13. **Investigation: Skip Counting and Scaling**

When we initially built our number line, we used each mark to indicate one unit. The number line then corresponded to the integers 1, 2, 3, … . For larger numbers, we let the marks represent bigger lengths. So if each mark represents 5 units, then the marks correspond to the multiples of 5, and we have 5, 10, 15, 20, … using "skip counting" by 5's.

a. Build a number line where each mark represents 3 units, skip counting by 3's. What is the 10^{th} number to the right of 0? What is the 10^{th} number to the left of 0? 30 $^{-}30$

b. Build a number line where each mark represents 10 units, skip counting by 10's from $^{-}40$ to 40.

c. Make a table of the numbers you get when skip counting by 2's to 20, skip counting by 3's to 30, skip counting by 4's to 40, … and skip counting by 10's to 100.

d. Now skip count by 2's, 3's, 4's, … up to 10's in the opposite direction. Make a table of the numbers you get when skip counting by 2's to $^{-}20$, by 3's to $^{-}30$, by 4's to $^{-}40$, etc.

e. Do you notice any patterns when you skip count? For example, the 5^{th} number when skip counting by 3's is 15. This is the same as the 3^{rd} number when you skip count by 5's. Does this kind of symmetry always hold? See TE.

Lily Pad Race

Objective: Adding / Subtracting Integers

Materials:

Integer cards from the CD (copy on card stock and cut out)
Lily Pad board game from the CD (one per group)
Coins or some other small items to use as game pieces
Paper / pencil (if necessary to help with solving integer problems)

Activity Instructions:

Students should be divided into small groups and given the following set of instructions for playing the game.

1) Pick a card and solve problem.
2) Move the number of spaces as your answer.
3) If your answer is:
    a. Positive = move forward
    b. Negative = move backwards
4) The player who reaches a Lily Pad first on either end of the board game wins the game.

Section 1.7 - Variables and Expressions

Big Idea:
Understanding the meaning of a variable within an expression

Key Objectives:
- Adding, subtracting and comparing values (multiplying and dividing not included here)
- Translating numerical expressions into words
- Translating words into numerical expressions
- Using the symbols for "is greater than" and "is less than"

Pedagogical/Orchestration
- A variable is a letter/symbol that represents a number.
- Emphasize that $a < b$ means the number a is to the left of the number b on a horizontal number line.

Materials:
No extra materials needed.

Activity:
Use the Jeopardy Game (found on CD) at any time, but may be best after completing Section 6.2. "Match My Algebra"

Vocabulary:
variable, expression

TEKS:
6.1(A,C); 6.5; 6.11(A,B); 6.12(A,B); 6.13(A,B)

New: 6.1(A, B, D, F, G); 6.7(B)

Launch for Section 1.7:

Tell your students, "I'm curious to find out how many languages each of you know." Ask students to raise their hands if they speak one language, two, three, etc. Have just a short discussion on what languages they know. Then tell them that this year, they will be learning a new language called "algebra," and this language is used for problem solving. Tell students, "Every language has words and phrases. What is the difference between a word and a phrase?" Allow for responses and conclude launch with, "Today we will be discussing how words and phrases are used in algebra. Pay close attention to the lesson and see if you can pick up on what a "phrase" is called in algebra, and what the "words" of algebra would be." Write "Word" and "Phrase" on the board leaving room to add additional examples throughout the lesson. Use the listing of words and phrases that would be associated with is greater than, is less than, more than, less than (e.g.increase, older, taller)

You might remember the term "unknown" instead of "variable" for x. To use "unknown" suggests both that x is a specific number and that its identity is the reason for algebra. Algebra today emphasizes the role of function of a variable x that can take many forms and be many possible numbers.

SECTION 1.7 VARIABLES AND EXPRESSIONS

Numbers give us a way to describe different quantities. The number 5 might be 5 marbles, or it might be 5 units on a number line. When we write 5, we have in mind a definite amount or quantity. Often, however, we will want to represent an unknown quantity. To do this, we use variables. For example, Amy has some marbles, but we don't know how many she has. We could then write:

M = the number of marbles Amy has.

A **variable** is a letter or symbol that represents a quantity or number. A variable may represent an unknown quantity, or it may represent general numbers.

We use numbers, variables, and mathematical operations to form **expressions**. For example, Lisa has one more marble than Amy. We could then write "$M + 1$" to describe how many marbles Lisa has. Expressions are mathematical phrases like "$M + 1$" that we use to describe quantities mathematically. If two phrases represent the same quantity, then we say that the two expressions are **equivalent**. For example, "M+1" and "1+M" are equivalent expressions. Similarly, 3+6 and 6+3 and 9 are also equivalent expressions.

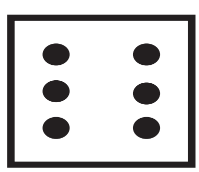
The variable M above represents the number of marbles Amy has, even though we do not know what the number is. Variables give us a convenient way to describe properties and ideas because variables can represent many different numbers.

EXAMPLE 1

Translate "three more than six" into a mathematical expression. Represent this with both concrete models as well as pictorially using a number line. Find equivalent expressions that your models represent.

SOLUTION

To find three more than six with a concrete model, we begin by representing six as



PROBLEM 1

"Five more than two" translates as 2 + 5.

Notice that we would get the same answer if we calculated 5 + 2, but that would really represent the expression "two more than five."

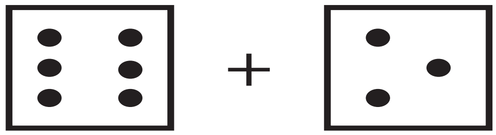
Recall back in 1.2 that we looked at the relationship "Five is more than two" but "Five more than two" is clearly different.

Emphasize that although the answers are the same in the "more than" expression in Example 1, it is still important to write the expression correctly, as we see in the next example. Remind your students about the fact that position in addition doesn't change the sum. Ask them to recall the Commutative Property for Addition. As a last thought, you will always have students who call the property the "Communitive." Remind them that numbers don't talk, or communicate. They can, however, switch positions, or commute. Talk about commuters moving from their home to work and back.

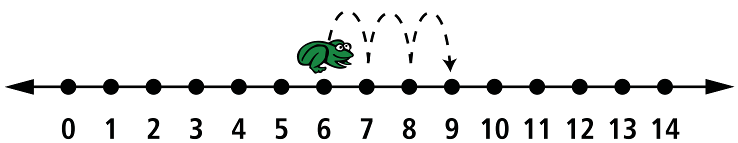
EXAMPLE 2

Emphasize the importance of order. In fact, the order is changed in expressions like "less than" or "greater than." In Example 2, find the difference between 2 – 5 and 5 – 2. Talk about the fact that the two answers are opposites of each other. Notice that 2 – 5 will result in a negative number. This foreshadows integer subtraction and while the students have seen negative numbers they have not yet looked at the operation of subtraction yielding negative differences.

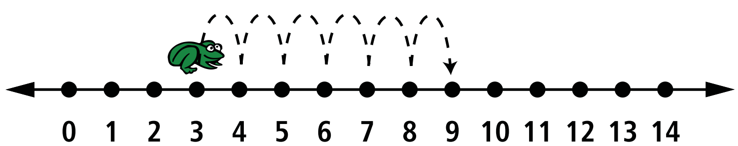
In order to find three more than 6, we must add 3. So our concrete model would be



We could represent this as 6 + 3 or equivalently as 3 + 6, or simply add and obtain 9 altogether. So the expressions 6+3, 3+6, and 9 are all equivalent, since they all have the same value. Even though these expressions are equivalent, our original expression "three more than 6" indicates that we should begin at 6, and then increase this amount by 3. So the most accurate translation would be "6 + 3. On this number line this would look like:



Use a number line to illustrate 3 + 6. Is your work similar to the representation below?



In each case, the point we end up at is 9. So each of the expressions 6+3, 3+6, and 9 are all equivalent, since they all have the same value.

EXAMPLE 2

Determine which of the expressions below are equivalent:

a. "two more than six" b. "three less than eleven" c. "one more than x"

Explain with concrete models, with the number line, and algebraically.

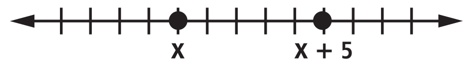
SOLUTION

a. With concrete models,

"two more than six" **&** •••••• + •• **&** 6+2

PROBLEM 2

"Five more than x" translates as $x + 5$. Notice that you get this answer no matter what the variable x represents.

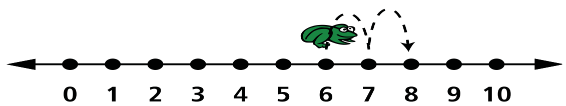


"three less than eleven" ••••••••••• – ••• & 11 – 3

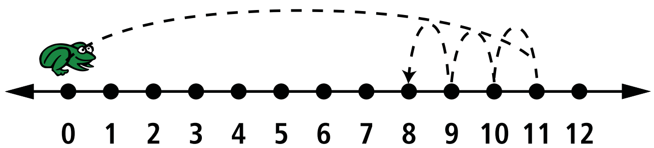
(in either case we are left with 8, so these expressions are equivalent.)

"one more than x" & x + 1. Since we do not know the value of x, this expression is not equivalent to the above.

b. With the number line, 6 + 2 ends up at 8



and 11 – 3 ends up at 8 as well



X + 1 ends up one unit to the right of X, but again, we do not know where X is. So the expressions 6+2, 11-3, and 8 are all equivalent. X + 1 is not equivalent, unless X has value 7.

PROBLEM 1

Translate "five more than two" into a mathematical expression.

EXAMPLE 3

Translate "five less than two" into a mathematical expression.

SOLUTION

"Five less than two" translates as 2 – 5.

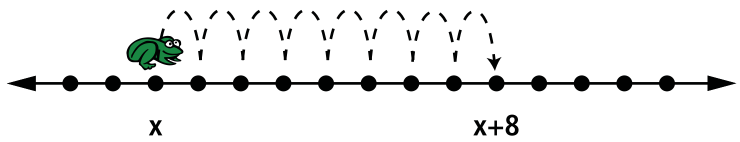
In this case, this is not the same as 5 – 2. Do you see the difference?

EXAMPLE 4

Translate "8 more than x " into a mathematical expression. Illustrate this on the number line with an arbitrary point with coordinate x.

SOLUTION

"8 more than x" can be written as x + 8. On the number line, we have an arbitrary point x and then go 8 units to the right as follows:



PROBLEM 2

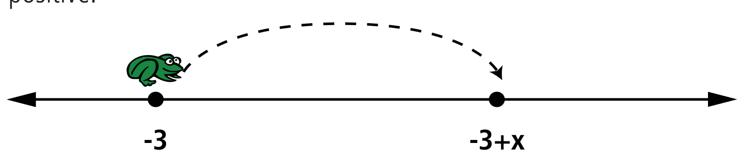
Translate "five more than x" into a mathematical expression. Illustrate this on the number line with an arbitrary point with coordinate x.

EXAMPLE 5

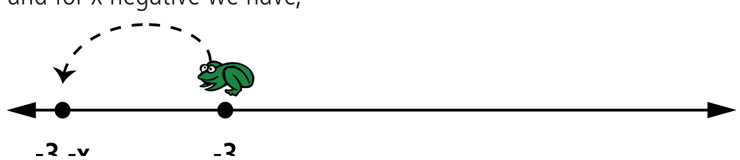
Translate "x more than – 3" into a mathematical expression. Illustrate this on the number line.

SOLUTION

"x more than – 3" can be written as – 3 + x. On the number line, we have for x positive:



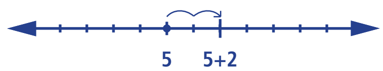
and for x negative we have,



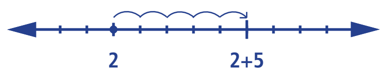
Remember that the symbol < is the "less than" symbol, and the symbol > is the "greater than" symbol. So, the inequality $x < 2$ says that "x is less than 2." Although we do not know what the variable x is, this inequality says that x is some number that is less than 2 and that x is to the left of 2. On the other hand, the expression "x less than 2" is written mathematically as 2 – x.

EXERCISES

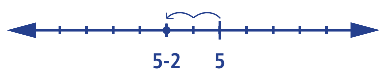
1. a. 5 + 2



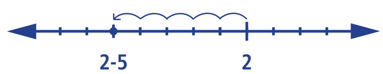
2 + 5



5 - 2



2 - 5



b. 8 - 3
3 - 8
8 + 3
3 + 8

c. 1+6
6+1
1-6
6-1

4. a. four less than nine
b. six more than negative three
c. three more than five
d. three less than negative four
e. five more than x
f. six less than r
g. m is less than seven
h. p is greater than zero

5. a. "three is less than nine" is a statement of inequality
"three less than nine" is an expression indicating subtraction
b. three is less than nine; 3 < 9
three less than nine; 9 - 3

6. a. 9 - 6
b. 8 > 4
c. 6 < 9
d. A > C
e. A < C
f. C - A

EXERCISES

1. Translate each of the following into a mathematical expression. Determine which of these are equivalent using concrete models. Check your answer algebraically.

   a. Two more than five b. Five more than two
   c. Two less than five d. Five less than two

2. Translate each of the following into a mathematical expression. Determine which of these are equivalent using the number line. Check your answer algebraically.

   a. Three less than eight b. Eight less than 3
   c. Three more than eight d. Eight more than three

3. Translate each of the following into a mathematical expression. Determine which of these are equivalent using the number line. Check your answer algebraically.

   a. Six more than 1 b. One more than six
   c. Six less than 1 d. 1 less than six

4. Translate each of the following into a mathematical expression. Determine which of these are equivalent using the number line. Check your answer algebraically
   a. Two more than X
   b. X less than two
   c. One less than a number three larger than x
   d. X larger than 2.

5. Write the following mathematically:

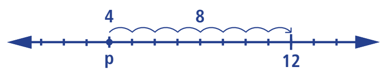
   a. 7 decreased by 5
   b. 9 less than 6
   c. 11 taken away from 20
   d. 3 years younger than a 12 year old
   e. 8 inches shorter than a six-foot-person

7. B = -1, 0, 1, 2, 3, 4, 5, or 6

8. negative four is less than two

9. 2 + 5

10. 12 = P + 8, P = 4



11. M - 4 = 1, M = 5

12. X + 4

13. 2y + 4 marbles

Spiral Review

14. Eric walks 255 feet and Emma walks 210 feet. So Eric has to walk farther.

15. a. numbers greater than -2 are to the right of -2. (i.e. -1, 0, 1, 2, 3, ...)
    b. numbers less than 1 are to the left of 1. (i.e. 0, -1, -2, -3, ...)
    c. -1 and 0

Ingenuity

Answers will vary with student choice of integer.

6. Translate each of the following into a mathematical expression:

   a. Two more than A — b. A more than two

   Two less than A — A less than two

   c. A more than B

   B more than

7. Translate the following expressions or inequalities into word phrases or sentences: See TE.

   Example: $^{-}3 - 6$ six less than negative three

   a. $9 - 4$ — e. $x + 5$
   b. $^{-}3 + 6$ — f. $r - 6$
   c. $5 + 3$ — g. $m < 7$
   d. $^{-}4 - 3$ — h. $p > 0$

8. a. Using words (not mathematical symbols) explain the difference between the statement "three is less than nine" and the statement "three less than nine." See TE.

   b. Translate the two statements from part **a** into mathematical expressions or inequalities. See TE.

9. Translate each of the following into a mathematical expression or inequality:

   a. six less than nine — d. A is greater than C
   b. eight is greater than four — e. A is less than C
   c. six is less than nine — f. A less than C

10. If B is less than 7 and greater than $^{-}2$, what integers could B represent?

11. Translate the inequality $-4 < 2$ into words. Answers may vary.

12. Translate the phrase "five greater than two" into a numerical expression.

13. Twelve is eight more than P. What integer does P represent? Show this on a number line.

14. Emily eats four cookies and has 1 cookie left. Let M = the number of cookies Emily had at the beginning. What integer does M represent?

15. Eddie and Sam like to play marbles. Eddie has x marbles. He buys 4 more marbles. Express the number of marbles Eddie has, in terms of x.

16. Eddie's friend, Miko, has y marbles. Sam's sister, Natasha has twice as many marbles as Miko. Natasha buys four more marbles. How many marbles does Natasha have now?

Spiral Review:

17. Emma walks 210 feet to get to Central Park. Eric walks 85 yards to get to Central Park. Who had to walk further to get to Central Park? Explain?

18. Use the number line to locate the numbers -2 and 1.

    a. Locate and identify two numbers greater than -2.

    b. Locate and identify two numbers less than 1.

    c. Locate and identify two numbers greater than -2 and at the same time less than 1.

19. **Ingenuity:**

    Choose a three-digit positive integer, and write this integer down. Then perform the following steps. After each step, write down the integer you get.

    a. Multiply your integer by 3.
    b. Add 2 to the integer you got in part a.
    c. Multiply the result from part b. by 2
    d. Subtract 1 from the integer you got in part c.
    e. Subtract your original integer from the result you got in part d.
    f. Finally, multiply the result from part e. by 2

    What do you notice about your answer? (If you don't notice anything at first, it may be helpful to try this process with a different integer.) Can you explain why this happens?

20. **Investigation:**

In each of the following equations, a number has been replaced with a question mark. Determine what number needs to go in the place of the question mark so that the equation is true.

Investigation

17. a. 7 b. 6 c. 6 d. 5 e. 21 f. 11

a. $5 + ? = 12$
b. $? - 4 = 2$
c. $15 - ? = 9$
d. $3 \cdot ? = 15$
e. $4 \cdot 4 = ? - 5$
f. $2 \cdot ? - 15 = 7$

Section 1.8 - Graphing on the Coordinate System

Big Ideas:
Develop and explore the coordinate system

Key Objectives:

- Locate, name and plot points on a coordinate plane.
- Identify corresponding quadrants with specific ordered pairs.

Pedagogical/Orchestration:
Make a big grid on the wall or floor so that students can interact with the coordinate plane. See Activities.

Activities:
Use Battleship Activity at the end of the section.

A fun and useful activity is to have the whole class build a large coordinate grid with yarn on the floor or wall. It will be useful in referencing vocabulary and plotting points.

Materials:
Rulers, Handouts of coordinate grids with axes (several per student for notes and exercises), Coordinate grid for whole-class demonstration from an overhead, a pull-down chart, or one created on chart paper, markers or map colors for exercises, Optional: dry-erase whiteboards with grids and dry erase markers for each student, globe(s) for the Investigation:

Internet Resources:
Game to review the coordinate plane: http://www.quia.com/cz/43437.html

Exercises:
12 (Investigation)—Have grid paper for this exercise.

Vocabulary:
This section has an extensive collection of vocabulary. As an activity/project, you may wish to have the students develop their own vocabulary foldable (booklet or flashcards) that contains the definitions, examples, and even pictures of each of the vocabulary word in this section. There will be an opportunity to define some of these words in the worksheets for this section also.

coordinate	quadrant	coordinate plane
ordered pairs	origin	x-coordinate
axes	y-coordinate	x-axis (horizontal axis)
lattice points	y-axis (vertical axis)	Cartesian coordinate system
point	line	plane
intersecting lines		

TEKS:

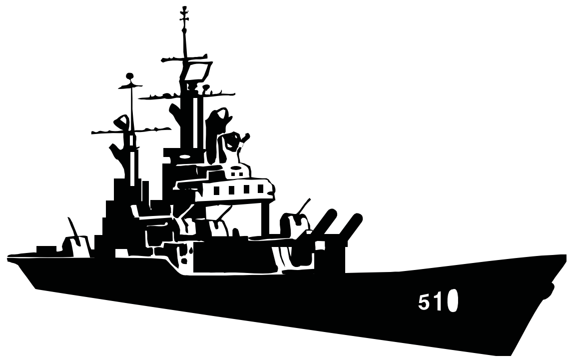
6.7; 6.11(A,C); 6.12(A,B); 6.13(A,B)

New: 6.11

Launch for Section 1.8:

Ask your students, "Have you ever laid in bed staring at the ceiling? Pretty boring, right? In the 1600s there was a man named Descartes who was doing just that, but entertained himself by watching a fly that was crawling around on the ceiling. After watching the fly for awhile he started wondering if there was a way to describe the location or position of the fly so that he could tell someone else where the fly was. It occurred to him that he could describe the fly's position by measuring its distance from the walls of the room. Maybe Descartes didn't realize it at the time, but he had just invented the coordinate plane! In fact, the coordinate plane is often called the Cartesian plane in his honor." Ask your students, besides determining the location of a fly, how could a system like this be useful in real life? Possible responses could be in locating a ship at sea, or a location on a map. Inform your students, "Today we will be discovering the Cartesian plane ourselves as a way to describe the position of points." Lead your students through the description of the coordinate plane found in the beginning of Section 1.7.

Battleship



Objective: The students will play Battleship to practice graphing and locating points on a coordinate system.

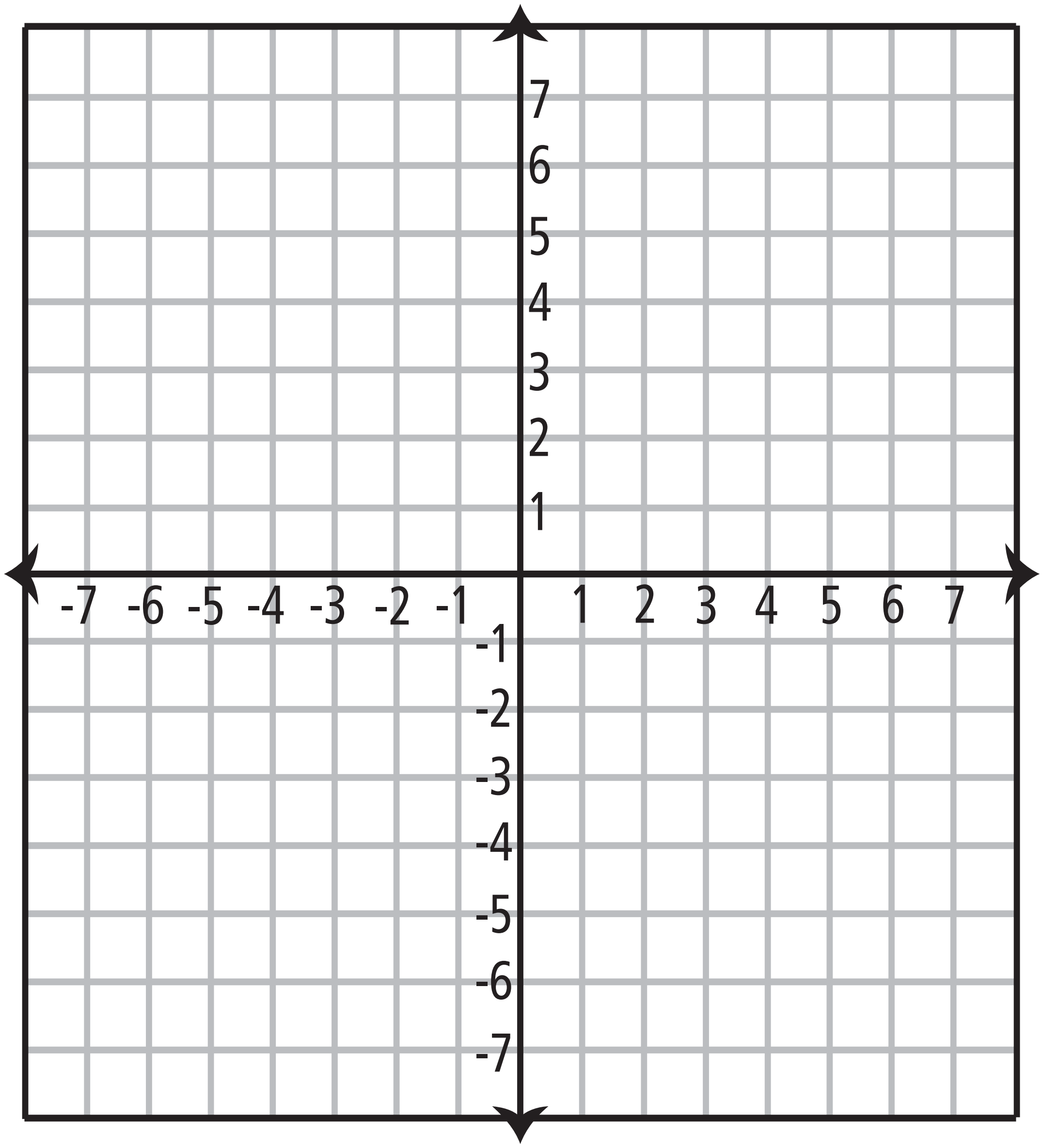
Materials:
Battleship Folders (glue instruction on the front, coordinate planes inside folder)
Dry erase marker

Activity Instructions:

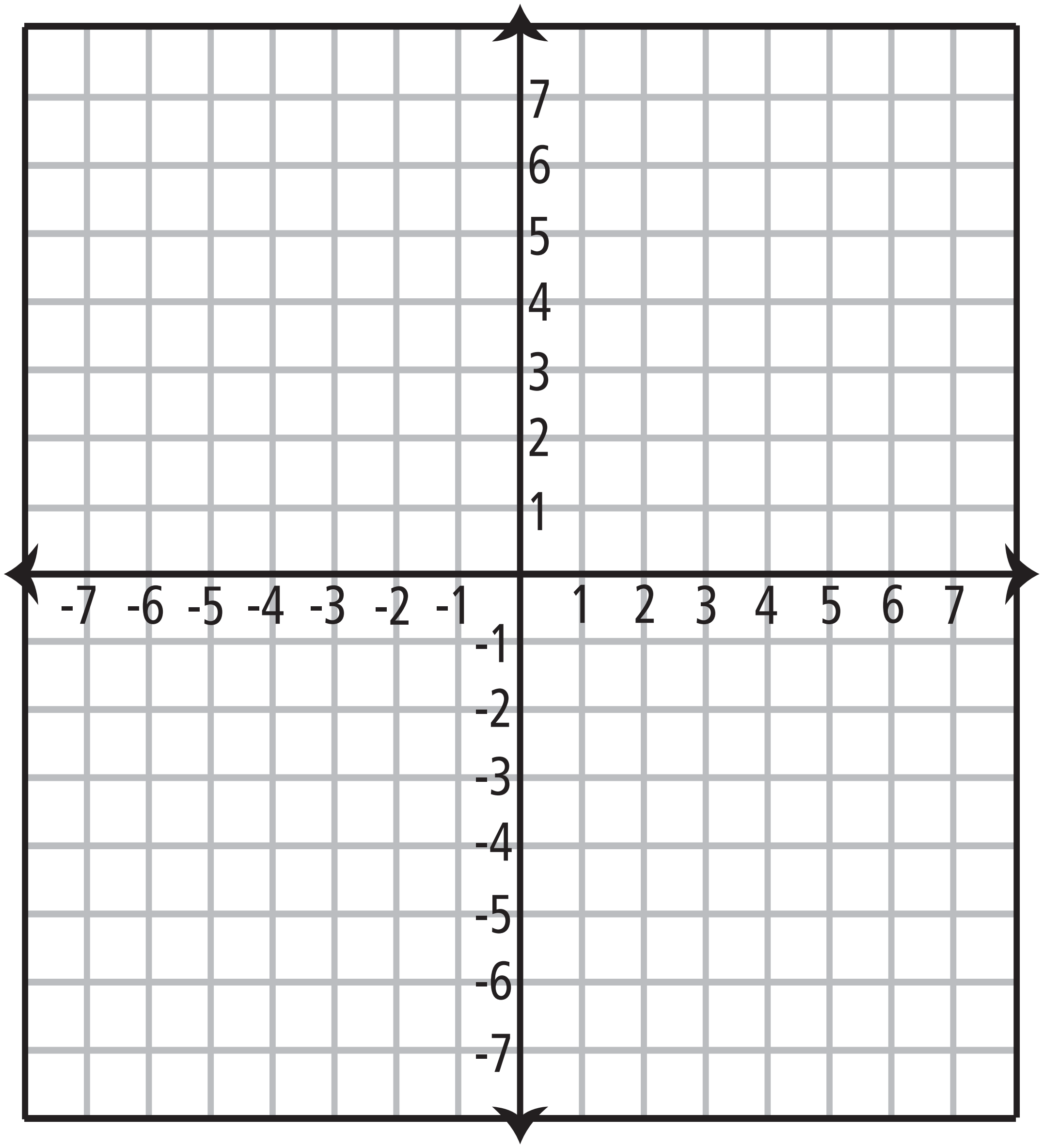
1. The students will choose a partner and play Battleship.
2. Each player will place 4 dots in a straight line to represent a ship- vertical, horizontal, or diagonal on their worksheet, "My Ship and Your Hits". Place 4 ships on any of the four quadrants.
3. The students will take turns guessing ordered pairs to try to find their opponent's battleships.
4. The first player calls out a coordinate pair, marking an "X" on the side MY GUESSES.
5. If it is a hit, the other player marks an X on the YOUR HIT side of the folder. If it was a miss there is no need to mark the coordinate. Each time inform your opponent if it was a hit or a miss.
6. The game continues until one player has found all of their opponents four ships.
7. The teacher can choose the level of difficulty for the game, based on the needs of the class:

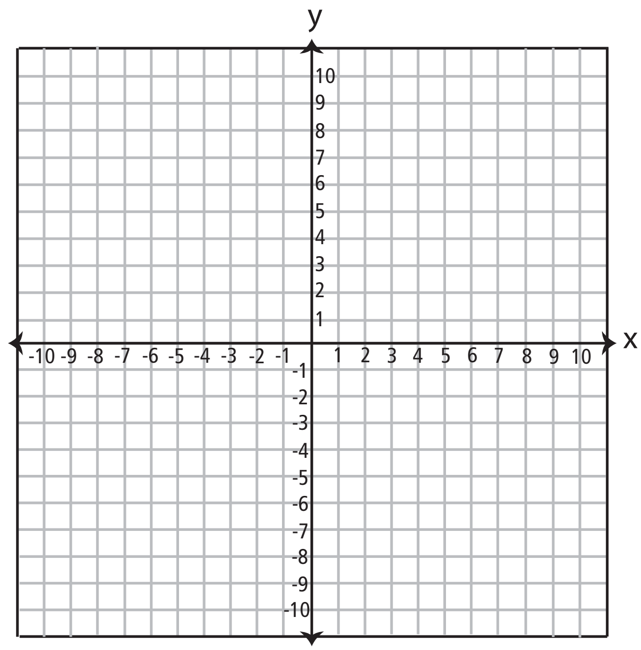
Easy……………..Quadrant I only.
Medium…………Quadrants I and II.
Hard……………..All four Quadrants.

My Guesses



My Ships and Your Hits

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8
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6
5
4
3
2
1
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1
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-2
-3
-4
-5
-6
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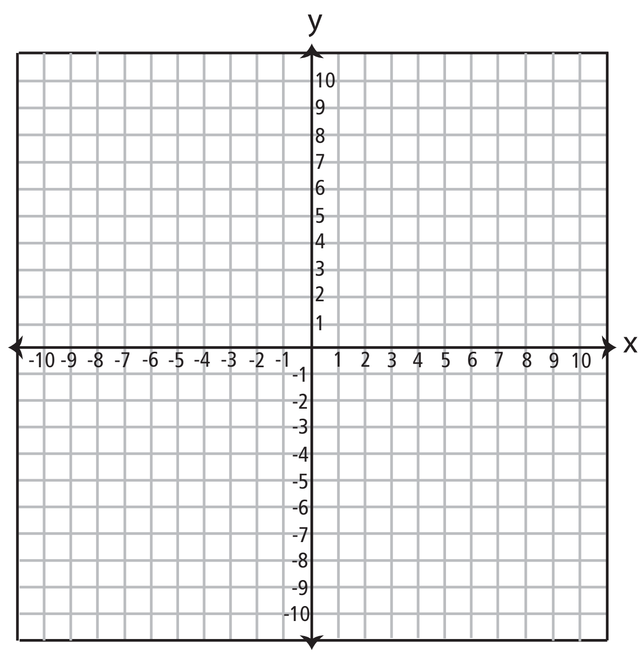
-9
-10


y
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-1
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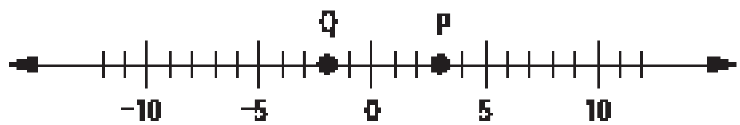
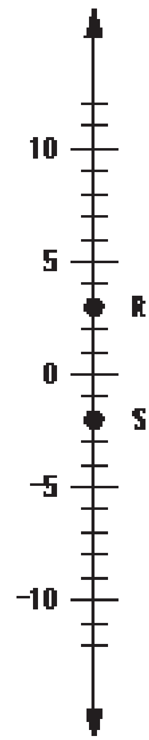
-9
-10

Teachers, note that this section includes a lot of vocabulary that might be new to your students. You may want to use butcher or chart paper to make a coordinate plane to display in your room. You can label the various aspects of the plane for students to reference as the year continues. You can also use sticky notes to make coordinate points for instruction. Many teachers have enjoyed using small, student-sized whiteboards with the coordinate plane printed on it. This allows students to erase mistakes easily and they can hold their boards up in unison for quick assessment.

SECTION 1.8 GRAPHING ON THE COORDINATE SYSTEM

We use the number line to represent numbers as locations. To each point on the number line we associate a number, or **coordinate**, which is the location of that number.





For example, the number line above shows points *P* and *Q*. To graph, or plot, a point *P* with coordinate 3 on the number line, we graph the point 3 units to the right of 0. Because point *Q* has coordinate ⁻2, we graph the point 2 units to the left of 0 on the number line.

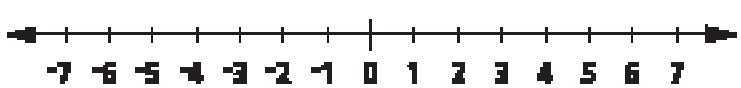
If we draw our number line horizontally as above, then positive numbers are located to the right of 0 and negative numbers are located to the left of 0.

Suppose we draw our number line vertically like a thermometer. In this case, positive numbers are above zero, and negative numbers are below zero.

On the vertical number line, we could locate a point *R* with coordinate 3 by graphing the point 3 units above 0. The point *S* with coordinate ⁻2 would be located 2 units below 0. On a number line, each point corresponds to a number.

If we want to plot points on a plane, we will need to use two numbers, again called coordinates, to locate the point. A **coordinate plane** is constructed as follows:

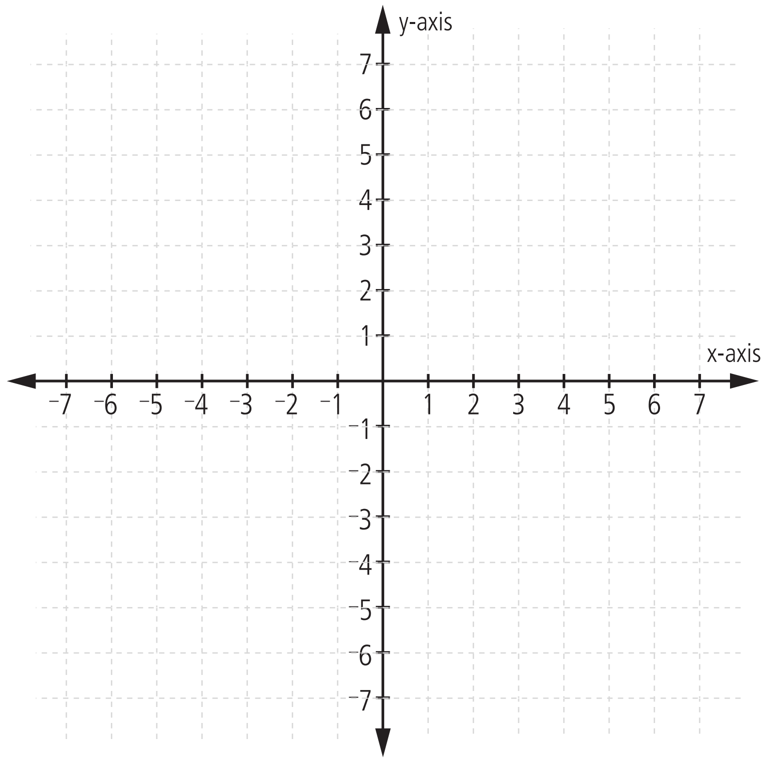
We begin by drawing a horizontal number line and locating the zero point, which is called the **origin**:



Ask your students in what order around the origin are the quadrants numbered.

Answers may vary for these questions. Students might research the answers to these questions online or in the library. Encourage them to research the questions.

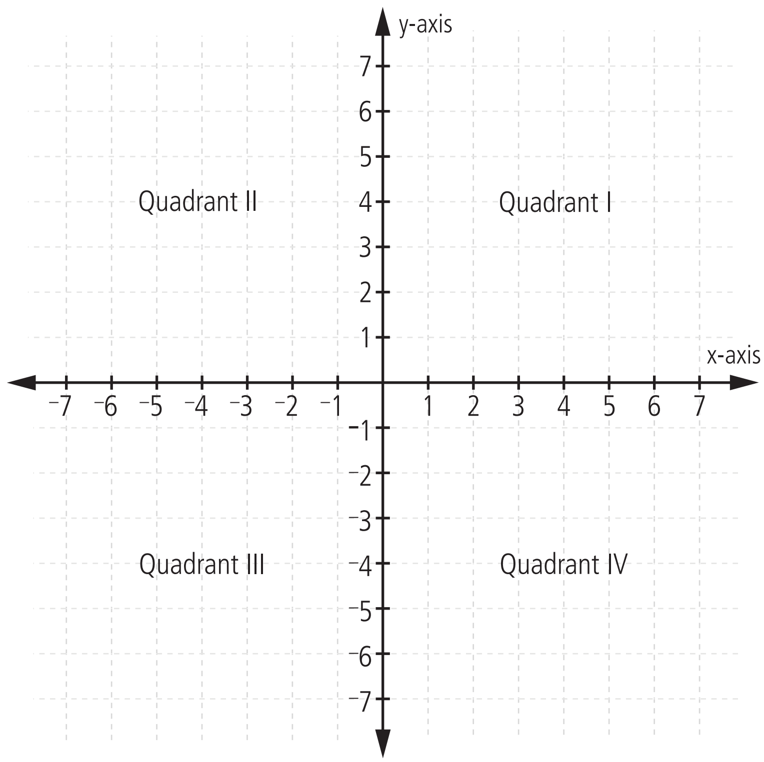
Next, draw a vertical number line through the origin of the horizontal number line so that the two zero points coincide:



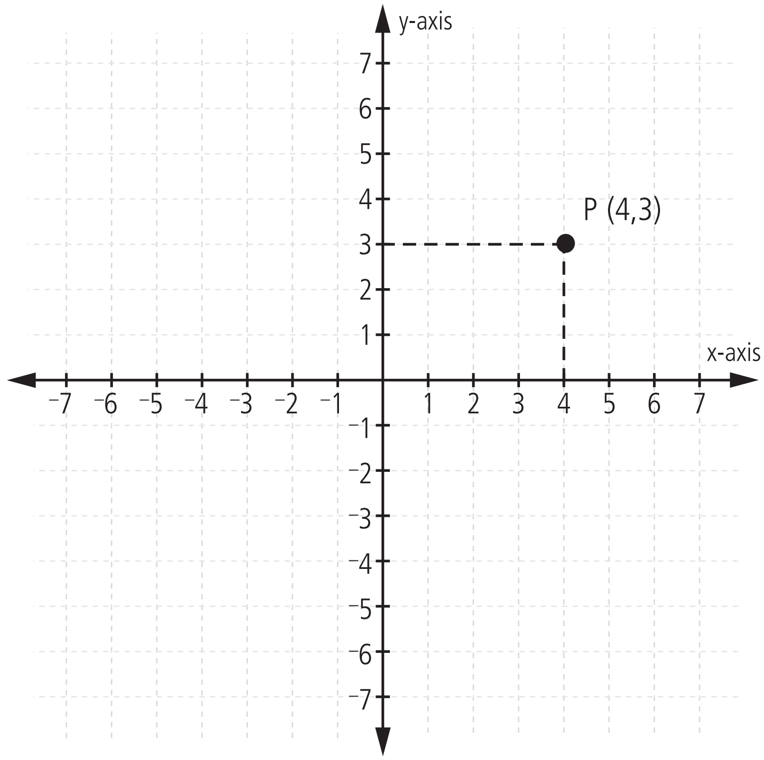
The horizontal number line is called the **horizontal axis,** or the **x-axis**; the vertical number line is called the **vertical axis,** or the **y-axis**.

Remember, the point at which the two number lines, or **axes**, meet is called the origin and is the zero point on both number lines.

The x- and y-**axes** divide the plane into four regions. Because there are four of them, we call each region a **quadrant**. By convention, we number the quadrants **counterclockwise**, starting with the upper-right quadrant. The axes don't belong to any quadrants but rather are their boundaries.


y-axis
x-axis
Quadrant II
Quadrant I
Quadrant III
Quadrant IV
7
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−7
−7 −6 −5 −4 −3 −2 −1
1 2 3 4 5 6 7

You may wish to discuss with the students lattice structures that they have seen in a garden. The term cross grids may also require a visual accompaniment. Grid paper can provide a good example.



The coordinates for the points in the coordinate plane are always **ordered pairs** of numbers, (first coordinate, second coordinate). The first coordinate is called the **x-coordinate**; the second is the **y-coordinate**.

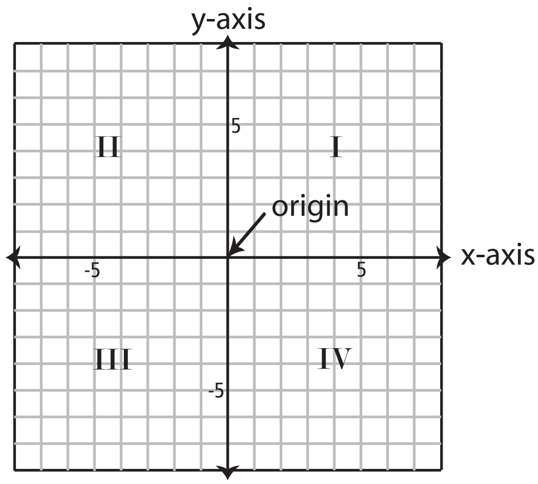
Consider the point *P* on the coordinate plane above. To identify point *P*, we begin at the origin. First we move 4 units to the right on the *x*-axis, then we move up 3 units. We arrive at the point *P*. Therefore, the point *P* is identified as (4, 3), where 4 represents the *x*-coordinate, and 3 represents the *y*-coordinate.

Notice that the coordinates of the origin are (0, 0) because the origin lies at the zero point of both number lines. The points that contain integers, like (4, 3) or (5, ⁻2), are called **lattice points** because they fall on the cross grids, which look like a lattice.

This coordinate system with horizontal and vertical axes is called a **Cartesian coordinate system**. It is named after René Descartes, the French mathematician and philosopher who invented it.

If time is an issue, consider having groups work on exercises 3-9 and share their findings.

1.



2.

A	(3, 4)	*G*	(0, -2)
B	(4, 2)	*H*	(6, 0)
C	(-4, -1)	*I*	(-1, 4)
D	(-4, 6)	*J*	(-4, 1)
E	(3, -2)	*K*	(-4, -4)
F	(2, 1)	*L*	(0, 0)

EXERCISES

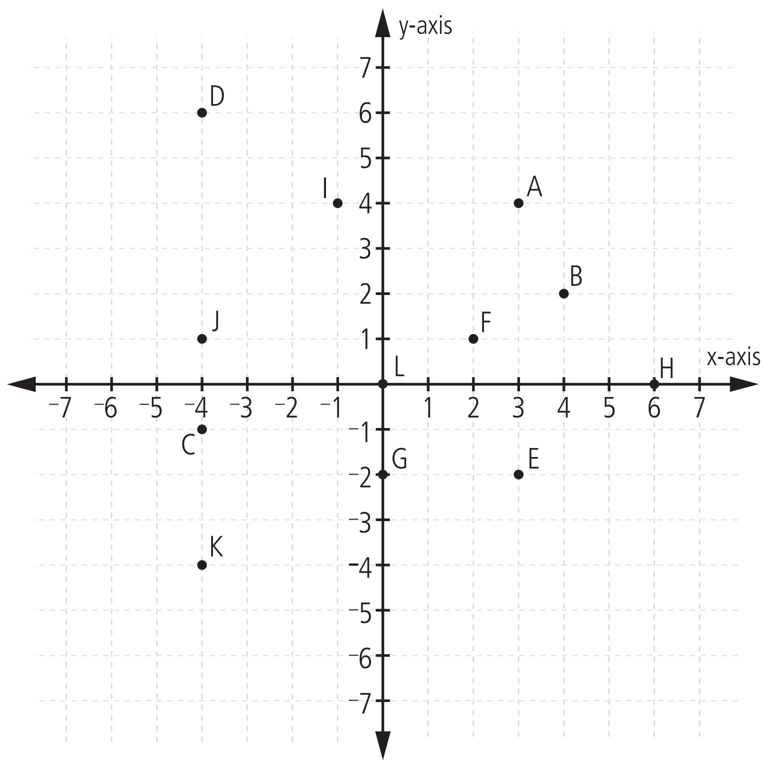
For most of these exercises, you will need to draw on coordinate planes.

1. a. Draw a coordinate plane and number each axis from 5 to -5.

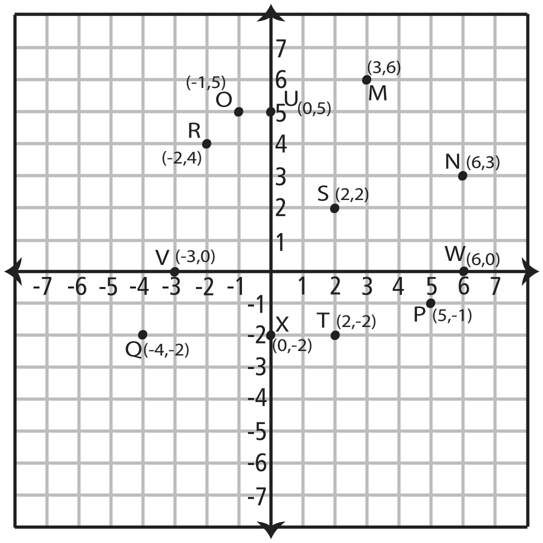
   b. Label the following parts of the coordinate grid from part a:

x-axis	quadrant I	quadrant III
y-axis	quadrant II	quadrant IV
origin		

2. Write the coordinates for each of the points *A* to *L* shown on the coordinate plane below. See TE.



3.



For the next few questions, it may help to have the students use a different color for each section (i.e. a is red, b is blue, c is green, etc.)

4. Answers will vary. One example of a student's response:
   a. (−1, 2), (−2, 4), (−5, 10)
   b. (1, −2), (2, −4), (5, −10)
   c. (1, 2), (2, 4), (5, 10)
   d. (−1, −2), (−2, −4), (−5, −10)
   e. All of the points from part (a) are in Quadrant II, all of the points from part (b) are in Quadrant IV, all of the points from part (c) are in Quadrant I and all of the points from part (d) are in Quadrant III.
5. Answers will vary. One example of a student's response:
   a. (0, 2), (0, 4), (0, 10)
   b. (0, -2), (0, -4), (0, -10)
   c. (0, 2), (0, 0), (0, -10)
   d. (1, 0), (0, 0), (-5, 0)
   e. All of the points from part (a) are on the positive part of the y-axis, and all points from part (b) are on the negative part of the y-axis. All of the points on part (c) are on the y-axis, but could be anywhere along the y-axis. All of the points from part (d) are on the x-axis.
6. Answers will vary. One example of a student's response:
   a. (1, 2), (2, 4), (5, 10), (-2, -4), (3, 6)
   b. (2, 1), (4, 2), (10. 5). (-4. -2). (6. 3)
   c. The points in part (a) form one line. Points in part (b) form another line. Plot these on a large coordinate system by students going to the board or floor. Plot enough of them to help them see a pattern.

3. Create a coordinate plane on graph paper, and plot and label the following points on a coordinate plane. See TE.

M (3, 6)	*Q* (⁻4, ⁻2)	*U* (0, 5)
N (6, 3)	*R* (⁻2, 4)	*V* (-3, 0)
O (⁻1, 5)	*S* (2, 2)	*W* (6, 0)
P (5, ⁻1)	*T* (2, -2)	*X* (0, ⁻2)

4. Find and plot 3 points that meet the following conditions: See TE.
   a. Each point has a negative *x*-coordinate and a positive *y*-coordinate.
   b. Each point has a positive *x*-coordinate and a negative *y*-coordinate.
   c. Each point has positive *x*-and *y*-coordinates.
   d. Each point has negative *x*-and *y*-coordinates.
   e. What do you notice about the points in each situation?
5. Find and plot 3 points that satisfy the following conditions: See TE.
   a. Each point has the *x*-coordinate equal to 0 and a positive *y*-coordinate.
   b. Each point has the *x*-coordinate equal to 0 and a negative *y*-coordinate.
   c. Each point has the *x*-coordinate equal to 0, but is different from part a.
   d. Each point has the *y*-coordinate equal to 0.
   e. What do you notice in each situation?
6. Find and plot 5 points that satisfy the following conditions: See TE.
   a. Each point has a *y*-coordinate that is double the *x*-coordinate.
   b. Each point has an *x*-coordinate that is double the *y*-coordinate.
   c. What do you notice about the points in each situation?
7. Find and plot 5 points that meet the following conditions: See TE.
   a. Each point has the *y*-coordinate equal to 1.
   b. Each point has the *y*-coordinate greater than 1.
   c. Each point has the *y*-coordinate less than 1.
   d. What do you notice?

7. Answers will vary. One example of a student's response:
   a. (-1, 1), (0, 1), (5, 1), (4, 1), (-2, 1)
   b. (0, 2), (2, 4), (-5, 10), (0, 3), (9, 7)
   c. (-1, -2), (0, -4), (5, -10), (0, -5), (-9, -17)
   d. The points in part (a) fall on a line. The points in part (b) are all above the line formed by the points in part (a), more particularly the y = 1 line. These points are also in Quadrants I and II. The points in part (c) are all below the line formed by part (a).

8. Answers will vary. One example of a student's response:
   a. (-3, -3), (-3, 0), (-3, 5), (-3, 4), (-3, -2)
   b. (-1, 0), (-1, 2), (-1, -5), (-1, 4), (-1, 9)
   c. (-2, -2), (-2, -4), (-2, -10), (-2, -5), (-2, -17)
   d. The points in part (a) fall on a line, the x = -3 line. The points in part (b) all fall on a line as well, the x = -1 line. The points in part (c) all fall in the region between the line from part (a) and the line from part (b), this also happens to lie in Quadrants III and IV.

9. Answers will vary. One example of a student's response:
   a. (-3, -3), (0-, 0), (5, 5), (4, 4), (-2, -2)
   b. (0, 1), (2, 3), (-5, -1), (-4, 1), (-90, -10)
   c. (-1, -2), (0, -4), (5, -10), (0, -5), (-9, -17)
   d. The points in part (a) fall on a line, the y = x line. The points in part (b) all fall above the line from part (a). The points in part (c) all fall below the line from part (a).

Spiral Review

10. David (3 hrs), Diane (4.25 hrs or 4 hrs), Robert (4 hrs), Karen (4.5 hrs or 4 hrs)

11. Correct Answer: C) 12 x 5

INGENUITY:

12. (R, U, L, D)

8. Find and plot 5 points that meet the following conditions: See TE.
   a. Each point has the x-coordinate equal to $^-3$.
   b. Each point has the x-coordinate equal to $^-1$.
   c. Each point has x-coordinates greater than $^-3$ and less than $^-1$.
   d. What do you notice?

9. Find and plot 5 points that satisfy the following conditions: See TE.
   a. Each point has the same y-coordinate as the x-coordinate.
   b. Each point has the y-coordinate larger than the x-coordinate.
   c. Each point has the y-coordinate smaller than the x-coordinate.
   d. What do you notice?

Spiral Review:

10. The table below shows the time it took 4 people to row up the river. Order the rowers in order from fastest to slowest.

Rowing Time

Rower	Time
Robert	4 hours 23 minutes
Karen	4.5 hours
Diane	4 hours 15 minutes
David	180 minutes

11. Tamika cut 5 cucumbers for a salad. She cut each cucumber into 12 pieces,. Which number sentence can be used to find C, the total number of cucumber pieces Tamika cut?
   a. C = 12 + 5   c. C = 12 x 5
   b. C = 12 - 5   d. C = 12 ÷ 5

12. **Ingenuity:**

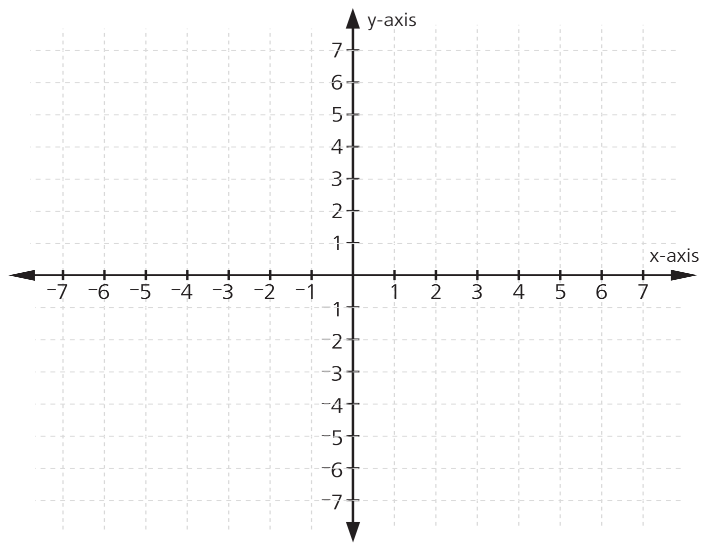
   Phyllis stands at the point (-2, 3) in the coordinate plane. She travels a distance of 7 units in one of the cardinal directions (up, down, left, or right), then travels 4 units in one of the cardinal directions, then travels 2 units in one of the cardinal directions, then travels 1 unit in one of the cardinal directions. After making her four moves, Phyllis is at the point (3, 6). In which direction did Phyllis travel on each move?

Investigation

13. a. 12
    b. 22
    c. The product of the lengths of sides of the rectangle gives the area.
    d. Use two of the triangles to form a unit square.
    e. 18

13. **Investigation:**

Draw a coordinate plane, and label each axis from -7 to 7. Draw grid lines in the coordinate plane as shown below. Notice that grid lines divide the plane into 1 x 1 squares, which we call "unit squares".



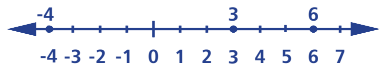
a. Locate the points (1, 4), (-2, 4), (-2, 0), and (1, 0). Connect these four points to form a rectangle. How many unit squares lie inside this rectangle?
b. Locate the points (-6, 5), (5, 5), (5, 7), and (-6, 7). Connect these four points to form a rectangle. How many unit squares lie inside this rectangle?
c. Find the lengths of the sides of the rectangles you created in (a) and (b) Do you see a connection between the lengths of the sides and the areas of the rectangles?
d. Locate the points (4, -1), (1, -4), (4, -7), and (7, -4). Connect these four points to form a quadrilateral. This quadrilateral happens to be a square. Suppose we want to count the number of unit squares that lie inside this square. What should we do with the unit squares that only partially lie inside the square?
e. How many unit squares lie inside the square you constructed in part (d)?

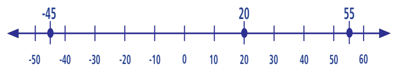
Challenge Problem

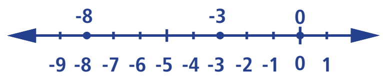
14. Total number of ways: 35 different ways.

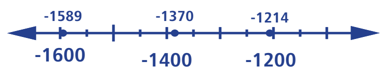
14. **Challenge Problem:**

Alan, an ant, starts at the origin in the coordinate plane. Every minute he can crawl one unit to the right or one unit up, thus increasing one of his coordinates by 1. How many different paths can Alan take to the point (4,3)?

1. a 

b 

c 

d 

2. The temperature rose by 13°C.

3. The temperature rose 20°F.

4. 1, 2, 3

5. The Illiad was written 40 years earlier than the Odyssey.

6. a. -5

   b. 28

7. a. 457 > 81

   b. -23 > -32

   c. 191 > -3

   d. |-11| > 8

   e. |-23| < |-32|

   f. -(-5) < |-6|

REVIEW PROBLEMS

1. For each part below, draw a number line with the three given integers marked:

   a. 3, ⁻4, 6
   b. 20, ⁻45, 55
   c. ⁻8, 0, ⁻3
   d. ⁻1214, ⁻1589, ⁻1370

2. At 7:00 A.M., Chicago's O'Hare Airport has a temperature of ⁻9 ˚C. At 11:00 A.M. that same day, the temperature reads 4 ˚C. Did the temperature rise or fall? Can you determine by how many degrees?

3. At 4 P.M. in London, the temperature was 77 ˚F. At 6 A.M. the same day, the temperature was 57 ˚F. Did the temperature rise or fall? By how much?

4. Maggie locked all of her money inside a safe and forgot the combination. Luckily, Maggie left this note for herself: "The combination to open this safe is three positive integers. These positive integers are represented, in order, by the variables a, b, and c. c is three, $b < c$, and $a < b$." What is the combination to open the safe?

5. Homer was a Greek poet who produced several well-known epics during his lifetime. He wrote *The Odyssey* around 680 B.C.E. and *The Iliad* around 720 B.C.E. Which of these two works was written first? How many years passed between these two dates?

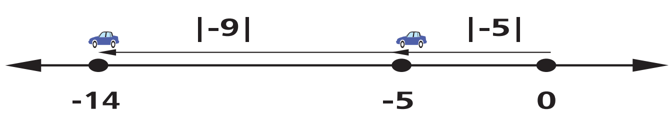
6. a. Which of the following numbers is farthest from 0: ⁻2, 3, or ⁻5?
   b. Which of the following numbers is closest to 14: 28, ⁻1, or ⁻5?

7. Compare the pairs of numbers below and place the appropriate symbols between them. Use < or >.

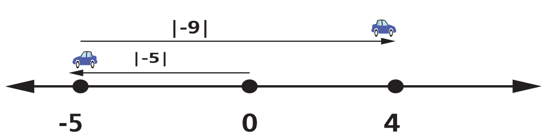
   a. 457 ☐ 81 b. ⁻23 ☐ ⁻32 c. 191 ☐ ⁻3
   d. |⁻11| ☐ 8 e. |⁻23| ☐ |⁻32| f. ⁻(⁻5) ☐ |⁻6|

8. a. 1
   b. 8
   c. 14
   d. 14
   e. 28
   f. 28
   g. 30
   h. 20
   i. 40
   j. 40
   k. 50
   l. 0

9. a. hottest: Venus, coldest: Neptune/Uranus
   b. closest: Mercury, farthest: Neptune
   c. With the exception of Venus, a planet's average temperature decreases as its distance from the Sun increases.



10. a. -5 + -9 = -14
    b. -5 – (-9) = 4



    c. -9 – 5 = -14
    d. -9 – (-5) = -4

11. 73 – 58 = 15

8. Draw a number line for each problem, and find the distance between the following pairs of numbers.

   Example: The distance between -2 and 9 is 11.

   a. 0 and -1
   b. -8 and 0
   c. 0 and -14
   d. 14 and 0
   e. -12 and 16
   f. -4 and 24
   g. -20 and 10
   h. -20 and -40
   i. 20 and -20
   j. 25 and -15
   k. -25 and 25
   l. 17 and 17

9. Answer the following questions using the table of information for each planet in the solar system. Justify your answers.

Studying the Planets		
Planet	Distance from Sun (million km)	Average temperature (°C)
Mercury	57.9	427
Venus	108.2	464
Earth	149.6	15
Mars	227.9	-63
Jupiter	778.3	-144
Saturn	1427	-176
Uranus	2869.6	-215
Neptune	4496.6	-215

   a. Which planet is the hottest? Which is/are the coldest?
   b. Which planet is closest to the sun? Which is the farthest from the sun?
   c. Is there a relationship between a planet's distance from the sun and its average temperature? Explain.

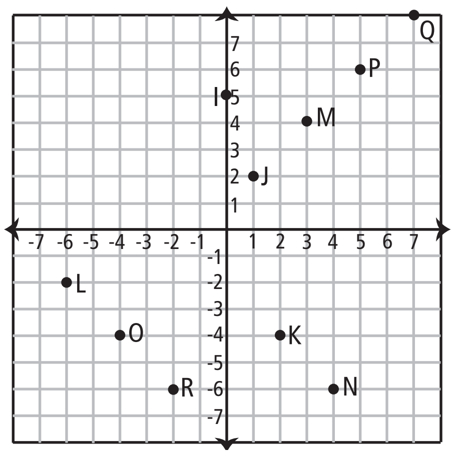
10. Calculate the sums and differences. Use the car model as needed.

   a. -5 + -9
   b. -5 – (-9)
   c. -9 – 5
   d. -9 – (-5)

11. Ryu has 58 cents and wants to buy a toy that costs 73 cents. How much more money does he need?

12. a. (-5, 7) e. (-3, 0) i. (0, 5)
 b. (4, 4) f. (-5, -2)
 c. (-4, 4) g. (3, -2)
 d. (1, 2) h. (-3, -3)

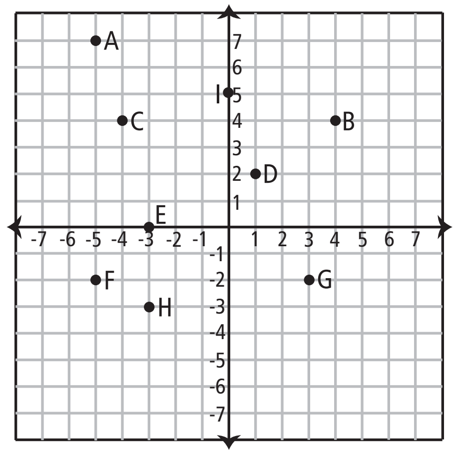
13.



14. Answer will vary. Samples:
 a. (3, 0)
 b. (0, 3)
 c. (5, 3)
 d. (-4, 3)
 e. (-5, -2)
 f. (6, 1)

15. Answers will vary. Samples:
 a. (0, 3), (0, 6)
 b. (4, 0), (9, 0)
 c. (-2, 1), (-2, -2), (-2, 5)
 d. (2, -2), (3, -2), (-5, -2)

12. Write the coordinates for each point shown on the coordinate plane below.



13. Draw a coordinate plane and plot the following points:

J. (1,2)	K. (2, -4)	L. (-6, -2)
M. (3, 4)	N. (4, -6)	O. (-4, -4)
P. (5, 6)	Q. (7, 8)	R. (-2, -6)

14. Plot the following points as specified on the coordinate plane:
    a. A point on the x-axis. Identify its ordered pair.
    b. A point on the y-axis. Identify its ordered pair.
    c. A point in the first quadrant.
    d. A point in the second quadrant.
    e. A point in the third quadrant.
    f. A point in the fourth quadrant.

15. Plot the following points that satisfy the stated characteristics. Identify the ordered pairs.
    a. Two points with the first coordinate 0.
    b. Two points with the second coordinate 0.
    c. Three points with the first coordinate -2.
    d. Three points with the second coordinate -2.

16. a. (3, -2), (6, -3), (5, -1)
    b. (-6, 1), (-9, 3), (-2, 4)

17. 8 feet below surface

18. -3°F

19. -2°C

16. Plot the following points, and identify the ordered pairs that satisfy the following properties:
    a. Three points with the first coordinate positive and the second coordinate negative.
    b. Three points with the first coordinate negative and the second coordinate positive.
17. Marci is snorkeling in the San Marcos River 5 feet below the surface. She dives 3 feet deeper. How many feet below the surface is Marci now?
18. The temperature in Anchorage, Alaska is $^{-}15$ °F. The temperature rises 12 °F during the day. What is the new temperature? (It may help to draw a vertical number line).
19. The temperature in Seattle, Washington is 13 °C. The temperature then dips 15 °C. What is the temperature now? (It may help to draw a vertical number line).

MATH EXPLORATIONS

Multiplying and Dividing

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Section 2.1 connects whole number multiplication that most students already know as a skill to a model that can represent the action of the operation using a number line. The linear model can later be extended to represent multiplication with not only whole numbers but with the set of integers. The area model introduced in section 2.2 is another way of representing multiplication. This model is particularly useful in modeling two-digit multiplication and identifying the partial products involved in the multiplication process. The distributive property is also introduced as an important property both in arithmetic and in algebra. Section 2.3 revisits the number line model and uses it to represent division. Division is defined using a missing-factor model connecting division to multiplication. Section 2.4 presents the area model to also represent the division process and to visually identify the quotient and the remainder. The Division Algorithm is a formal way of relating the dividend, divisor, quotient, and remainder that can now be represented with the area model. This also leads to the next section, section 2.5, on long division. Students may already know the algorithm of long division but may not connect the procedure to any important concepts that justify the process. The scaffolding method is intended to help students make sense of the procedure that can then be made more efficient and more like the standard algorithm.

Section 2.1 - Skip Counting with Integers

Big Idea:

Using linear models to multiply integers

Key Objectives:

- Connect multiplication to the number line.
- Discover multiplication rules for whole numbers

Materials:

Number Line handouts, Graph paper, Graphing Calculators

Pedagogical/Orchestration:

- Teacher will guide students through Explorations 1 and 2 and have class discussion to connect multiplication and addition.
- It may be necessary to review skip counting at the start of this chapter.

Activity:

"Leap Frog", "Skip Counting Race", "Addition and Subtraction Activity"

Vocabulary:

factor, multiplication, skip counting model, frog model

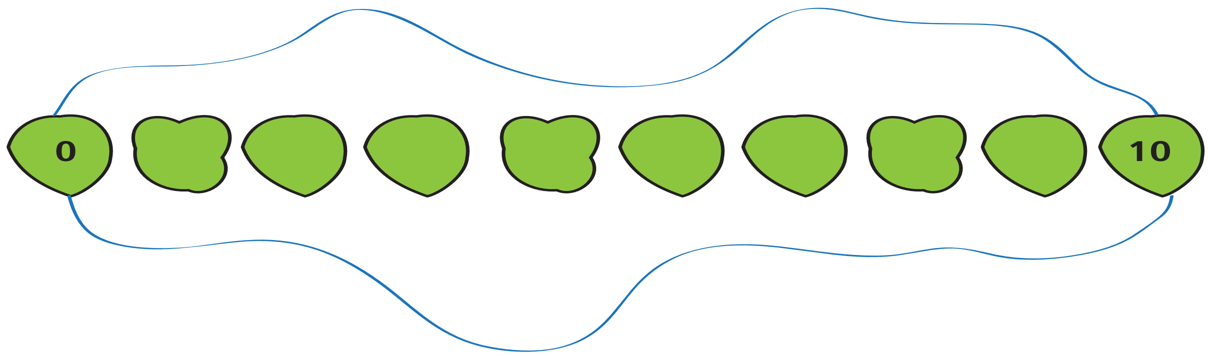
TEKS:

6.2(C,D); 6.8(B); 6.11(B,C,D)

New: 6.3(D, F)

Launch for Section 2.1:

See if you can figure out where Fernando the Frog is. To cross his pond, he jumps from lily pad to lily pad. The lily pads are in a straight line and equally spaced across the pond. Draw a picture of the pond and 11 lily pads with a lily pad on one shore labeled 0 and the last lily pad on the opposite shore labeled 10. (Draw pond and lily pads on the board. For each example, have a student use a frog cutout and model the jumping frog on the board.)



Fernando is sitting on lily pad 0 and facing right. He jumps to the right from one lily pad to the next four times. Then he makes 3 more jumps to the right. On which lily pad is Fernando sitting? (Lily pad 7) What is an addition problem we could write to represent this? (4+3=7) What if Fernando is on lily pad 0 again and jumps 3 times to the right and then jumps 6 more times to the right. Which lily pad is Fernando on now? (Lily pad 9) What is the addition problem? (3+6=9) Try this "challenge" problem. Let Fernando start back at lily pad 0 and make some jumps to the right to end up on lily pad 5. Then he makes some more jumps and ends up on lily pad 9. What is the addition problem that represents this? (5+4=9)

Let's have Fernando try something different. He starts at lily pad 0 facing right, and jumps 7 times to the right. He then decides to jump backwards and makes 5 backward jumps (to the left). On which lily pad is Fernando sitting now? (Lily pad 2) What operation is this representing? (Subtraction) What would be an equation that represents this? (7 – 5 =2) (Some more examples can be done like this such as 8 – 3 = 5 and so forth.) What math model do you think the lily pads represent? (A number line) In this launch we have actually been using a number line to do addition and subtraction and will continue this concept in our lesson.

Skip Counting Race

Objective: Students will become more flexible with numbers and the number line. It will help reinforce their skip counting strategies using the number line.

Materials:
Number line

Activity Instructions:
The teacher will remind students that when you multiply: 5*7 the first factor tells the direction of the jump and the length, the second factor tells the number of jumps. Since 5 is positive the direction is positive and the length is 5, then, 7 says to jump that many jumps. Students record this on a number line written on paper; the answer is 35.

Students will continue to try other combination of numbers and to use the number line and jumping frog strategies.

The teacher pre-writes the number lines on the board. Two or three pairs of students go up to the board. One student is the Coach and the other is the player. The Coach prepares the number line numbered from any number to any number. The Coach will help only if needed and will not be allowed to write on the board. The Player will solve the problem.

Each Coach gives the Player a multiplication problem, after teacher counts off: "1,2,3". The pair who answers the problem correctly first wins.

Leap Frog

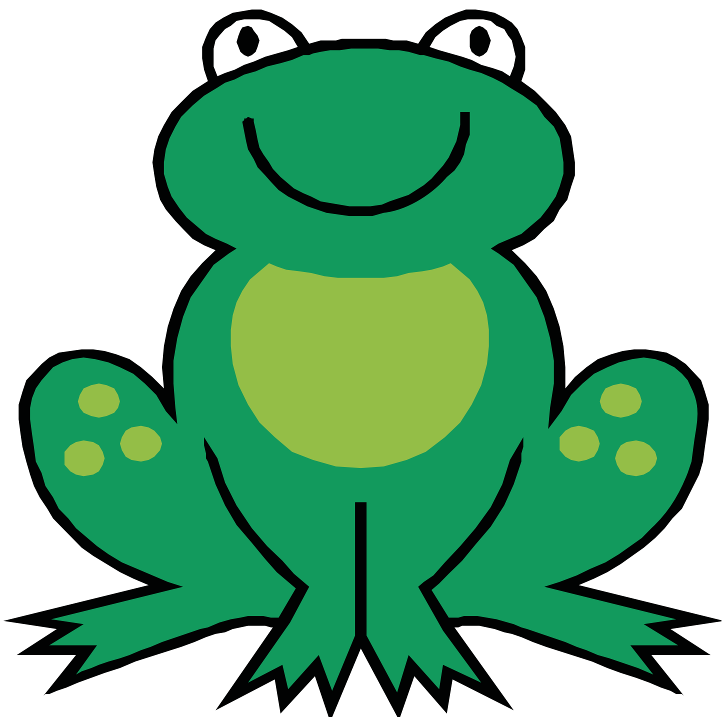
Objective: This activity will help the students visualize the motion of the frog as it leaps on the number line.

Materials:

An evenly spaced number line on the board marked from 0 to 100
The frog cut-out below
Magnetic Tape

Activity Instructions:

Make your number line on the board.
Cut out the frog below, and decorate him however you see fit.
Attach the magnetic tape to the back of the cut-out so the frog will stick to the board.
Use the number line and the frog to demonstrate the motion of several examples of integer multiplication problems. Let the students come to the board and move the frog if time allows.



Consider using the number line for addition and subtraction as the launch for this section. This will either review or newly present this model for addition and subtraction before introducing multiplication and division on the number line.

Addition and Subtraction Activity

Objective: This activity is a review of addition and subtraction of integers on the number line. The motion of the frog leaps is one way to visualize addition and subtraction.

Materials:

An evenly spaced number line on the board marked from -100 to 100
The frog cut-out on the previous page
Magnetic Tape

Activity Instructions:

Make your number line on the board.
Cut out the frog from the previous page, and decorate him however you see fit.
Attach the magnetic tape to the back of the cut-out so the frog will stick to the board.
Use the number line and the frog to demonstrate the motion of several examples of integer addition and subtraction problems. Let the students come to the board and move the frog if time allows. Use the number line to demonstrate the following addition and subtraction problems: 5 + 3 and 5 - 3.

The following activity may be useful for preparing the students for skip counting.

Investigation: Skip Counting and Scaling

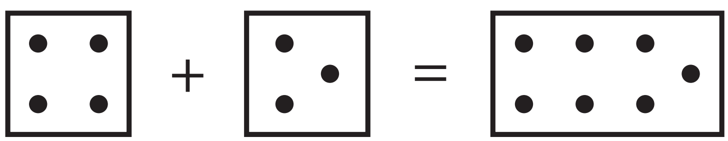
When we initially built our number line, we used each mark to indicate one unit. The number line then corresponded to the integers 1, 2, 3, … . For larger numbers, we let the marks represent bigger lengths. So if each mark represents 5 units, then the marks correspond to the multiples of 5, and we have 5, 10, 15, 20, … using "skip counting" by 5's.

a. Build a number line where each mark represents 3 units, skip counting by 3's. What is the 10th number to the right of 0? 30
b. Build a number line where each mark represents 10 units, skip counting by 10's from 0 to 40.
c. Make a table of the numbers you get when skip counting by 2's to 20, skip counting by 3's to 30, skip counting by 4's to 40, … and skip counting by 10's to 100.

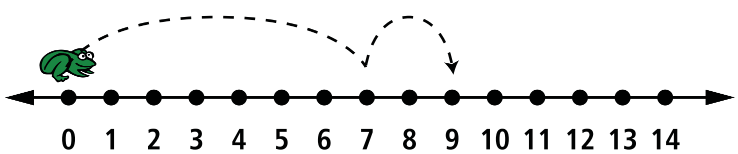
2 MULTIPLYING AND DIVIDING

SECTION 2.1 SKIP COUNTING WITH INTEGERS

Addition is a mathematical operation for combining integers. Pictorially, using the "set model," when we add two integers we are combining the sets. To add 4 and 3 we draw the picture below:

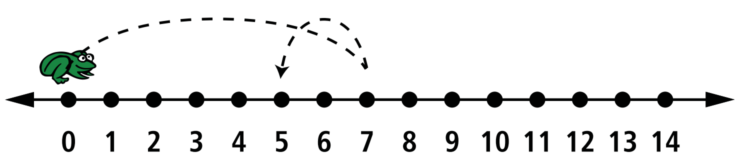


We also use our number line model to describe addition and subtraction. For example, we model the addition problem 7 + 2 on the number line, with cars or with frog leaps.



How might 7 - 2 look on the number line?

Does it look like this?

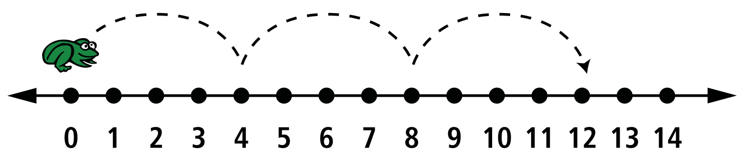


We know that skip counting by 3's generates the list 3, 6, 9, 12, 15, 18, 21, 24, 27, . . ., which continues indefinitely. Skip counting provides a model for multiplication that we can represent on a number line.

On the number line, the frog's jumps correspond to the numbers we are adding. In order to multiply, we can think of a frog that jumps along the number line. For example, when you multiply $4 \cdot 3$,

- the result of the multiplication is called the **product**
- the first **factor** indicates which direction the frog should face and the length of each jump
- the second factor indicates the number of jumps

The picture below models the multiplication $4 \cdot 3 = 12$. Notice the frog is facing in the positive direction because the first factor, 4, is positive. The frog takes 3 jumps, and each jump is 4 units long. The final location is the product 12.



EXPLORATION 1: FROG JUMP MULTIPLICATION

Copy and fill Table 2.1a in which each jump is 4 units long.

Table 2.1a

Length of Jump (factor)	Number of Jumps (factor)	Frog's Location (product)
4	0	0
4	1	4
4	2	8
4	3	**12**
4	4	**16**
4	5	**20**
4	6	**24**
4	10	**40**
4	20	**80**
4	n	**4n**

In groups, have students explain and show their thinking in filling the table using the number line. What direction is the frog facing on each jump? How long is each jump?

Length of Jump	Number of Jumps	Frog's Location
7	0	0
7	1	7
7	2	**14**
7	3	**21**
7	4	**28**
7	5	**35**
7	6	**42**
7	10	**70**
7	20	**140**
7	*n*	**7n**

PROBLEM 1

a. 48 b. 40 c. 24 d. 56

Does this table look familiar? You might recognize these numbers from a multiplication table of 4's where the pattern is $4 \cdot 1 = 4$; $4 \cdot 2 = 8$; $4 \cdot 3 = 12$; $4 \cdot 4 = 16$. You can think of (4)(3) as (4 units per jump)(3 jumps) = 12 units.

Copy and fill the skip counting Table 2.1b as you did in Table 2.1a, but this time use jumps of directed length 7.

Table 2.1b

Length of Jump	Number of Jumps	Frog's Location
7	0	0
7	1	7
7	2	**14**
7	3	**21**
7	4	**28**
7	5	**35**
7	6	**42**
7	10	**70**
7	20	**140**
7	n	**7n**

Multiplication of 7 and 12, often written as 7×12, can also be written as $7 \cdot 12$, $7 * 12$, or (7)(12).

PROBLEM 1

Compute the following products. Explain how you arrive at your answer.

a. (8)(6)

b. (8)(5)

c. (8)(3)

d. (8)(7)

Let's summarize the frog model:

> The first factor tells us the length of each jump, and the second factor tells us the number of jumps.

Directed Length of Jump	Number of Jumps	Frog's Location
15	0	**0**
15	1	**15**
15	2	**30**
15	3	**45**
15	4	**60**
15	6	**90**
15	10	**150**
15	20	**300**
15	100	**1500**
15	n	**15n**

a. (15)(7) = **105** c. (15)(12) =**180**

b. (15)(30) = **450** d. (15)(1000) = **15000**

Although numbers that are multiplied need a sign or punctuation to distinguish (3)(2) from 32, in the case of multiplication with variables, the common notation has no sign or punctuation. When your students see 4*x*, they should understand that means "4 times *x*" or "4 *x*'s."

EXPLORATION 2

a.

Time	1 A.M.	2 A.M.	3 A.M.	4 A.M.	5 A.M.	6 A.M.	7 A.M.	8 A.M.	9 A.M.	10 A.M.	11 A.M.	12 P.M.	1 P.M.
x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
T(x)	54° F	57° F	60° F	63° F	66° F	69° F	72° F	75° F	78° F	81° F	84° F	87° F	90° F

b. 60 ˚F
c. The x-value which corresponds to 12 pm is x = 5. The temperature at 12 pm is 87˚F. It will take 4 hours for the temperature to rise 12 degrees. The time will be 11 am.
d. The temperature is 63 ˚ F at 4 am.
e. Yes; 72 + 3(-4); 72 + 3(5)

Using the frog model, complete the following table using the length 15.

Table 2.1c

Directed Length of Jump	Number of Jumps	Frog's Location
15	0	**0**
15	1	**15**
15	2	**30**
15	3	**45**
15	4	**60**
15	6	**90**
15	10	**150**
15	20	**300**
15	100	**1500**
15	n	**15n**

Use this table to compute the following products:

a. (15)(7) **105**

b. (15)(30) **450**

c. (15)(12) **180**

d. (15)(1000) **15000**

EXPLORATION 2

In McAllen, Texas, the temperature rises an average of 3°F per hour over a twelve hour period from 1 a.m. to 1 p.m. The temperature at 7 a.m. is 72°F. Let x be the number of hours after 7 a.m.

a. Make a table that shows a relationship between the time and temperature over the twelve hours. (Hint: make a row for time, a row for the number of hours since 1 a.m. and a row for the temperature).

b. What was the temperature at 3 a.m.? **60°F**

c. What x-value corresponds to 12 p.m.? What is the temperature at 12 p.m.? How many hours will it take for the temperature to rise 12 degrees? What time will that be?

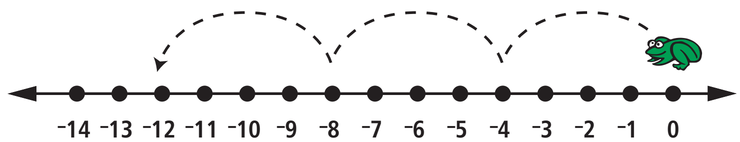
d. When is the temperature 63°F?

Table 2.1d

Directed Length of Jump	Number of Jumps	Frog's Location
-4	0	0
-4	1	-4
-4	2	(-8)
-4	3	(-12)
-4	4	(-16)
-4	5	(-20)
-4	6	(-24)
-4	10	(-40)
-4	20	(-80)
-4	*n*	-4n

e. Is it possible to use multiplication to help determine the temperature in parts b and c? If so, explain how.

You learned how to add positive and negative integers in the first chapter. Is there a way to think about multiplying a negative integer times a positive integer? You can use the frog model to multiply $^-4 \cdot 3$, as shown below.



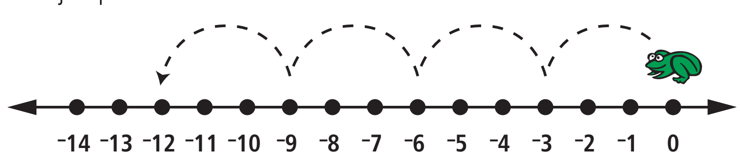
Copy and fill the skip counting Table 2.1d as you did in Table 6.1a, but this time use jumps of directed length $^-4$.

Table 2.1d

Directed Length of Jump	Number of Jumps	Frog's Location
$^-4$	0	0
$^-4$	1	$^-4$
$^-4$	2	
$^-4$	3	
$^-4$	4	
$^-4$	5	
$^-4$	6	
$^-4$	10	
$^-4$	20	
$^-4$	n	

Using the pattern demonstrated in this table, compute the product $^-3 \cdot 4$, or $^-3 \cdot 4$.

The picture below models the product $^-3 \cdot 4$. The first factor tells us which direction the frog should face and the length of each jump; the second factor tells us the number of jumps.



The frog is facing left because we are modeling a jump of $^-3$ units per jump.

Use the number line to compute the following products:

a. (⁻3)(6) **⁻18**

b. (⁻3)(5) **⁻15**

c. (⁻3)(3) **⁻9**

d. (⁻3)(1) **⁻3**

Table 2.1e

Directed Length of Jump	Number of Jumps	Frog's Location
3	-6	**⁻18**
3	-5	**⁻15**
3	-4	**⁻12**
3	-3	**⁻9**
3	-2	**⁻6**
3	-1	**⁻3**
3	0	0
3	1	3
3	2	6
3	3	**9**

Use the number line to compute the following products:

a. $(^-3)(6)$
b. $(^-3)(5)$
c. $(^-3)(3)$
d. $(^-3)(1)$

How can we make sense of the product $(3)(^-4)$? This is the first example where the second factor is negative.

The first number, 3 or $^+3$, gives the length of each jump, and the direction the frog is facing. Because the number is positive, the frog faces right.

The second factor gives the number of jumps. What do we mean by the number $^-4$ as the number of jumps? If we think of the jumps taking place at equal time intervals, we can imagine the frog jumping along a line.

We pick one location, call it 0, and name the time as the "0 jump." When the frog takes its first jump, jump 1, the frog lands at location 3. When the frog takes its second jump, jump 2, the frog lands at location 6.

Let's go back to the 0 location and ask where the frog was on the jump before it arrived at 0. We call this jump $^-1$. Because the frog jumps 3 units to the right every jump, the frog must have been at location $^-3$, which is 3 units to the left of 0. Two jumps before reaching 0, the frog was at location $^-6$. We can now copy and fill the table below.

Table 2.1e

Directed Length of Jump	Number of Jumps	Frog's Location
3	$^-6$	
3	$^-5$	
3	$^-4$	
3	$^-3$	
3	$^-2$	
3	$^-1$	
3	0	0
3	1	3
3	2	6
3	3	

Table 2.1f

Directed Length of Jump	Number of Jumps	Frog's Location
-3	-6	(18)
-3	-5	(15)
-3	-4	(12)
-3	-3	(9)
-3	-2	(6)
-3	-1	(3)
-3	0	0
-3	1	-3
-3	2	-6
-3	3	(-9)

It is now possible to answer the earlier question. What do we mean by $^{-}4$ jumps? This means we jump backward in time, or simply jump backward.

Use the number line to compute the following products. Verify that your answers agree with the table.

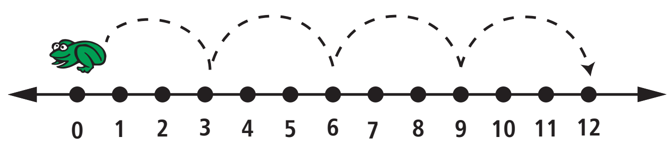
a. $(3)(^{-}6)$ **$^{-}18$** c. $(3)(^{-}3)$ **$^{-}9$**

b. $(3)(^{-}5)$ **$^{-}15$** d. $(3)(^{-}1)$ **$^{-}3$**

Let's summarize the frog model:

- The first factor tells us which direction the frog should face and the length of each jump.
- The second factor tells us the number of jumps and the direction of the jump. When the second factor is positive, the frog jumps forward; when the second factor is negative, the frog jumps backward.

Using the frog model, compute the product $(^{-}3)(^{-}4)$. The directed length of each jump is $^{-}3$. Determine what happens when the frog jumps backward in time.



Copy and fill the following table, starting at the bottom and working up.

Table 2.1f

Directed Length of Jump	Number of Jumps	Frog's Location
$^{-}3$	$^{-}6$	
$^{-}3$	$^{-}5$	
$^{-}3$	$^{-}4$	
$^{-}3$	$^{-}3$	
$^{-}3$	$^{-}2$	
$^{-}3$	$^{-}1$	
$^{-}3$	0	0
$^{-}3$	1	$^{-}3$
$^{-}3$	2	$^{-}6$
$^{-}3$	3	

1.	a.	-4 · 2	TE -8	d.	-6 · 3	TE -18	g.	- 13 · 3	TE -39
	b.	7 · -5	TE -35	e.	11 · -4	TE -44	h.	40 · -2	TE -80
	c.	(6)(-7)	TE -42	f.	(-5)(8)	TE-40	i.	(10)(-12)	TE -120

j. Students should observe that the product is always to the left of zero when you multiply a positive number times a negative number or a negative number times a positive number

2. Students should observe that the product is always to the right of zero when you multiply a negative number times a negative number.

3.	a.	(5)(7)	TE 35	b.	(-6)(8)	TE -48
		(5)(-7)	TE -35		(-6)(-8)	TE 48
		(-5)(7)	TE -35		(6)(-8)	TE -48
		(-5)(-7)	TE 35		(6)(8)	TE 48

4. a. The product is negative. The simplest rule is the product is negative if the signs of two factors are different (or if the number of negative factors is odd).
   b. The product is negative. The simplest rule is the product is negative if the signs of two factors are different (or if the number of negative factors is odd).
   c. The product is positive. The simplest rule is the product is positive if the signs of two factors are the same (or if the number of negative factors is even).

5.	a.	16(-14)	TE -224	d.	(-115)(-4)	TE 460
	b.	(-24) 32	TE - 768	e.	223(-13)	TE - 2,899
	c.	- 13 · 25	TE -325	f.	(- 125) (-7)	TE 875

Use the table to compute the following products:

a. (⁻3)(⁻6) 18 c. (⁻3)(⁻3) 9

b. (⁻3)(⁻5) 15 d. (⁻3)(⁻1)

EXERCISES

1. Use the frog model on the number line to compute the following products. As you multiply, visualize the process to verify the accuracy of the products.

   a. ⁻4 · 2**6** d. ⁻6 · 3 g. ⁻13 · 3

   b. 7 · ⁻5**15** e. 11 · ⁻4 h. 40 · ⁻2

   c. (6) (⁻7) f. (⁻5) (8) i. (10) (⁻12)

   j. What patterns do you observe? In your own words, describe the patterns using complete sentences.

2. Use the number line to demonstrate (⁻2) (⁻3). Do the same for (⁻4) (⁻6). What patterns do you see? In your own words, describe the pattern using complete sentences.

3. Use the number line frog model to compute the following products.

   a. (5)(7) b. (⁻6)(8)

   (5)(⁻7) (⁻6)(⁻8)

   (⁻5)(7) (6)(⁻8)

   (⁻5)(⁻7) (6)(8)

4. We know from experience that when we multiply a positive number by another positive number we will always get a positive number. Using the patterns you discovered in exercises 1-3:

   a. Find a rule for the product of a negative number and a positive number.

   b. Find a rule for the product of a positive number and a negative number.

   c. Find a rule for the product of two negative numbers.

5. Evaluate the following products:

   a. (16)(⁻14) d. (⁻115)(⁻4)

6. Pedro has between 1000 and 2000 tater tots or approximately (17) (80) = 1360. Answers will vary in how students arrived at their conclusion.

7. After his deposit, Johnny has $200 in his account because
0 + 25 + 25 + 25 + 25 + 25 + 25 + 25 + 25 = 200 or 25 x 8 = 200.

8. Andrew must pay (7 tickets) ($9 per ticket) = $63

9. a

hr. #	0	1	2	3	4	5	6	7
temp.	70 °F	66 °F	62 °F	58 °F	54 °F	50 °F	47 °F	42 °F

b. 70 + (-4)(7) = 42˚F

10. a.

time	10 a.m.	9 a.m.	8 a.m.	7 a.m.	6 a.m.	5 a.m.
hr. #	5	4	3	3	1	0
temp.	88 °F	85 °F	82 °F	79 °F	76 °F	73 °F

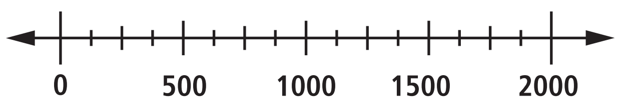
b. 88 + (-5) (3) = 73˚F . Encourage students to multiply by a negative integer to reach the solution

b. $(^{-}24)\,32$ e. $223\,(^{-}13)$

c. $^{-}13 \cdot 25$ f. $(^{-}125)\,(^{-}7)$

For problems 6 - 8, solve each word problem. Write your answers in a complete sentence.

6. Pedro has 17 bags of tater tots that have approximately 80 tots in each bag. Place a point on the number line at the approximate location that indicates about how many tater tots Pedro has. Explain how you arrived at your estimate.



7. Johnny is opening a checking account today. He deposits 8 checks he got for his birthday, each of \$25. How much money will he have in his account after making these deposits? Show two different ways to solve this problem.

8. Andrew is buying tickets to the movies for himself and six friends. Each ticket costs \$9. How much does he have to pay for the tickets?

Write your answers in complete sentences for problems 9 - 12.

9. On a November day, a cold front blew into town. The temperature was 70 ˚F before the temperature dropped an average of 4 ˚F an hour. What was the temperature after 7 hours?

   a. Create a table to solve this problem.

   b. Write an expression you can use to solve this problem using addition and multiplication.

10. One morning in San Antonio, the temperature rises for five hours, from 5:00 AM to 10:00 AM The temperature rises an average of 3 ˚F per hour. The temperature at 10:00 AM is 88 ˚F. What was the temperature at 5:00 AM?

   a. Create a table to solve this problem.

   b. Write an expression you can use to solve this problem using addition and multiplication

11. a

time	6 a.m.	7 a.m.	8 a.m.	9 a.m.	10 a.m.	11 a.m.	12 p.m.
hr. #	0	1	2	3	4	5	6
temp.	10 °F	14 °F	18 °F	22 °F	26 °F	30 °F	34 °F

b. 4 (hr. #) + 10. So, at 12 p.m. the temperature is 34 °F.

12. Have your students act out on a number line or explain using a number line;
a. 18 ft. b. 20 seconds

13. **Spiral Review (7.1A)**
Saturday's low: -5 °C; Sunday's low: -10 °C
Sunday's temperature was colder.

14. **Spiral Review (5.6)**
b. s = 46 -15

15. 70 + 70 = 140

11. On a cold day in Roanoke, Virginia, the temperature at 6:00 AM is 10 ˚F. The temperature increases 4 ˚F per hour for the next six hours. What will the temperature be at 12:00 PM?

    a. Create a table to solve this problem.

    b. Write an expression you can use to solve this problem using addition and multiplication.

12. A bee flies by Tommy traveling east at 6 feet per second. Assuming the bee is traveling in a straight line:

    a. How far is the bee from Tommy after 3 seconds?

    b. How many seconds will it take the bee to travel 60 feet?

Spiral Review:

13. The low temperature on Saturday was 5 degrees below zero Celsius. On Sunday, the low temperature was 10 degrees below zero Celsius. Represent each day's low temperature as an integer. On which day was it colder?

14. Jack and Jill collect stamps. Jack has 46 stamps in his collection. He has 15 more stamps than Jill does. Which of the following equations can be used to find *s*, the number of stamps Jill has in her collection?

    a. s = 46 + 15

    b. s = 46 - 15

    c. s = 54(15)

    d. s = 54 ÷ 15

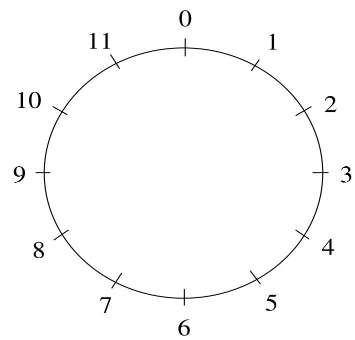
15. **Ingenuity:**

    Euclid Middle School put on a play to raise money for the school's drama club. The school sold adult tickets for $5 each and child tickets for $3 each. The total value of the tickets sold was $560. If the school sold exactly as many child tickets as it sold adult tickets, how many tickets did it sell overall?

16. a. 8 b. 3 c. 11 d. 5 e. 44

16. **Investigation:**

Draw a circle and label twelve points on the circle from 0 to 11 as shown below:



Suppose a frog begins at the point 0, and makes several jumps, jumping the same number of units clockwise each time. For example, the frog might make 3 jumps, jumping 2 units each time. If it did this, it would land at the points 2, 4, and finally 6.

a. Suppose the frog started at 0 and made 2 jumps, jumping 4 units clockwise each time. Where would it finally land?

b. Suppose the frog started at 0 and made 3 jumps, jumping 5 units clockwise each time. Where would it finally land?

c. Suppose the frog started at 0 and made 5 jumps, jumping 7 units clockwise each time. Where would it finally land?

d. Suppose the frog started at 0 and made 7 jumps, jumping 11 units clockwise each time. Where would it finally land?

e. Suppose we instead marked 60 points on a circle, and labeled them 0 through 59. If the frog started at 0 and made 16 jumps, jumping 59 units clockwise each time, where would it finally land?

Section 2.2 - Area Model for Multiplication

Big Idea:
Using area models to represent multiplication

Key Objectives:
- Model one- and two-digit multiplication with areas.
- Discuss the use of place value in the Area Model.
- Use the distributive property to find partial products.
- Apply the area model to algebraic problems.

Materials:
Grid paper, Rulers, Algebra tiles could be used with area model (optional)

Pedagogical/Orchestration:
- Tie in the area model with the distributive property.

Internet Resources:
Matching multiplication review game: http://www.quia.com/mc/66145.html

Flash cards multiplication review: http://www.quia.com/jg/65626.html

Jeopardy multiplication review game: http://www.quia.com/cb/212909.html

Activities:
"Floor Plan Design"

Vocabulary:
Linear model, Area model, Distributive property of multiplication, Partial product, Commutative property of multiplication, Associative property of multiplication, Multiplicative identity property.

TEKS:
6.2(C,D); 6.8(A,B); 6.11(B,C,D)

New: 6.1(A, D, E, F); 6.3(D, F); 6.7(D)

Launch for Section 2.2:
Pose the following situation. You have a rectangular floor design 2 feet by 3 feet. How many square tiles will you need to fill in the design (6 square tiles). Students most likely are going to mention addition by counting the tiles or adding the tiles, but some may mention multiplication. What if the same design was going to be enlarged to be 24 feet by 36 feet. How many square tiles would you now use? What method would be the most efficient way to find the answer?

Floor Plan Design

Objective: The students will apply the area model of multiplication to design a floor plan for a room with specific area requirements.

Materials:
Notebook paper, plain white paper, or Grid paper
Rulers

Activity Instructions:

1. The teacher will explain to the class that they are to design a floor plan. Their floor plan will have 4 rooms with a total area of 620 square feet. The four rooms must all be of different size.
2. The students will turn in a final product that shows their floor plan design with all dimensions labeled. The students will also provide a mathematical explanation of how they can prove that their floor plan design meets the 620 sq. feet requirement.

Shikaku-Rectangle Puzzles

Objective: The goal of a Shikaku puzzle is to generate rectangles whose area is the same as the number encompassed in each rectangle.

Material:

Copy of puzzle sheet

Activity Instructions:

1. Each of the following Shikaku puzzles needs to be sectioned into rectangles (and squares) along the grid lines, so that the number in each rectangle refers to the area of that rectangle.
2. Only one number can appear in each rectangle.
3. Each square on the gird is used in exactly one rectangle (that is, no rectangles may overlap)

4		2			
	9				
				10	
			5		6

9			12			5			
								6	
8		6	8						
				6		8		12	
4									
		3		9				4	

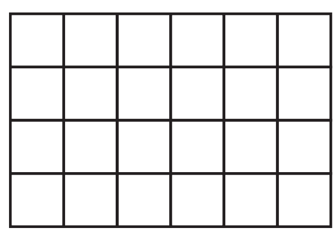
SECTION 2.2 AREA MODEL FOR MULTIPLICATION

In the previous section, we explored multiplication using the frog model of skip counting on the number line, or repeated addition. This is also called the **linear model** for multiplication. In addition to the linear model, we can represent multiplication as area.

EXPLORATION

A bird refuge is in the shape of a rectangle 4 miles long and 6 miles wide. Draw a visual representation of this refuge on grid paper, using 1 centimeter = 1 mile, and use it to determine the area of this rectangle. Explain how you use the grid to compute the area. Multiply 4 miles by 6 miles using the traditional algorithm only after you have an answer using the visual representation.

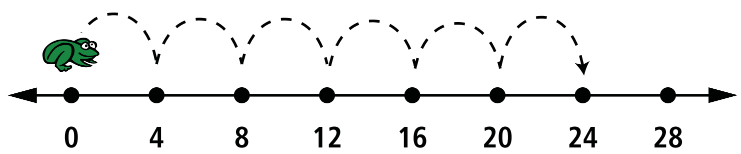
To multiply 4 by 6, consider the picture of the rectangle below. The area of a rectangle is the number of unit or 1×1 squares that it takes to cover the figure with no overlaps and no gaps. What is the area of the rectangle below assuming that each square in the grid has area 1 square unit?



The area is 24 square units. The dimensions of this rectangle, which is 4 units wide and 6 units long, are 4 units in width and 6 units in length. Sometimes the dimensions are abbreviated and used to describe the rectangle: the rectangle is 4 units wide and 6 units long or there are 4 rows and 6 columns. The area can be computed by summing the areas of the columns: $4+4+4+4+4+4 = 4\cdot 6 = 24$. We can also think of this area as the sum of the area of the rows: $6+6+6+6 = 6\cdot 4 = 24$. The rectangle is called a 4 by 6, or a 6 by 4, rectangle because the area is computed as the product $4\cdot 6 = 6\cdot 4 = 24$. We call this relationship the **commutative property of multiplication**.

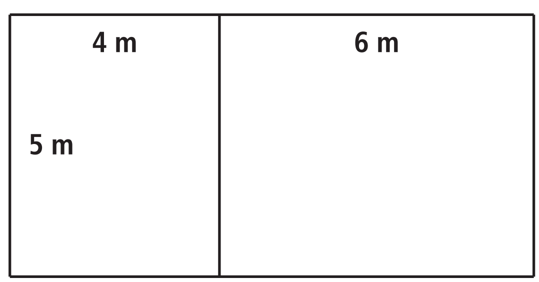
PROPERTY 2.1: COMMUTATIVE PROPERTY OF MULTIPLICATION
For any numbers x and y, $$x \cdot y = y \cdot x.$$

Remember, when we add 4 to itself 6 times, it is the same as when a frog jumps 6 times on a number line, with each jump 4 units long. The visual representation of area above is another model that describes multiplication.



EXAMPLE 1

The Elliots are constructing a small building that is one room wide and two rooms long. Each room is 5 meters wide. The front room is 4 meters long, and the back room is 6 meters long. What is the floor space of each room? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? The floor plan below shows the layout:

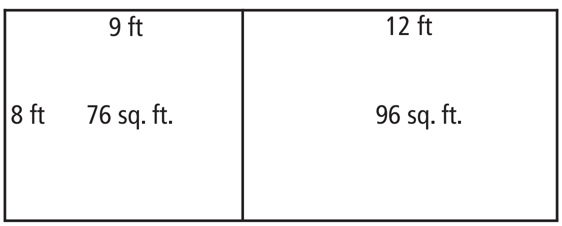


SOLUTION

The area of the room on the left is calculated by (5m)(4m) = 20 square meters, or 20 sq. m. The area of the room on the right is (5m)(6m) = 30 square meters. The area of each room is called the **partial product**. Adding the partial products gives you the total area. The total area is the sum of the areas of the two rooms:

20 square meters + 30 square meters = 50 square meters.

PROBLEM 1

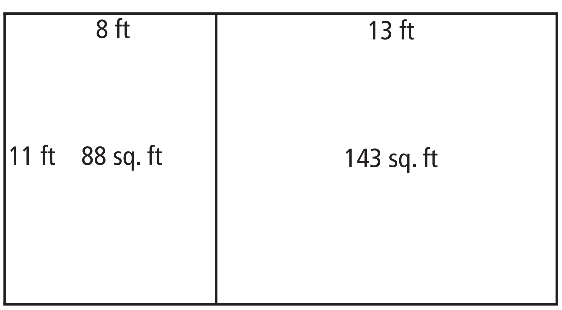


$8 \cdot 9 = 72$ $8 \cdot 12 = 96$

8ft (9ft + 12ft) = (8ft)(9ft) + (8ft)(9ft) = 72 + 96 =168 square feet

The length of the building is 9 + 12. So, we can write the area as 8ft (9ft + 12ft) = (8ft)(21ft) = 168 square feet

PROBLEM 2



$11 \cdot 8 = 88$ $11 \cdot 13 = 143$

11ft (8ft + 13ft) = (11ft)(8ft) + (11ft)(13ft) = 88 + 143
= 231 Sq. ft.

The length of the building is 8 + 13. So, we can write the area of the building floor as 11ft (8ft + 13ft) = 11(21)
= 231 sq. ft.

Another way to compute the total area is to consider the larger rectangle and its width and length:

(5 meters)(4 meters + 6 meters) = (5 meters)(10 meters) = 50 square meters.

Notice that this is the same area.

PROBLEM 1

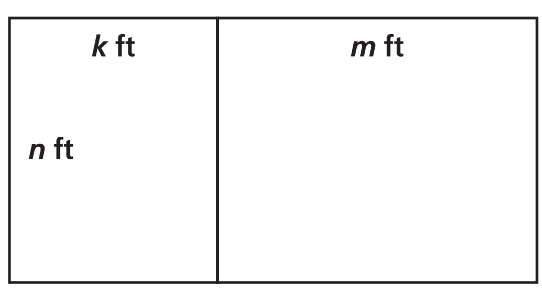
The Redfield family is constructing a small building that is one room wide and two rooms long. Each room is 8 feet wide. The front room is 9 feet long, and the back room is 12 feet long. What is the floor space of each room? What is the length of the building? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? Create the floor plan that shows this situation.

PROBLEM 2

The Gonzalez family is constructing a small building that is one room wide and two rooms long. Each room is 11 feet wide. The front room is 8 feet long, and the back room is 13 feet long. What is the floor space of each room? What is the length of the building? What is the floor space of the building? How are the areas of the two rooms related to the area of the building? Create the floor plan that shows this situation.

EXAMPLE 2

Suppose the dimensions of the Elliots' building have not been decided yet. We need a formula for the areas. Call the width of the building *n* feet and the lengths of rooms 1 and 2, *k* and *m* feet respectively. Find the area of each room and the building's total area.



SOLUTION

The area of room 1 is $(n \text{ ft})(k \text{ ft}) = n \cdot k$ square ft.

We often abbreviate square feet with sq. ft.

The area of room 2 is $(n \text{ ft})(m \text{ ft}) = n \cdot m$ sq. ft.

The area of the building is $n(k+m)$ sq. ft.

Remember, the area of the building can also be computed as the sum of the areas of the two rooms, $(n \cdot k + n \cdot m)$ sq. ft.

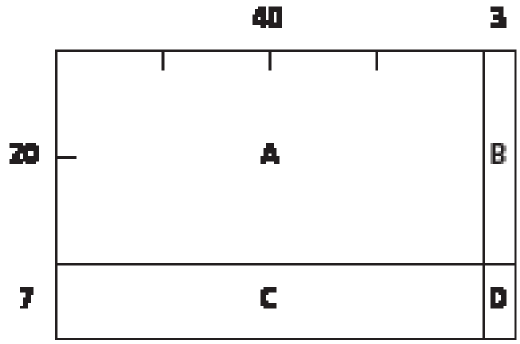
So, $n(k+m) = n \cdot k + n \cdot m$. We call this relationship the **distributive property**. This property tells us how addition and multiplication interact.

PROPERTY 2.2: DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION
For any numbers k, m, and n, $$n(k+m) = n \cdot k + n \cdot m.$$

You have already learned to multiply two-digit and three-digit numbers. Now you can use the area model and the distributive property to explore this process carefully. Begin by modeling the product of a one-digit number and a two-digit number. To multiply $6 \cdot 37$, use place value to write the product $6 \cdot 37$ as $6(30+7)$. By the distributive property, $6 \cdot 37 = 6(30+7) = 6 \cdot 30 + 6 \cdot 7 = 180 + 42 = 222$.

	30	**7**
6	**180**	**42**

Visualize the product of 27 x 43 as area with the picture below:



Area of A = $20 \cdot 40 = 800$;
Area of B = $20 \cdot 3 = 60$;
Area of C = $7 \cdot 40 = 280$;
Area of D = $7 \cdot 3 = 21$. The total area is $800 + 60 + 280 + 21 = 1161$.

You can extend the same process to multiply 27 by 43 using the distributive property, or in a vertical format.

$$\begin{aligned} 27 \cdot 43 &= (20 + 7)(40 + 3) \\ &= 20(40 + 3) + 7(40 + 3) \\ &= 20 \cdot 40 + 20 \cdot 3 + 7 \cdot 40 + 7 \cdot 3 \\ &= 800 + 60 + 280 + 21 = 1161 \end{aligned}$$

$$\begin{array}{r} 27 \\ \underline{\times 43} \\ 21 \\ 60 \\ 280 \\ \underline{+800} \\ 1161 \end{array}$$

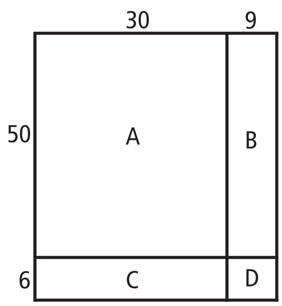
EXAMPLE 3

Find the product of 68 and 47 using the area model. Identify each rectangular region into the partial products found using the distributive property.

Multiply 68 by 47 using the distributive property.

$$\begin{aligned} 68 \cdot 47 &= (60 + 8)(40 + 7) \\ &= 60(40 + 7) + 8(40 + 7) \\ &= 60 \cdot 40 + 60 \cdot 7 + 8 \cdot 40 + 8 \cdot 7 \\ &= 2400 + 420 + 320 + 56 = 3196 \end{aligned}$$

PROBLEM 3



Area Model:

Area of A = 50 x 30 = 1500

Area of B = 50 x 9 = 450

Area of C = 6 x 30 = 180

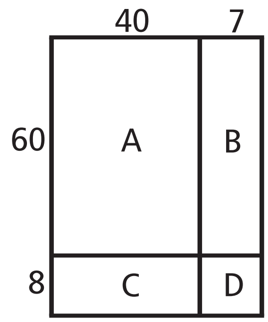
Area of D = 6 x 9 = 54

Distributive Property:

56 x 39= (50 + 6)(30 + 9) = 50(30 + 9) + 6(30 + 9) = 50 x 30 + 50 x 9 + 6 x 30 + 6 x 9

= 1500 + 450 + 180 + 54 = 2184

Visualize the product as area with the picture below:



Area of A = $60 \cdot 40 = 2400$;
Area of B = $60 \cdot 7 = 420$;
Area of C = $8 \cdot 40 = 320$;
Area of D = $8 \cdot 7 = 56$.

The total area is $2400 + 420 + 320 + 56 = 3196$.

PROBLEM 3

Find the product of 56 x 39 using the area model. Identify each rectangular region as one of the partial products found using the distributive property.

Suppose you are multiplying three numbers such as 2, 3, and 4. How can you multiply $2 \cdot 3 \cdot 4$? One way is to first multiply $(2 \cdot 3)$ then multiply the product 6 with 4. Notice the result of $(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$. Another way is to look at $3 \cdot 4$ first and then multiply the product 12 to 2 so that you have $2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$. Notice that the two ways, $(2 \cdot 3) \cdot 4$ and $2 \cdot (3 \cdot 4)$ give the same product 24.
Try multiplying the three numbers, $3 \cdot 4 \cdot 6$ as $(3 \cdot 4) \cdot 6$ and as $3 \cdot (4 \cdot 6)$. Were your two products equal? Your products using either computation is 72. This relationship is formally referred to as the associative property of multiplication.

PROPERTY 2.3: ASSOCIATIVE PROPERTY OF MULTIPLICATION
For any numbers k, n, and m, $$(k \cdot m) \cdot n = k \cdot (m \cdot n) = (k \cdot m) \cdot n = k \cdot (m \cdot n)$$

We also state the special property that 1 has in multiplication,

EXERCISES

1. a. 15 c. 104
   b 12 d. 888

2. a. 1386 b. 3072 c. 18432
   d. 11344 e. 8310 f. 1682

3. Be sure to tell the students this problem needs to be drawn to scale using a ruler. It will take up quite a bit of room on their paper to draw the rectangle with length 25 cm, but it will fit. The rectangles have the same area because 25(12) = 12(25) = 300.

4. $950, the actual value is $945.82

PROPERTY 2.4: MULTIPLICATIVE IDENTITY PROPERTY
For any number m, $(m \cdot 1) = 1 \cdot m = m$. We call 1 the **multiplicative identity**.

EXAMPLE 4

Identify the multiplication property that is used to relate the equivalent expressions.

1. $4 \cdot (7 + 9) = 4 \cdot 7 + 4 \cdot 9$
2. $100 \cdot 25 = 25 \cdot 100$
3. $267 \cdot 1 = 1 \cdot 267 = 267$
4. $(7 \cdot 25) \cdot 100 = 7 \cdot (25 \cdot 100)$

EXERCISES

0. Use the indicated property and write an equivalent number expression.

   a. Use the commutative property of multiplication to write an equivalent number expression to (-2)(-5). What number does this equal?

   b. Use the associative property of multiplication to write an equivalent number expression to (-3 • 2) • 4. What number does this equal?

1. Use the area model and the distributive property to compute the following products. Indicate the area of each interior part in your model.

   a. (3)(5) **15** c. (26)(4) **104**

   b. (4)(3) **12** d. (37)(24) **888**

2. Compute the following: **See TE.**

   a. (42)(33) b. (128)(24) c. (512)(36)

   d. (709)(16) e. (831)(10) f. (58)(29)

3. Draw rectangle A with length 25 cm and width 12 cm. Draw rectangle B with length 12 cm and width 25 cm. Explain why these rectangles have the same area. **See TE.**

4. Ramses has 38 framed posters he plans to sell for \$24.89 each. Estimate the amount of money Ramses will make.11. Give two examples to illustrate each of the following multiplication properties:

5. (12 boxes) (52 cards per box) = 624 cards
(29 boxes) (52 cards per box) = 1198 cards
(36 boxes) (52 cards per box) = 1872 cards
(48 boxes) (52 cards per box) = 2496 cards

6. 460 pencils

7. a. A = (3)(2) = 6; B = (3)(4) = 12; C = (3)(3) = 9; D = (3)(5) = 15
b. 42
c. The area of the large rectangular pen is equal to the sum of the areas of the smaller rectangular compartments.

8. 1472 square units.

9. 36

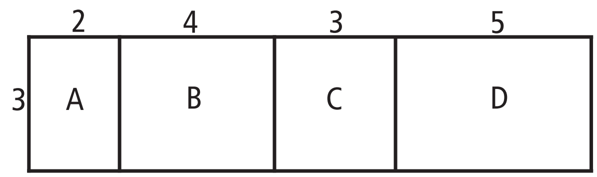
10. 72

11. Answers may vary.

5. Eddie ordered 12 boxes of playing cards. There are 52 cards in each box. How many cards did Eddie order? What if he ordered 29 boxes? 36 boxes? 48 boxes? **(12)(52) cards**

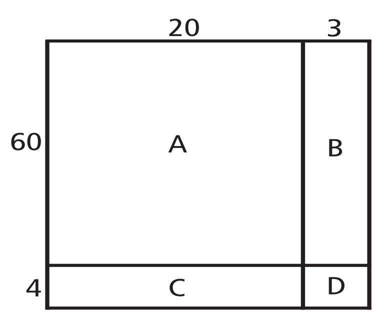
6. Mrs. Guerra purchased 18 packages of 24 pencils. She already had 28 pencils. How many pencils does she now have?

7. Ronnie has a large rectangular area that is fenced to create 4 smaller rectangular sections. The plans below show the dimensions of the area.



   a. Find the partial products of each section.
   b. What is the total area?
   c. How is the total area related to the areas of the smaller sections?
   d. Write equivalent expressions for the area using the distributive property.

8. Compute the area of the large rectangle below.



9. Calculate the sum $1 + 2 + 3 + \cdots + 6 + 7 + 8$. **36**

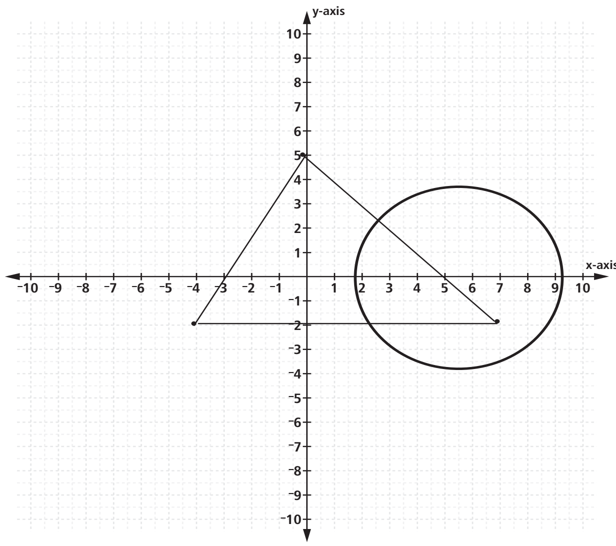
10. Calculate the sum $2 + 4 + 6 + \cdots + 12 + 14 + 16$. How does problem 9 help?

11. (0, 0), (1, 2); Answers may vary.

12. d.

Spiral Review

11.



Name 2 points that are inside the triangle but outside the circle.

12. Melissa made cookies. The recipe required less than $\frac{1}{3}$ cup of nuts. Which of the following fractions is less than $\frac{1}{3}$?

    a. $\frac{1}{2}$ b. $\frac{3}{4}$ c. $\frac{2}{5}$ d. $\frac{1}{6}$

13. **Ingenuity:**

    Recall that when we multiply the expression $(a + b)(x + y)$, we get $ax + ay + bx + by$. This expression has four terms: ax, ay, bx, and by.

    a. Multiply the expression $(a + b + c)(x + y)$. How many terms does your answer have? Draw a rectangular area model for the product.

    b. Multiply the expression $(a + b + c)(x + y + z)$. How many terms does your answer have?

    c. Multiply the expression $(a + b)(m + n)(x + y)$. To do this, first find the product $(a + b)(m + n)$, and then multiply the result by $(x + y)$. How many terms does your answer have?

    d. Without multiplying the expression $(a + b + c)(k + m + n)(x + y + z)$, is it possible to tell how many terms the product has? Explain your reasoning.

Ingenuity

13. Find the areas of the smaller rectangles and add them together to compute the area of the larger rectangle.
    $ac + ad + bc + bd = (a + b)(c + d)$

a. $(a + b + c)(x + y) = ax + ay + bx + by + cx + xy$, thus the expression $(a + b + c)(x + y)$ has 6 terms.

	x	**y**
a	**ax**	**ay**
b	**bx**	**by**
c	**cx**	**cy**

b. $(a + b + c)(x + y + z) = ax + ay + az + bx + by + bz + cx + cy + cz$, thus the expression $(a + b + c)(x + y + z)$ has 3^2 or 9 terms.

c. $(a + b)(m + n)(x + y) = amx + amy + anx + any + bmx + bmy + bnx + bny$, thus the expression $(a + b)(m + n)(x + y)$ has 2^3 or 8 terms.

d. The expression $(a + b + c)(k + m + n)(x + y + z)$ has 27 terms. Considering the case where we have only 2 parentheses for each part of the rectangular model we must pick one element of the first parentheses and one element of the second parentheses for each small rectangle in the larger rectangle model. So, how many ways are there to choose 1 element out of the first parentheses multiplied by how many ways there are to choose an element out of the second parentheses multiplied by how many ways there are to choose an element out of the third parentheses will yield 3 times 3 times 3, or 3^3, which is 27.

   Yes. Multiply the number of terms in each factor. You will get $3(3)3 = 27$.

INVESTIGATION:

14. a. 10506
    b. 10920
    c. 10908
    d. 11663

14. **Investigation:**

    Use the area model to compute the following products:

    a. 102 x 103

    b. 104 x 105

    c. 101 x 108

    d. 107 x 109

Discuss any connections you see between the factors given and the products you found in parts (a) through (d).

Section 2.3 - Linear Model for Division

Big Idea:
Dividing using the linear model

Key Objective:
- Understand and use the vocabulary of division.
- Understand what the missing factor looks like in the linear model.
- Understand the relationship of multiplication and division through skip counting.

Materials:
Number line handout, graph paper, straight edge, preferably NO calculators

Pedagogical/Orchestration:
- This section is a review of basic long division with the linear model. This method is for students who never "got" division, and will encourage them to look for patterns and develop an understanding of how division works.

Activity:
"Big Water Bottle"

Exercises:
#10—examines the relationships among addition, multiplication, and division.

#13—factorials foreshadowed.

Vocabulary:
Divisor, Quotient, Dividend, Factor, Factorial, Missing factor model, Remainder

TEKS:
6.2(C,D,E); 6.11(A,B,C,D); 6.12(A,B); 6.13(A,B)

New: 6.1(A, D); 6.3(D, F)

Warm-Up:
Mrs. Valero's class has 32 students. She wants to purchase a new computer, which costs $1,176. If she pays for it in 28 equal payments what will be the dollar amount of each payment? Write your answer in the grid provided. (Teachers will need an answer grid that is similar to the grid format used in the STAAR test)

Launch for Section 2.4:
Lead your students through the Big Water Bottle Activity found on the next page.

Big Water Bottle

Objective: The students will use the linear model of division to solve a problem.

Materials:
Notebook paper

Activity Instructions:
The teacher will give the students the following problem:

A 32 oz. bottle of water needs to be evenly distributed between five kids. How much water will each child get, and how much will be left over?

The students will show the solution to this problem using all three models (Linear Model, Area Model, and Long Division.)

As a challenge, the teacher could change the size of the bottle or the number of kids that will be sharing the water. Students could display their solutions on poster size paper, and then share their solutions with the class.

*This model is often referred to as the Measurement Model or Repeated Subtraction Model for Division.

ACTIVITY: Models for Division

Objective: Explore models of division.

This is a class activity. Ask 1 or 2 students to make and implement a plan to divide the class into 3 equal groups for a game. The best number of students is a multiple of 3. Notice that we are modeling a division problem, for instance, 21 ÷ 3. If the number of students in the class is not a multiple of 3, the class will have to decide what to do. For instance, if the number of students is not a multiple of 3, there will be a remainder of either 1 or 2. If there is a remainder of 1, the class might appoint the extra person to be a game director. If there is a remainder of 2, the class might also have a scorekeeper.

Reflect on the models your students used. Ask if there are any other methods to divide, excepting long division with pencil and paper. Make sure they discover the first two methods below.

1. Partitive Model: Create piles or groups of objects by placing one object in each pile on each round and then count the number of rounds it takes to use all the objects, as when a dealer deals the cards in a card game.
2. Repeated Subtraction: Remove 3 objects at a time and count the number of removals it takes to remove the original objects.
3. Linear Model (See discussion below): Students skip count by 3 from the total toward zero*. This is an additive model. Be sure your students do not confuse the linear model with the skip counting method of multiplication.
4. Area Model: Ask your students to organize the piles of 3 objects into columns, create as many columns as possible and line them up to form a rectangular shape. In doing so, they have discovered how to reverse the area model of multiplication to divide.

The **divisor** can be either the length of each jump or the number of jumps, depending on the problem. Remind your students factor is another word for divisor when the remainder is 0, and, in that case, we can use them interchangeably.

*If zero cannot be reached the left over is called the remainder.

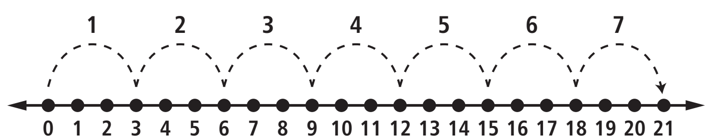
SECTION 2.3 LINEAR MODEL FOR DIVISION

Just as with multiplication, we will explore the operation of division. We will start by looking at some models to better understand how division works.

ACTIVITY: MODELS FOR DIVISION

Your teacher will now lead the class in an activity that reviews models of division.

You divided a class of 21 by 3 in the activity. One method involved subtracting 3 objects at each step from the original group and counting the number of times it takes to distribute all 21 objects. Another way to think about this problem is to add groups of 3 until you have 21. Skip count by 3's to accumulate objects until you have the desired number, 21. The number of skips that it takes to get to 21 is the result 21 divided by 3.



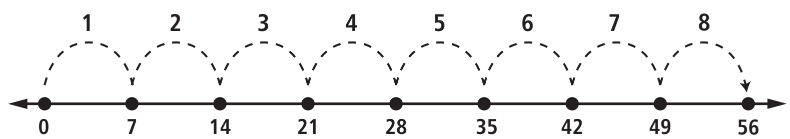
To skip count by 3's, count 3, 6, 9, 12, 15, 18, 21, and so on. You know that $21 = 3 \cdot 7$ because we must skip count 7 steps by 3's to get to 21. The inverse of the multiplication statement is $21 \div 3 = 7$, which means when you divide 21 by 3 the result is 7, because 21 is decomposed into 7 skips of 3 units per skip. This is equivalent to 7 groups of 3. We call 3 the **divisor**, 7 the **quotient**, and 21 the **dividend**.

When the divisor divides evenly into the dividend, or the remainder is zero, the word **factor** is used interchangeably with divisor. Looking for the quotient when 21 is divided by 3 is the same as looking for the missing factor x that satisfies $3 \cdot x = 21$. The x that satisfies this equation is the quotient and represents the number of skips of length 3 it takes to reach 21. We call this the **missing factor model**. It is the reverse of the multiplication process.

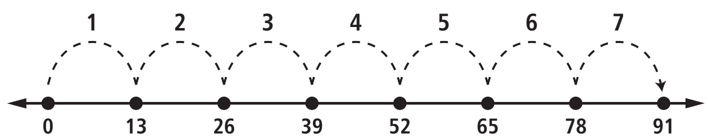
PROBLEM 1

For part (a), set up the number line with tick marks 7 units long, or set up a number line to 55, with 5 units marked.

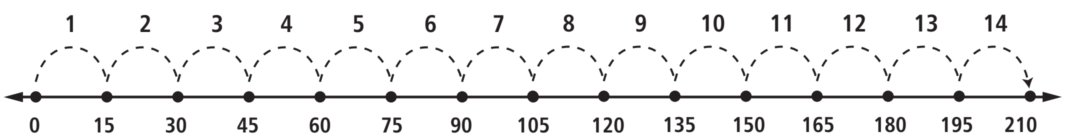
a. 8



b. 7



c. 14



Remind your students that the portion "leftover," which is always smaller than the divisor, is called the remainder.

We should note here that the word "factor" can be a noun that means divisor, as above, where 3 is a factor of 21. It can also be a verb. When we say, "Factor 21," we mean write 21 as a product of two or more positive integers. In this case, write $21 = 3 \cdot 7$ to factor 21 into a product of two numbers, 3 and 7, which are both factors of 21. We will talk more about this in Chapter 3.

PROBLEM 1

Use a number line with the appropriate scale and the skip counting model to compute the following quotients: See TE.

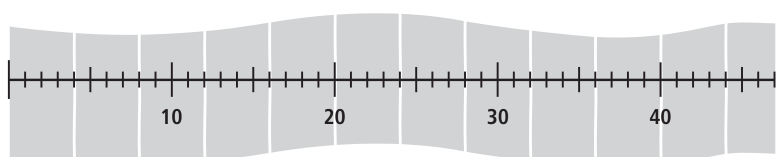
a. $56 \div 7$ b. $91 \div 13$ c. $210 \div 15$

EXAMPLE

Robin has 47 feet of ribbon. She wants to cut this ribbon into 4-foot strips for decorations. How many 4-foot strips of ribbon can she make? How much ribbon will be left over, if any?

SOLUTION

In order to make the 4-foot strips, Robin rolls out all of the ribbon and marks off 4-foot lengths. She then skip counts the number of pieces she needs to cut and finds that $4 \cdot 11 = 44$. Therefore, 47 feet divided into 4-foot pieces equals 11 pieces with 3 feet of ribbon left. In other words, $47 \div 4$ is 11 with a remainder of 3.



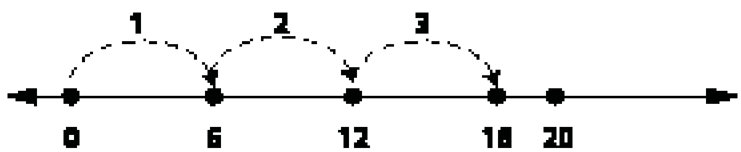
Notice that when using the remainder, the solution is $47 = 4 \cdot 11 + 3$. For now, any number left after dividing you may leave as a remainder.

Have students think about this question and then discuss in small groups. Notice that this problem is different from the previous problem in two ways. We are asking students to partition or equally share the 20 pieces of candy among 6 children without telling them how much to give each child. In the measurement model we specified how much to give each child, we just did not know how many children we could give that amount of candy to. Listen also to see how they deal with the 2 left-over pieces.

EXPLORATION

Mr. Garza has 20 pieces of candy. He wants to divide the candy equally among 6 children. How should he distribute the candy?

One way to distribute the candy is to think of this process in steps. In step 1, give each child 1 piece of candy. This means Mr. Garza has $20 - 6 = 14$ pieces of candy left. In step 2, Mr. Garza gives each child a second piece of candy. He now has $14 - 6 = 8$ pieces of candy left. In step 3, Mr. Garza gives each child a third piece of candy. He now has $8 - 6 = 2$ pieces of candy left. He can no longer give an equal number of pieces to each of the 6 children, so he stops. It took 3 steps to equally distribute as many pieces of candy as Mr. Garza could. That means each child received 3 candies. Write this as $20 = 3 \cdot 6 + 2$. Picture this as a linear model by skip counting to divide 20 by 6, which corresponds to the counting 3 skips of length 6: $3 \cdot 6 = 18$, 2 units short of 20.



In division, the problem involves the dividend and the divisor, and the task is to compute the quotient. In the linear model, the dividend is the total length. There are two possible cases:

(1) Know the length of each jump and call it the divisor. Find the quotient, which in this case is the number of jumps that equal the total length.

(2) Know the number of jumps and call it the divisor. Find the quotient, which in this case is the length of each jump.

In multiplication, the problem starts with the length of each jump and the number of jumps. The answer is the accumulated length of all the jumps. We can think of the division process as the reverse of multiplication.

EXERCISES

1. a. $6, 6 \cdot 7 = 42$ d. $9, 6 \cdot 9 = 54$ g. $33, 33 \cdot 7 = 231$
   b. $8, 8 \cdot 3 = 24$ e. $29, , 29 \cdot 1 = 29$ h. $59, 59 \cdot 11 = 649$
   c. $8, 8 \cdot 8 = 64$ f. $12, 12 \cdot 4 = 48$ i. $76, 76 \cdot 24 = 1824$

   Recall the commutative property of multiplication that states that $a \cdot b = b \cdot a$. Students may give the associated multiplication fact, for part (a) for example, as $6 \cdot 7$ or $7 \cdot 6$.

2. a. $4 \cdot 11 + 1 = 45$ d. $8, 8 \cdot 7 = 56$ g. 67, R 3; $67 \cdot 8 + 3$
   b. $7 \cdot 5 + 4 = 39$ e. 6, R 3; $6 \cdot 10 + 3$ h. 55, R 2; $55 \cdot 9 + 2$
   c. $5 \cdot 4 + 4 = 24$ f. 9, R 3; $9 \cdot 8 + 3$ i. 68, R 2; $68 \cdot 6 + 2$

3. a. 1 - 10 e. 100 - 1000.
   b. 10 - 100 f. 10 - 100
   c. 10 - 100
   d. 100 - 1000

4. 26 bags

5 a. 7 flowers
  b. 3 flowers

6. Erica and her friends can make 8 total bracelets with one bead leaft over.

EXERCISES

1. Evaluate the following quotients, and write the associated multiplication fact. Use the linear model or long division, if needed. See TE.

a. $42 \div 7$	d. $54 \div 6$	g. $231 \div 7$
b. $24 \div 3$	e. $29 \div 1$	h. $649 \div 11$
c. $64 \div 8$	f. $48 \div 4$	i. $1824 \div 24$

2. Write the associated multiplication fact, making the remainder as small as possible: See TE.

a. $45 \div 4$	d. $56 \div 7$	g. $539 \div 8$
b. $39 \div 7$	e. $63 \div 10$	h. $497 \div 9$
c. $24 \div 5$	f. $75 \div 8$	i. $410 \div 6$

3. Estimate each quotient to determine in which range it belongs: between 1-10, between 10-100, or between 100-1000.See TE.

a. $48 \div 6$	c. $272 \div 4$	e. $1491 \div 7$
b. $272 \div 16$	d. $964 \div 9$	f. $1190 \div 17$

Solve each problem from 4 -10. Write your answers in complete sentences.

4. Peter has 624 oranges. He wants to place the oranges into equal amounts in bags. Each bag will contain 24 oranges. Estimate the number of bags he can make with the number of oranges he has. See TE.

5. Audrey invited 5 friends to a dinner party. She wants to place a small vase with flowers in front of each dinner plate. She has 38 flowers.

   a. How many flowers can she use for each vase so that each has the same number of flowers?

   b. Will she have any flowers left to add to the centerpiece? If so, how many?

6. Erica and her friends are making bead bracelets. It takes 13 beads to make each bracelet. If they have 105 beads, how many bracelets will they be able to make? Will any beads be left over? If so, how many?

7. 30; 2 cards left over

8. a. The bag could have contained at most 95 pencils. In this case, each student gets 4 pencils
   b. bag = 74 pencils; students = 3 pencils; bag = 53 pencils; students = 2 pencils
   bag = 32 pencils; students = 1 pencils

9. Uncle Bill can cut 7 sections of PVC pipe with 5 in left over..

10. a. 15 b. 214 c. 31
When there are 6 of the same integer added then divided by 6, the quotient is an integer.
' d. 824 = 2 • 412 e. 174 = 3 • 58

11. Spiral Review (6.2C)
   7 • 5 + 5 • 8 = 35 + 40 = 75 people maximum

7. Madison and her two friends are playing a card game that contains 92 cards. The game requires her to deal out all the cards so that each player gets an equal amount. What is the maximum (most) number of cards each of them could be dealt? How many cards, if any, would be leftover in this case?See TE.

8. There are 21 students in Mrs. Padron's 3rd period class. One day she comes to class with a bag of colorful pencils. She gives each student an equal number of pencils and discovers that she has 11 pencils left over. She knows that the bag had less than 100 pencils to begin with.

   a. What is the largest possible number of pencils the bag could have contained originally? How many pencils would she have given each student in that case?

   b. What other possible solutions could this problem have?

9. Uncle Bill is installing a sprinkler system in his backyard. The system requires 13-inch sections of PVC pipe. He purchases one long 8-foot piece of PVC pipe. How many 13-inch sections can she cut from this long piece? How much pipe will be left over after he cuts the sections? (Hint: 1 ft = 12 in).

10. Evaluate: See TE.

   a. (15 + 15 + 15) ÷ 3

   b. (214 + 214 + 214 +214) ÷ 4

   c. (31 + 31 + 31 + 31 + 31 + 31) ÷ 6

   What patterns do you notice?

   Use the pattern you noticed to complete the next two expressions.

   d. (412 + 412 + 412 + 412) ÷ 2

   e. (58 + 58 + 58 + 58 + 58 + 58) ÷ 2

   What pattern do you notice now?

Spiral Review

11. At the Elk lodge there are a total of 12 tables in the dining room. 7 of the tables seat 5 people each, and 5 of the tables seat 8 people each. What is the maximum number of people who can sit at the tables in the dining room?

12. Spiral Review (6.2D)

    30 • 30 = 900 student approximately

13. **Ingenuity:**

    25

12. Sam counted 27 buses in the school parking lot. If the buses hold between 30 and 36 students, which is the best estimate of the total number of students on the buses?

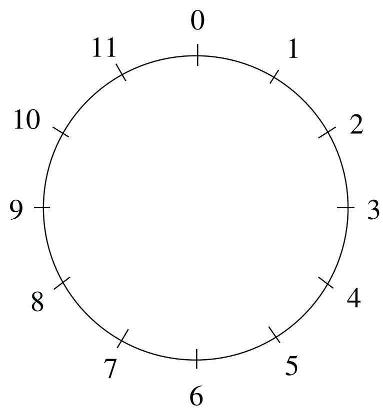
    a. 800 b. 809 c. 900 d. 975

13. **Ingenuity:**

    A certain auto race consisted of 250 laps around an oval track. The only drivers who led laps during the race were Dale, Jeff, and Jimmie. Jeff led three times as many laps as Dale, and Jimmie led twice as many laps as Jeff. In how many laps did Jimmie lead?

14. **Investigation:**

    Draw a circle, and label twelve points on the circle from 0 to 11, as in the Investigation in section 2.1.



Suppose a frog begins at the point 0, and hops clockwise around the circle, hopping one unit at a time. For example, if the frog hops 14 times, then it will make one complete lap around the circle, and finally land at the point 2.

a. Suppose the frog hopped 17 times. How many laps around the circle would it make, and where would it finally land?

b. Suppose the frog hopped 28 times. How many laps around the circle would it make, and where would it finally land?

c. Suppose the frog hopped 45 times. How many laps around the circle would it make, and where would it finally land?

14. **Investigation:**
    a. 1 lap; land at 5
    b. 2 laps; land at 8
    c. 3 laps; land at 9
    d. 8 laps, land on 4; Yes, divide 100 by 12 to determine the quotient (laps) and remainder (landing)

15. **Challenges:**
    423

d. Suppose the frog hopped 100 times. How many laps around the circle would it make, and where would it finally land? Is there a way to figure this out without actually keeping track of all 100 hops on the circle?

e. Suppose we instead marked 60 points on a circle, and labeled them 0 through 59. If the frog started at 0 and made 314 hops, where would it finally land?

15. **Challenge:**

Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left over. They put the coins back, throw one pirate overboard, and split the coins again, only to find that there are 3 coinc left over. So, they throw another pirate over and try again. This time, the coins split evenly. What is the least number of coins there could have been?

Section 2.4 - The Division Algorithm

Big Idea:
Dividing using the area model

Key Objectives:
- Understand the Division Algorithm.
- Discover how area relates to division, including remainders.

Materials:
Calculators for checking number sense, grid paper

Pedagogical/Orchestration:
This section re-introduces division using the area model, with remainders. It provides a classic long division method.

Activity:
"Division Algorithm"

Vocabulary:
Division, Divisor, Quotient, Remainder, Dividend, Algorithm

TEKS:
6.1(A); 6.2 (C,D); 6.5

New: 6.1 (E, F); 6.3(D, F)

Launch for Section 2.4:
Ask students to draw a model to represent and solve the following problem: "There are 125 students going on a field trip and each bus holds 40 students. The school district wants to use as few school busses as possible, so the principal will bring his 10-passenger van to hold any extras. How many busses do they need to order, and how many students will have to ride with the principal in the van?" Look around to see what models different students are using, and discuss as a class. Ask "What do the extra students riding in the van represent?" (The remainder.) Tell your students that today they will use area models to solve problems like the school bus problem. (If no one came up with an area model to represent this problem, you can revisit it after the lesson and have students draw the area model for the problem. The area model will actually resemble 3 long school buses with one short van to hold the 5 students.)

Division Algorithm

Objective: Students will increase their ability and understanding of the area model of division and the division algorithm.

Materials:
Tiles
Grid paper
Times tables (optional)
Index cards with division problems

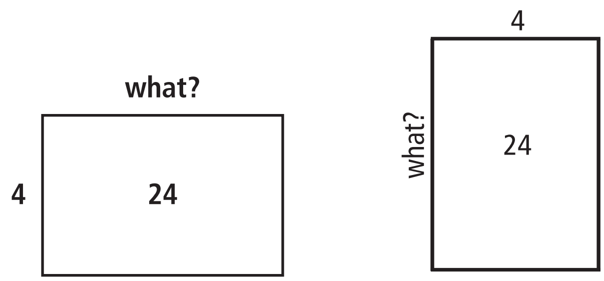
Activity Instructions:
Teacher will provide each student with at least 35 tiles. If teacher does not have enough tiles to give out 35 tiles per student, students will share tiles in pairs.

Students will form rectangles using tiles to show the division algorithm. For example: In the problem $33 \div 4$, students may think of 4x8=32; then form a rectangle with 4 rows of tiles and 8 columns of tiles; then students will add 1 more tile.

Each student will draw each answer on grid paper. Begin process again using other problems written on index cards provided by the teacher.

SECTION 2.4 THE DIVISION ALGORITHM

Another way of thinking of division is by using the area model. This is similar to the missing factor model. To divide 24 by 4, draw a length of 4 and ask what the width x is to equal a total area of 24. What you are doing is looking for the missing factor: $24 = 4 \cdot$ (what?), and $24 =$ (what?) $\cdot\ 4$. This is an example of the commutative property of multiplication.



You know that division is the reverse operation for multiplication, just as subtraction is the reverse operation for addition. What do we mean by this?

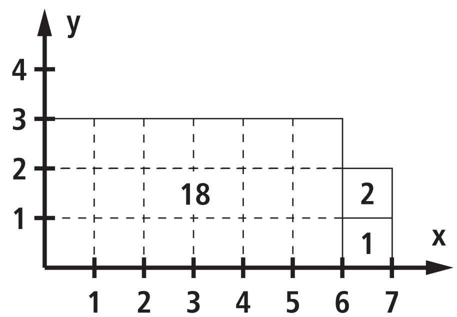
Begin with the number 12. Add 3 to get 15. To undo the addition, you need to subtract 3 from 15 and return to the original number 12. Similarly, in the example above, you found the number 6. Multiply by 4 to obtain 24. That is, $24 = 6 \cdot 4$. To undo this multiplication, divide 24 by 4 and return to the start because $24 \div 4 = 6$.

EXAMPLE 1

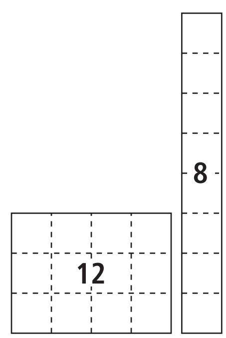
Using the area model, what is $20 \div 3$?

SOLUTION

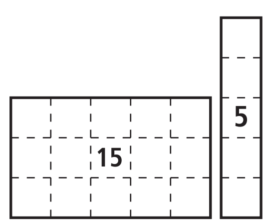
Begin with a length of 3 on the *y*-axis. If we mark off a length of 6 on the *x*-axis, the area of the rectangle is 18. We compute this as $18 = 3 \cdot 6$. To get an area of 20, we must add 2 more square units to the end of the rectangle. That means $20 \div 3$ has quotient 6 with remainder 2 because this corresponds to the calculation $20 = 3 \cdot 6 + 2$.



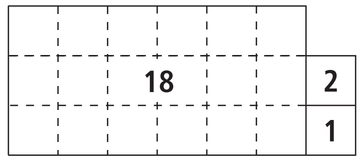
Why is the quotient 6 and the remainder 2? Why not say the quotient is 5 and the remainder is 5? Why not say the quotient is 4 and the remainder 8 since $20 = 3 \cdot 4 + 8$? How do we decide between the different quotients and remainders?



If we think of 20 as $3 \cdot 5 + 5$,



the picture shows that we could break up the last column into pieces of lengths 3 and 2.



Adding this extra 3 to the rectangle is represented by the calculation $3 \cdot 6 + 2$. The picture on the previous page shows we can break up that last column into two pieces one of length 3 and another piece of length 2. By adding these extra pieces of length 3 to the rectangle, we have the same calculation $3 \cdot 6 + 2$.

By using the area model, we write a in the form of the calculation $a = d \cdot q + r$, where d is the height of the rectangle, q is the length of the rectangle along the x-axis, and r is the remainder, or the height of the column, added to equal a. This is the division algorithm where d is the divisor, q is the quotient, and r is the remainder.

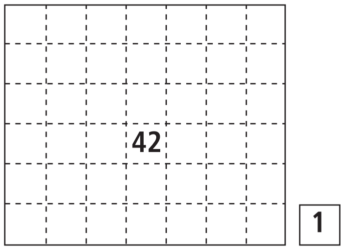
Are there any restrictions on r and d? Yes! When examining the above example, 20 can be written in several different ways:

$$20 = 3 \cdot 4 + 8$$
$$20 = 3 \cdot 5 + 5$$
$$20 = 3 \cdot 6 + 2$$

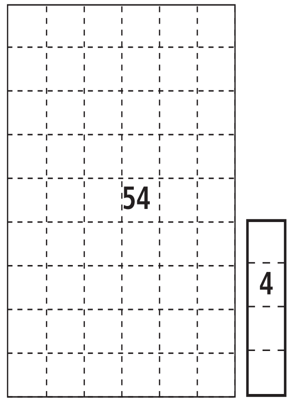
The smaller the values of r the closer we are to seeing if 20 is a multiple of 3. Only when we get to $r = 2$, do we see that 20 is not a multiple of 3. There will be some remainder, namely 2. So, one condition to put on r and d is that $r < d$, and r must be greater than or equal to zero ($r \geq 0$).

The divisor d must be positive because r is not negative, and d is greater than r. We write this with our inequalities as follows: Because $r < d$, then $d > r$. And because $r \geq 0$, then $d > r \geq 0$ and $d > 0$. With this added restriction, we write $20 = 3 \cdot 6 + 2$.

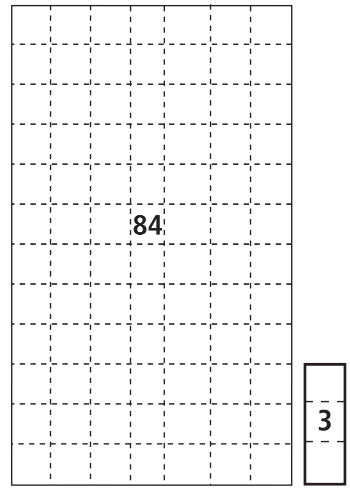
a. 43 = 6(7) + 1



b. 58 = 6 (9) + 4

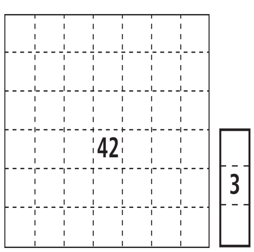


c. 87 = 12(7) + 3



EXERCISES

1. a. 4 r 5
   b. 7 r 1
   c. 5 r 0
   d. 5 r 7
   e. 7 r 5
   f. 13 r 6
2. a. between 10 and 100
   b. between 1 and 10
   c. between 100 and 1000
3. 87 mp3 files, 4 MB left over
4.



5. Pamela gets $15 dollar bills and has 3 quarters left.

We state the formal division algorithm:

THEOREM 2.1: DIVISION ALGORITHM

Given two positive integers a and d, we can always find unique integers q and r such that $a = dq + r$ and $0 \leq r < d$. We call a the **dividend**, d the **divisor**, q the **quotient,** and r the **remainder**.

In our previous example with $20 = 3 \cdot 6 + 2$, the dividend $a = 20$, the divisor $d = 3$, the quotient $q = 6$, and the remainder $r = 2$.

Compute the following division problems by writing the corresponding division algorithm and sketching a picture that explains what the algorithm represents.

a. $43 \div 6$ b. $58 \div 9$ c. $87 \div 12$

EXERCISES

1. Draw the area model for each of the following, then use the division algorithm to compute the quotient. **See TE.**

   a. $29 \div 6$ d. $47 \div 8$

   b. $38 \div 5$ e. $68 \div 9$

   c. $80 \div 16$ f. $110 \div 8$

2. Using the area model, predict whether the quotient is between 1 and 10, between 10 and 100, or between 100 and 1000 by estimating the length of the rectangle's base.

   a. $812 \div 40$ b. $217 \div 31$ c. $2401 \div 12$

3. If each mp3 file takes up 8 MB of space, how many files can you fit on a 700 MB CD? How much space, if any, will you have left over? **175 mp3 fi**

4. Use graph paper to model $45 \div 6$ using the area model. **See TE.**

5. Pamela has 63 quarters. She goes to the bank and trades them for dollar bills. How many dollar bills will she get? How many quarters, if any, will she have left?

6. Jeremy can make a path that is 16 tiles long and have 2 tiles left over.

7. Round 43 to 40. Round 1591 to 1600. 1600 ÷ 40 = 40. So, the number of bags is between 10 -50

8. This exercise is a fine time to reinforce the power of simplification in division, a practice that has inexplicably fallen into disuse. It is a powerful tool for computation and later for algebraic simplification.
   a. 7
   b. 6
   c. 12
   d. 5
   e. 6
   f. 1
   g. Since a number divided by itself is 1 anytime a number appears in the divisor and dividend, it "divides out"
   h. 8

9. One way to see that the two models are the same is to notice that we can convert from one to the other. For example, starting with the linear model, convert each jump to a column in the area model.

10. Spiral Review (5.8B)
    Translation

11. Spiral Review (6.11A)
    (30) (4) = 120 batteries needed. 120 ÷ 10 = 12 packages

6. Jeremy has 50 square tiles, each is 1 foot by 1 foot. He would like to construct a path that is 3 feet wide. How long can he make the path? How many tiles, if any, will he have left? **You can make a path 13 tiles long and have 1 tile left over.**

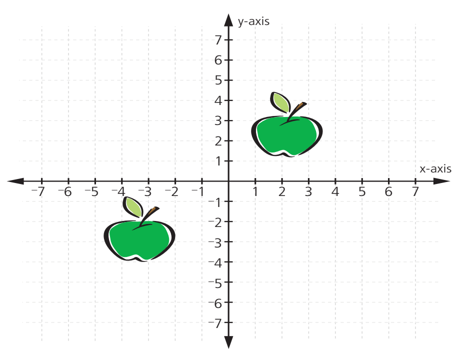
7. Danny wants to fill 43 bags with candy. He has 1591 pieces of candy he will use. Estimate which range best describes the number of bags he can fill: 1- 10, 10 - 50, 50 - 100, or more than 100. Explain how you reached this estimate. **See TE.**

8. Evaluate the following expressions:
   a. $(2 \cdot 7) \div 2$
   b. $(4 \cdot 3 \cdot 6) \div (4 \cdot 3)$
   c. $(6 \cdot 2 \cdot 4) \div 4$
   d. $(9 \cdot 3 \cdot 5) \div (9 \cdot 3)$
   e. $(6 \cdot 5 \cdot 4) \div (5 \cdot 4)$
   f. $(4 \cdot 3 \cdot 2) \div (2 \cdot 4 \cdot 3)$
   g. Write rules to describe any patterns you noticed in a - f. What causes these patterns?
   h. Using the patterns you observed above, compute the following: $(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3) \div (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)$.

9. Given two positive integers m and n, explain why the area and linear models of division from $m \div n$ give the same results.

Spiral Review

10. Which transformation is shown below?



See TE.

11. Mrs. Murphy needs to replace batteries in 30 calculators. Each calculator uses 4 batteries. The batteries are sold in packages of 10. How many packages of batteries does Mrs. Murphy need to buy?

Investigation

13. a. She will have 8 groups total, and 1 will be a 5 student group.
    b. 8 + 5 = 13 tables

Challenges:

14. 15

12. **Ingenuity:**

    Answer the following questions:

    a. Suppose we made a 4 x 4 square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of 3 tiles each, how many sets would we have?

    b. Suppose we made a 7 x 7 square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of 6 tiles each, how many sets would we have?

    c. Suppose we made a 77 x 77 square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of 76 tiles each, how many sets would we have?

    d. Suppose we made an n x n square of tiles, and removed one tile from the upper right corner. If we wanted to divide the remaining figure into sets of n - 1 tiles each, how many sets would we have?

13. **Investigation:**

    Use the area model to answer the following question:

    a. Ms. Reyes wants to divide her class of 33 students into groups of four or five students each, with as few five-student groups as possible. How many of these groups will be five-student groups?

    b. Later that day, Ms. Reyes goes to a teacher appreciation luncheon. There are 83 teachers present at the luncheon. The organizer of the luncheon wants to organize the teachers into tables of six or seven teachers each, with as few seven-teacher tables as possible. How many tables will there be, and how many of these tables will have seven teachers?

14. **Challenge:**

    When a number n is divided by 11, the quotient is 11 with a possible remainder. When n is divided by 10, the quotient is 12 with a possible remainder. When n is divided by 9, the quotient is 13 with a possible remainder. What is the quotient when n is divided by 8?

Section 2.5 - Long Division

Big Idea:
Modeling of long division using the scaffolding method

Key Objectives:

- Relate the long division process to the linear and area models of multiplication.
- Discover patterns in division and relate to multiplication.
- Understand the long division process.

Materials:
Graph paper, Calculators starting with Exercise 9

Pedagogical/Orchestration:

- This is a basic introduction to the fundamental meaning of division and the long division process.
- Have students check the answer as often as possible by performing the related multiplication to get the original dividend back.

Activity:
:"Bags of Stuff"(to be used at the end of this section) and "Division Decisions"

Exercises:
Until Exercise 9, discourage the use of calculators. The division problems are designed to review basic skills and build confidence through the recognition of patterns that aid computation.

Vocabulary:
dividend, quotient, divisor, remainder, scaffolding

TEKS:
6.2 (C,D); 6.11(C); 6.12(B) New: 6.1 (D); 6.3(D, F)

Launch for Section 2.5:
Tell your students, "Draw a model and see if you can figure out what number I am thinking about. This number divided by 4 equals 5 remainder 3. What is the number?" Allow students to use their own strategies to determine the answer. Once students have worked on the problem, show any correct visual models the students have drawn that show the answer to be 23. Ask students to come up with a situation that would model the problem. One possibility: "I am thinking of the number of pencils that a teacher started with if she divided up the pencils evenly amongst 4 students so that each student received 5 pencils and the teacher had 3 pencils left over." Tell your students, "The number you found is called the dividend of the problem, and we will be using this method throughout the lesson to check our division process and to understand the true meaning of division."

Bags of Stuff Activity:

Materials:

Long division exercises

Stories about division

Activity Instructions:

Teacher will guide students in filling in the boxes to solve long division application problems. Students will think of 539 oranges that need to be stored in bags of 4 oranges. They will figure out how many bags will be needed to store all the oranges. First write the dividend 'inside' the corner, and the divisor outside.:

$$4\overline{)539}$$

Hundreds	Tens	Ones
How many 4's fit in 539?	How many 4's fit in 139?	How many 4's fit in 19?
$\begin{array}{r} 100 \\ 4\overline{)539} \\ \underline{-400} \\ 139 \end{array}$	$\begin{array}{r} 30 \\ 100 \\ 4\overline{)539} \\ \underline{-400} \\ 139 \\ \underline{-120} \\ 19 \end{array}$	$\begin{array}{r} 4 \\ 30 \\ 100 \\ 4\overline{)539} \\ \underline{-400} \\ 139 \\ \underline{-120} \\ 19 \\ \underline{-16} \\ 3 \end{array}$

Division Decisions Activity:

Materials:

20 - 30 objects per group

Instructions:

Option 1 (class activity): Ask 1 or 2 students to make and implement a plan to divide the class into 3 equal groups for a game. The best number of students is a multiple of 3. Notice that we are modeling a division problem. If the number of students in the class is not a multiple of 3, the class will have to decide what to do. For instance, if the number of students is not a multiple of 3, there will be a remainder of either 1 or 2.

Option 2 (group activity): Give groups of 3 to 4 students a set of 20 to 30 objects. Ask them to make and implement a plan to divide the objects equally among group members. Emphasize the fact that it is necessary to understand the process used in distributing the objects.

Reflect on the models your students used. Ask if there are any other methods to divide, excepting long division with pencil and paper. Make sure they discover the first two methods below.

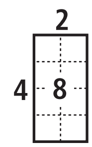
1. Partition Model: Create piles or groups of objects by placing one object in each pile on each round and then count the number of rounds it takes to use all the objects, as when the dealer deals the cards in a card game.
2. Repeated Subtractions: Remove 3 objects at a time and count the number of removals it takes to remove the original objects.
3. Linear Model: Students skip count by 3 until the total is reached. This is an additive model. Be sure your students do not confuse the linear model with repeated subtraction. Ask your students to organize the piles of 3 objects into columns, create as many columns as possible and line them up to form a rectangular shape. In doing so, they have discovered how to reverse the area model of multiplication to divide.

SECTION 2.5 LONG DIVISION

We have seen how closely related multiplication and division are. For example, we know $8 \div 4 = 2$ because $4 \times 2 = 8$. Also recall that in the long division form, the multiplication fact is rewritten as

$$\begin{array}{r} 2 \\ 4\overline{)8} \end{array}$$

The area model looks like this:

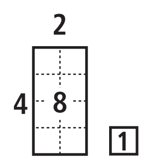


We have the **dividend** 8 "under" the **quotient** 2, and the **divisor** 4 is to the left of the dividend.

By changing the dividend to 9, our problem becomes $4\overline{)9}$. Because $8 \div 4 = 2$, $9 \div 4$ must be more than 2. In the long division form,

$$\begin{array}{r} 2 \\ 4\overline{)9} \\ \underline{-8} \\ 1 \end{array}$$

The area model looks like this:

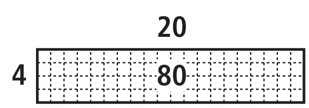


The quotient is 2, and the **remainder** is 1.

Consider the problem $4\overline{)80}$, where the dividend is not simply 8 but 8 tens. The quotient is then 2 tens, or 20, because $4 \times 20 = 80$. In the long division form,

$$\begin{array}{r} 20 \\ 4\overline{)80} \\ \underline{-80} \\ 0 \end{array}$$

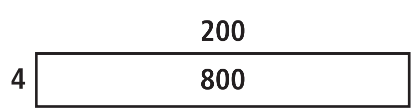
The area model is:



Use the long division form to evaluate the problem $4\overline{)800}$ with 8 hundreds. Do you agree the answer is 2 hundreds, or 200? In the long division form,

$$\begin{array}{r} 200 \\ 4\overline{)800} \end{array}$$

The area model is:



PROBLEM 1

Model the long division process.

a. $52 \div 4 =$ **13** b. $960 \div 6 =$ **160**

Note that the above picture is not to scale. If you were to draw it to scale and leave the height unchanged, imagine how long the rectangle would be.

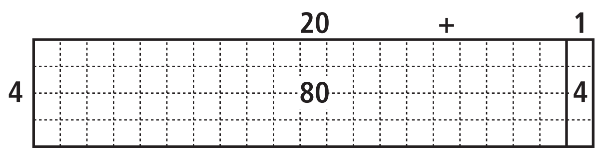
A more complex problem is $4\overline{)84}$. One way to think of this problem is to notice the place values and observe that $84 = 80 + 4$. You know that

$$\begin{array}{r} 20 \\ 4\overline{)80} \end{array} \text{ and } \begin{array}{r} 1 \\ 4\overline{)4} \end{array}.$$

Putting these together gives

$$\begin{array}{r} 21 \\ 4\overline{)84} \end{array}.$$

Here is the area model for this problem:



or

$$\begin{array}{r} 1 \\ 20 \\ 4\overline{)\ 84} \\ \underline{-80} \\ 4 \\ \underline{-4} \\ 0 \end{array}$$

This is called the **scaffolding** method because the different partial quotients are first computed and stacked, then combined, much like a scaffold is used in constructing a building.

PROBLEM 1

Use this scaffolding method to compute the following quotients. You may sketch a picture of the corresponding area model if it helps.

a. $52 \div 4$

b. $960 \div 6$

PROBLEM 2

a. 685 ÷ 11 **62 r 3**

b. 2153 ÷ 14 **153 r 11**

c. 378 ÷ 12 **31 r 6**

d. 1341 ÷ 16 **77 r 9**

e. 5642 ÷ 28 **201 r 14**

PROBLEM 2

Use the scaffolding method to compute the following quotients:

a. $685 \div 11$ b. $2153 \div 14$

c. $378 \div 12$ d. $1341 \div 16$

e. $5642 \div 28$

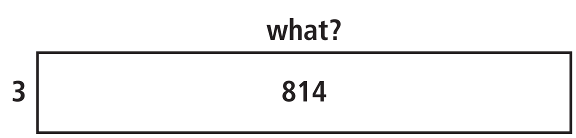
If you remember the common algorithms for adding, subtracting, and multiplying, you know to start working with the smaller place values and then work up to the larger place values. Think about working the following problems, and pay close attention to the place you start and the direction you move in your computation.

$$\begin{array}{r} 3108 \\ +\ \ 15 \\ \hline \end{array} \qquad \begin{array}{r} 3108 \\ -\ \ 15 \\ \hline \end{array} \qquad \begin{array}{r} 3108 \\ \times\ \ 15 \\ \hline \end{array}$$

In dividing, however, we know that it is more common to start with the largest place value to determine the quotient and then gradually include the smaller place values. Using these numbers, we will compute $3108 \div 15$. First, though, let's look at the division problem $814 \div 3$, or $3\overline{)814}$,and use it as an example to see

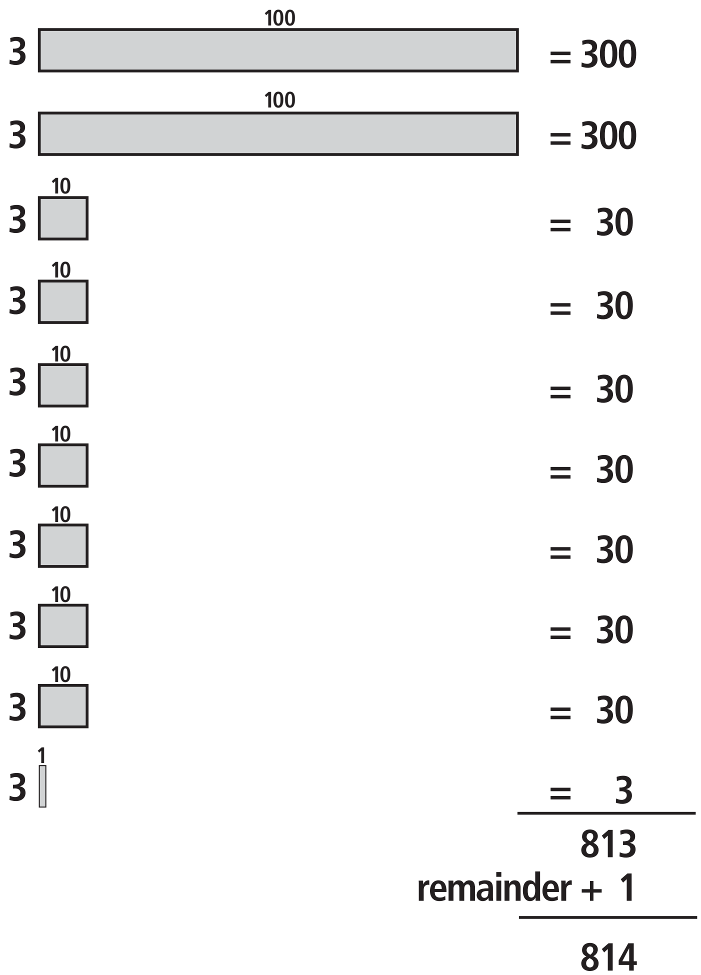
- how long division works,
- how it relates to multiplication, and
- what representation helps us to better understand the long division procedure.

One way to think about this problem is to consider the related multiplication statement. Because the division problem is, "What does $814 \div 3$ equal?" the related multiplication statement reads, "What times 3 equals 814?"



Ask your students to work this individually and share what they learned in groups. Building the rectangle, get 2 100x15 rectangles, 0 10x15 rectangles and 7 1x15 rectangle. Using the long division method, the quotient is stacked above the dividend. In the traditional long division method, 15 does not go into 3 but 15 does go into 31. Have your students make sense of how this step translates in the scaffolding method or in the area model. When we move from 3, in the thousands place, there are no 1000x15 rectangles so we look at the number of 100x15 rectangles we need when we consider 31 or 31 hundreds. This is a difficult place value concept. Another difficulty in this problem involves the fact that there are NO 10x15 rectangles. In the traditional method, we must put a 0 in the tens place. In the scaffolding method, the fact that there are no 10x15 rectangles is clear.

What does **814 ÷ 3 =** ? We look at the area models that go with the Scaffolding method.



The model has reached the sum 813, but because 1 more is needed to reach the dividend 814, 1 is the remainder. The result of the long division can be written as 271 r 1.

$$\begin{aligned} \mathbf{814 \div 3} &= \underbrace{\mathbf{100 + 100}}_{\mathbf{200}} + \underbrace{\mathbf{10 + 10 + 10 + 10 + 10 + 10 + 10}}_{\mathbf{70}} + \mathbf{1 \ \ r\,1} \\ &= \mathbf{200 + 70 + 1 \ \ r\,1} \\ &= \mathbf{271 \ \ r\,1} \end{aligned}$$

From our work on the division problem, 814 ÷ 3 has quotient 271 and remainder 1. Determine what 813 ÷ 3 equals. If you found the quotient 271, you are correct.

3108 ÷ 15 = **207r3**

EXERCISES

1. a. 9 b. 62 c. 32
   90 620 320

2. Anne is correct. 1230 ÷ 12 = 102 r 6. So, 103 buses are needed

3. a. 10 r 3
   b. 237 r 1
   c. 13 r 1
   d.. 174 r 0

4. a. 5 r 4
   b. 4 r 8
   c. 35
   d. 16 r 17
   e. 42
   f. 308 r 2

5. 8 movies.

6. 5 days

7. 11 days

Now carefully compute 3108 ÷ 15, showing the scaffolding method and the area model. Notice how close the scaffolding method is to the long division method.

EXERCISES

1. Compute the following quotients using scaffolding long division, if necessary. Verify your answer using multiplication. You might also want to check using your calculator, if you are unsure.

   a. 72 ÷ 8 b. 868 ÷ 14 c. 736 ÷ 23

      720 ÷ 8 8680 ÷ 14 7360 ÷ 23

2. There are 1230 people attending a concert. A shuttle bus that seats 12 people will transport them to the stadium. Arne says they will need buses in the hundreds. Barak says they will need buses in the thousands. Who is right and, why do you say so?
3. Compute the following quotients and remainders. Check your answer with a visual model. Write a relationship of dividend, quotient, divisor, and remainder using the division algorithm.

   a. 123 divided by 12

   b. 475 divided by 2

   c. 209 divided by 16

   d. 870 divided by 5

4. Use one of the division methods you have learned to complete the following quotients:

   a. 39 ÷ 7 d. 673 ÷ 41

   b. 68 ÷ 15 e. 1512 ÷ 36

   c. 315 ÷ 9 f. 8318 ÷ 27

5. Sara is going to rent some movies for a party. Each movie rental costs $9, and she has $75 to spend. How many movies can she rent?

6. Israel will drive 530 miles to McAllen from San Antonio this summer. If he is driving 125 miles per day, how many days will it take him to get to McAllen?

7. Melissa needs $425 to buy a new purse. If she earns $40 per day, how many days of work will it take Melissa to buy her purse?

8. 2400 seats

9. 28

10. yes, $25 left over

11. a. $12
    b. $3.00
    c. 33 cents

12. We want the students to determine the most number of games that can be purchased. She can buy 6 games and have $9 left.

13. a. 9
    b. 90
    c. 900
    d. 9000
    e. 6
    f. 60
    g. 3
    h. 300
    i. When the dividend is multiplied by a power of 10, the quotient is multiplied by the same power of 10.

8. A football stadium has 4800 seats. During the last game of the season, the stadium made \$28,800.00. How many seats were filled, if each seat sold for \$12.00?

9. Fossum Middle School has cafeteria tables that seat 10 students each. If there are 273 students going to lunch, how many tables will be needed?

10. Ana, Steven, Ruth, and Mark were raising money for a talent show. Ana raised \$45, Steven raised \$60, Ruth raised \$48, and Mark raised \$12. They needed to buy uniforms that cost \$35 each. Will they be able to buy 4 uniforms? If yes, how much extra money will be left? If not, how much more money do they need to raise?

11. a. Donna paid \$60 for five CDs of equal cost. How much did each CD cost?

    b. Shirley paid \$36 for twelve hot dogs. How much did she pay for each hot dog?

    c. Gary paid 99 cents for three pieces of candy. How much did he pay for each piece?

12. Rolinda has \$279 to spend on video games. Each game costs \$45. What is the greatest number of games Rolinda can buy? How much money will Rolinda have left?

13. Calculate the following:

a. $27 \div 3$	e. $24 \div 4$
b. $270 \div 3$	f. $240 \div 4$
c. $2700 \div 3$	g. $33 \div 11$
d. $27000 \div 3$	h. $3300 \div 11$

    i. Write rules to describe any patterns you noticed in **a–h**. What causes these patterns?

14. Cathy = \$5, Casey = \$12, Chasity = \$10. Total spent \$27

15. Part a

16. **Ingenuity:**

   a. A=3
   b. B=4
   c. C=7; D=9

16. **Investigation:**

   a. 28 pieces per student, and 17 for Mr. Jensen
   b. 30 per student, and 3 for Mr. Jensen

Spiral Review

14. Casey, Cathy, and Chasity each bought food for a faculty party. Casey spent \$7 more than Cathy. Cathy spent \$5 less than Chasity. Chasity spent \$10. What is the total amount of money spent on food?
15. Paige earns \$12 each week walking her friend's dog. Which of the following is the best estimate of how much money she will earn in 38 weeks of walking the dog?

    a. \$450 c. \$460 c. \$470 d. \$480

16. **Ingenuity:**

    Solve the following "missing digit" puzzles. In each puzzle, each letter represents a digit from 0 to 9.

    a. Suppose that AA2 ÷ 4 = 8A. What digit does A represent?

    b. Suppose that 1B28 ÷ 3B = B2? What digit does B represent?

    c. Suppose that 6CD ÷ C = DC? What digit do C and D represent?

16. **Investigation:**

    On Monday, Mr. Jensen brought a bag of small candies to school for his algebra class to enjoy. He divided the candies evenly among his 22 students. Each student got 9 candies, and Mr. Jensen was left with 13 candies, which he ate.

    a. Mr. Jensen realized that he did not bring enough candies to class on Monday, so on Tuesday, he brought three bags of candies. Each bag had the same number of candies as the bag he brought on Monday. If Mr. Jensen divided these candies evenly among his 22 students and kept the remainder for himself, how many candies did each student get, and how many did Mr. Jensen get? (Assume that Mr. Jensen always gives his students as many candies as he can, provided that each student has the same number.)

    b. Mr. Jensen was happy with the number of candies he brought on Tuesday, so he brought the same number again on Wednesday. This time, however, there were only 21 students in class. How many candies did each student get, and how many did Mr. Jensen get?

    c. Can you find a way to do parts (a) and (b) without figuring out how many candies are in a bag?

17. **Challenge:**

$$\begin{array}{r}
34 \\
5\overline{)173} \\
\underline{15} \\
23 \\
\underline{20} \\
3
\end{array}$$

17. **Challenge:**

While performing a trick of long division, a mathemagician made some of his digits disappear. Alas, he cannot reconstruct the missing digits... until he remembers that one of them is a 7. Fill in the blanks to complete the calculation.

```
      _4
  _)1_3
    __
    --
     __
     2_
    ---
      3
```

1. a. 75
   b. 128
   c. 105
   d. 99

2. a. 36
   b. 35

3. $180

4. 25 classes

5. 13440 gumballs

6. a. $(25 \cdot 4) \cdot 52 = 100 \cdot 52 = 5200$
   b. $(25 \cdot 12) \cdot 83 = 300 \cdot 83 = 24900$
   c. $(5 \cdot 20) \cdot 202 = 100 \cdot 202 = 20200$
   d. $(2 \cdot 50) \cdot 221 = 100 \cdot 221 = 22100$

7. a. $(100 - 8) \cdot 8 = 736$
   b. $(100 - 2) \cdot 36 = 3528$

8.

Rectangle	Length	Width	Area
A	**3**	**6**	**18**
B	**7**	**11**	**77**

REVIEW PROBLEMS

1. Use the number line frog to compute:
   a. 15 · 5
   b. (8)16
   c. 15 (7)
   d. (11) (9)
2. Use the area model to compute:
   a. 12 · 3
   b. 5 · 7
3. Jimmy bought 12 tickets to the carnival. Each ticket cost $15. How much did Jimmy pay?
4. There are 580 students at Fossom Middle School. Each math class will have at most 24 students. What is the minimum number of math classes at Fossum Middle School?
5. Lorianne is buying large bags of bubble gum for a gumball machine. The large bags contain 560 gumballs. She purchases 24 large bags. How many gumballs will she have?
6. Compute the following using multiplication properties using the commutative property to make the computation easier. Show your work.
   a. 25 · 52 · 4
   b. 25 · 83 · 12
   c. 202 · 20 · 5
   d. 221 · 50 · 2
7. Explain how to compute using the distributive property to make the computation easier.
   a. 92 · 8
   b. 98 · 36
8. Draw a 3 x 6 rectangle and call it rectangle A. Draw a 7 x 11 rectangle and call it rectangle B. Then fill in the table.

Rectangle	Length	Width	Area
A			
B			

9. a. 7 (r0) c. 35 (r3)
   b. 10 (r10) d. 3 (r10)

10. a. 4 (r0) c. 3 (r8)
    b. 7 (r8) d. 8 (r18)

12. a. 72 (r0) c. 25 (r19) e. 53 (r61)
    b. 52 (r2) d. 24 (r29) f. 98 (r42)

13. a. 103 (r2) c. 163 (r3) e. 110 (r2)
    b. 122 (r1) d. 284 (r4) f. 25 (r5)

14. 5 games

9. Use the linear model to find the quotient that makes the remainder as small as possible:

   a. $21 \div 3$
   b. $120 \div 11$
   c. $143 \div 4$
   d. $94 \div 28$

10. Use the area model to find the quotient that makes the remainder as small as possible:

    a. $24 \div 6$
    b. $71 \div 9$
    c. $44 \div 12$
    d. $210 \div 24$

11. Predict whether the quotients to the following division problems are between 0.1 and 1, between 1 and 10, between 10 and 100, or between 100 and 1000. Explain your prediction and then do the division to check your answer.

    a. $217 \div 37$
    b. $738 \div 33$
    c. $5521 \div 97$
    d. $364 \div 3$

12. Find the quotient and remainder using scaffolding.

    a. $216 \div 3$
    b. $210 \div 4$
    c. $819 \div 32$
    d. $989 \div 40$
    e. $3921 \div 73$
    f. $7882 \div 80$

13. Find the quotient and remainder using long division.

    a. $311 \div 3$
    b. $611 \div 5$
    c. $2122 \div 13$
    d. $3412 \div 12$
    e. $3412 \div 31$
    f. $1202 \div 48$

14. Nama has \$150 and wants to buy video games. Each video game costs \$27. How many video games can she buy?

15. Perform the following operation and calculate the sum, difference, product, or quotient as appropriate

    a. $13 + (-4)$
    b. $-203 + 4$
    c. $13 \cdot (-4)$
    d. $13 - (-4)$
    e. $-4 \cdot (-13)$
    f. $20 \div 4$
    g. $203 \div 4$
    h. $-203 \cdot 4$
    i. $-203 - 4$
    j. $-203 + 4$

[redacted]

MATH EXPLORATIONS

Factors and Multiples

CH. 3 FACTORS AND MULTIPLES **111**

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Section 3.1 uses the Possible Rectangle Activity to relate factors, divisors, multiples, and division and for students to also use those terminologies. Prime and composite numbers are distinguished through the number of rectangles that can be formed with whole number dimensions. Square numbers are also introduced and represented as special rectangles along a number line. The Sieve of Eratosthenes is another activity that is intended to deepen students' understanding of number patterns, factors, multiples, primes and composites. Divisibility rules are presented as convenient ways of checking divisibility by certain one digit numbers and whether numbers are prime or composite numbers. Section 3.2 discusses the order of operations through the creation of a flip chart that can be used as a reference during the year. Exponential notation is presented as a convenient way of writing certain forms of multiplication. The emphasis is not on the properties of exponents but using the definition of exponents and powers to simplify expressions with exponents with the same base. Section 3.3 develops the understanding that any integer greater than 1 is either prime or can be written as a unique product of prime factors. Factor trees are used to decompose the composite numbers. Sections 3.4 and 3.5 use various methods for finding the Greatest Common Factor and Least Common Multiple. Our book uses the prime factorization method, T-chart, listing, and Venn Diagrams as four primary ways to find GCF and LCM.

Section 3.1 - Factors, Multiples, Primes and Composites

Big Idea:

- Discovering relationships of factors, divisors, multiples and division
- Understanding and identifying prime and composite numbers

Key Objectives:

- Introduce the prime numbers through exploration
- Introduce formal definitions of prime and composite numbers.

Materials:

Graph paper, Blackline masters for Possible Rectangles and Sieve of Eratosthenes, Map colors or Markers for Sieve activity

Pedagogical/Orchestration:

Two big activities in this section: Possible Rectangles and Sieve of Eratosthenes.

- Exploration 1, step 2, third sentence talks about "number of rectangles possible with area *n*." Remind the students about what area is. The concept of area is taught in 5th grade, maybe earlier. We count a 2x3 rectangle and a 3x2 rectangle only once.
- Exploration 1, step 9 foreshadows primes: numbers that have exactly 2 factors; the factor 1 and the factor itself.
- Exploration 2, defines perfect squares.
- Another pattern is factor pairs. The dimensions created in the Possible Rectangle Dimensions column are called factor pairs. When these factor pairs are multiplied, they result in the value of column *n*.

Activities:

"Factor Bingo", "Sieve of Eratosthenes", "Product Bingo", "Divisibility Rules Booklet", and "Square Number Model" and "Shikaku-Rectangles"

Exercises:

Exercise 5 foreshadows common multiples.
Exercise 7 foreshadows least common multiple.
Exercise 12 good for group activity and class discussion.

Internet Resources:

Battleship game to review prime, composite, and divisibility rules: http://www.quia.com/ba/34873.html
KenKen Puzzles: http://www.kenken.com/misc/classroom
and http://www.nytimes.com/ref/crosswords/kenken.html

Vocabulary:

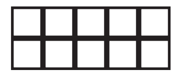
divisible, factor, multiple, factor pair, product, common factor, common multiple, square numbers, prime number, composite number

TEKS:

6.1(E,F); 6.13(A); 5.4(A)

Launch for Section 3.1:

Can you draw a rectangle that has an area of 5 square units and a length of 5 units? (Have a student draw one, and correct it if it is not somewhat drawn to scale. ). What is the width of the rectangle? (1 unit) What is a multiplication problem we can write to represent this? (5 X 1 = 5)

Can you draw another rectangle that has an area of 10 square units and a length of 5 units? 

What is the width of the rectangle? (2 units) What is a multiplication problem we can write to represent this? (5 X 2 = 10) What would be the width of a rectangle that has an area of 15 square units and a length of 5 units? (3 units) (Have a student draw this rectangle on the board) What is the multiplication problem? (5 X 3 = 15)

The following are some multiplication problems we just modeled:

5 X 1 = 5
5 X 2 = 10
5 X 3 = 15

What are some more equations we could write using this pattern?

5 X 4 = 20
5 X 5 = 25
5 X 6 = 30

What are some patterns you notice in these problems? (Possible answers: 5 is being multiplied by increasing natural numbers, or the products are increasing by 5)

The products or areas are the results of a natural number being multiplied by 5. Therefore, the products or areas are known as multiples of 5. In other words, the multiples of 5 are 5, 10, 15, 20, 25, 30, and so forth.

Can you make some rectangles with areas that are multiples of 6? Make sure the lengths of the sides are natural numbers. (Have students draw some rectangles.) What are some multiples of 6? (6, 12, 18, 25, 30, and so forth)

We have just learned the word "multiple" which represents the areas of our rectangles with natural number side lengths. Listen to the lesson for another vocabulary word that describes the side lengths of our rectangles. (On the board write the following sentence and tell students to listen to the lesson and fill in the blank with a vocabulary word they will learn in the lesson: "The side lengths of our rectangles are the ________________ of the area." (The word that fits in the blank is "factors.")

Factor Bingo

Objective: This game is designed to reinforce factors, multiples and prime & composite numbers; it will also help students review multiplication facts.

Materials:
One Factor Bingo card per student
One set of number cards 1-36
Timer/stop watch
Color pencils

Activity Instructions:

1. Make number cards. Write numbers 1-36 on a set of index cards, one set per group.
2. Play the game with 2, 3, 4, or 5 players.
3. The dealer shuffles the deck of number cards and deals the pack equally, facing down.
4. The time is set for 1 – 2 minutes
5. Each player starts at the same time and goes through as many cards as they can, marking off the factors of their number cards on their Factor Bingo card.
6. The winner is the player who crosses off the most factors by the end of the 10 minutes.
7. To break a tie, players may re-shuffle number cards.

Variation: To continue reinforcing multiplication facts, students may play again for speed by decreasing the amount of minutes allowed for play.

Factor Bingo Cards

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

36	35	34	33	32	31
30	29	28	27	26	25
24	23	22	21	20	19
18	17	16	15	14	13
12	11	10	9	8	7
6	5	4	3	2	1

19	11	20	7	23	1
12	21	8	24	2	28
22	9	25	3	29	13
10	26	4	30	14	33
27	5	31	15	34	17
6	32	16	35	18	36

1	22	7	27	11	30
17	2	23	8	28	12
33	18	3	24	9	29
14	34	19	4	25	10
31	15	35	20	5	26
13	32	16	36	21	6

1	7	13	19	25	31
2	8	14	20	26	32
3	9	15	21	27	33
4	10	16	22	28	34
5	11	17	23	29	35
6	12	18	24	30	36

21	26	30	33	35	36
15	20	25	29	32	34
10	14	19	24	28	31
6	9	13	18	23	27
3	5	8	12	17	22
1	2	4	7	11	16

31	25	19	13	7	0
32	26	20	14	8	2
33	27	21	15	9	3
34	28	22	16	10	4
35	29	23	17	11	5
36	30	24	18	12	6

16	15	13	14	17	18
10	9	7	8	11	12
4	3	1	2	5	6
22	21	19	20	23	24
28	27	25	26	29	30
34	33	31	32	35	36

Product Bingo

Objective: This game is designed to reinforce factors, multiples and prime & composite numbers; it will also help students review multiplication facts.

Materials:
One Product Bingo card per student
2 sets of number cars with the numbers 1-10 written on them
Timer/stop watch
Color pencils

Activity Instructions:

1. Make number cards. Write numbers 1-10 on a set of index cards, two sets per group.
2. Play the game with 2, 3, 4, or 5 players.
3. The dealer shuffles the deck of number cards and deals the pack equally, facing down.
4. Decide who goes 1^{st}, 2^{nd}, 3^{rd}, etc.
5. Each player selects a card from his/her hand and places it face up on the table for all to see. Each card will be the 1^{st} factor for each individual.
6. Player 1 gets to select one card from his/her hand to be the 2^{nd} factor. Players multiply their own individual factor to the common factor and make their product which they can then mark off if it is on their Bingo Cards. (Note that players are allowed to see each others Bingo Cards in order to help strategize what factors to select).
7. Play progresses to the next player, repeating steps 5-7 and re-shuffling and re-dealing as needed.
8. The winner is the player who crosses out four numbers connected in a row vertically, horizontally, or diagonally.

Product Bingo Cards

2	22	53	14	5	60
27	8	9	10	11	12
33	44	55	66	77	88
59	20	42	52	83	54
25	26	27	28	29	30
30	32	33	34	95	100

6	22	24	34	44	56
76	12	88	84	25	90
98	84	14	5	68	28
100	42	82	16	20	56
12	11	10	9	8	7
32	8	15	33	27	20

2	22	53	14	5	60
27	8	9	10	11	12
33	44	55	66	77	88
59	20	42	52	83	54
25	26	27	28	29	30
30	32	33	34	95	100

16	20	24	21	28	36
20	25	30	35	40	45
24	30	36	42	48	54
28	35	42	49	56	63
32	40	48	56	64	72
36	45	54	63	72	81

15	45	35	65	75	85
12	20	18	16	42	22
16	28	42	49	70	63
56	64	8	40	32	48
54	81	36	27	9	27
36	60	54	30	50	100

16	20	24	21	28	36
20	25	30	35	40	45
24	30	36	42	48	54
28	35	42	49	56	63
32	40	48	56	64	72
36	45	54	63	72	81

4	6	8	10	12	14
6	9	22	15	18	21
8	32	16	20	24	21
10	15	20	25	30	42
14	21	28	35	63	81
50	30	40	60	80	100

10	12	14	16	28	36
15	18	21	28	40	30
24	35	42	49	56	63
27	9	24	60	100	20
8	7	50	55	63	81
90	32	16	20	70	72

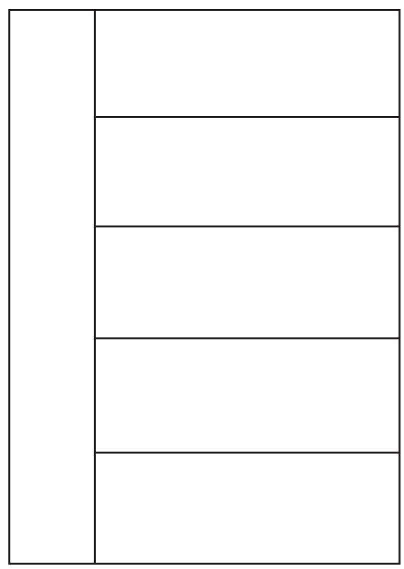
Divisibility Rules Booklet

Objective: Students will explore and use the divisibility rules for 2,3,5,9, and 10. (Rules for 4 and 6 are optional)

Materials:
a sheet of colored paper per student
scissors

Activity Instructions:

1. Have students fold a sheet of colored paper in half widthwise. Place the folded side on the right and the open side on the left. (See example below)



2. On the left side, have students write "Divisibility Rules." On the right side, create a window for each of the following numbers 2,3,5,9, and 10. Be sure students cut only the top layer of the top fold when creating the number windows. Write one number in front of each window.
3. Guide students through each divisibility rule from below. They will copy the rule on the inside left part of the opened window. On the right side, they will write examples of the rule's use and a non-example.
4. Continue with each window until all rules are addressed.

Divisibility Rules:
A number is divisible by 2 if the number is even.
A number is divisible by 3 if the sum of its digits is divisible by 3.
A number is divisible by 5 if the number ends in 5 or 0.
A number is divisible by 9 if the sum of its digits is divisible by 9.
A number is divisible by 10 if the number ends in 0.

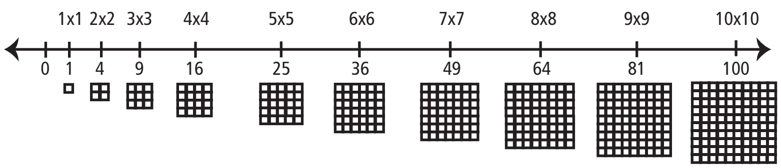
Square Number Model

Objective: Understand square numbers from 0 to 100 and use the number line to organize and draw square numbers from 0 to 100

Materials:
Grid paper
Rulers
Colors

Activity Instructions:
Each student has a sheet of grid paper, a ruler, and colors. Teacher guides the students in constructing a number line depicting square numbers from 0 to 100 by following these steps:

1. Use ruler to draw a number line with numbers from 0 to 100; enlarge the following numbers: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100; teacher explains that these enlarged numbers have a special name and definition.
2. Under each enlarged number, students draw the corresponding rectangle ("square") for it; for example: for #1 draw a 1by 1 square; for #4 draw 2by2; #9 draw a 3by3… etc. At the end, the teacher will remind students that since for each enlarged number there exists a pair of factors which forms a square, these numbers are called square numbers. Teacher may also review that a square number has an odd number of factors because the repeated factor is a pair but only counts as one factor.



   Students may color each square drawn and label the factors/dimensions for each square drawn under each square number. Later this square number model may be used to write the exponents corresponding each square number; for example: 2x2=4 or 2^2
3. Remind students that they need to keep this square number model to be used later with the section on exponents.

Sieve of Eratosthenes (answer key)

to be used with Exploration 2

~~1~~	**2**	**3**	~~4~~	**5**	~~6~~	**7**	~~8~~	~~9~~	~~10~~
11	~~12~~	**13**	~~14~~	~~15~~	~~16~~	**17**	~~18~~	**19**	~~20~~
~~21~~	~~22~~	**23**	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	**29**	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	**37**	~~38~~	~~39~~	~~40~~
41	~~42~~	**43**	~~44~~	~~45~~	~~46~~	**47**	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	**53**	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	**59**	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	**67**	~~68~~	~~69~~	~~70~~
71	~~72~~	**73**	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	**79**	~~80~~
~~81~~	~~82~~	**83**	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	**89**	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	**97**	~~98~~	~~99~~	~~100~~

*Prime numbers are bolded; composite numbers are crossed out. 1, which is neither prime nor composite is also crossed out.

Sieve of Eratosthenes (student copy)

to be used with Exploration 2

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Possible Rectangles KEY

n	Number of Possible Rectangles	Possible Rectangle Dimensions *	Possible Side Lengths (in increasing order)
1	1	1 × 1	1
2	1	1 × 2	1, 2
3	1	1 × 3	1, 3
4	2	1 × 4, 2 × 2	1, 2, 4
5	1	1 × 5	1, 5
6	2	1 × 6, 2 × 3	1, 2, 3, 6
7	1	1 × 7	1, 7
8	2	1 × 8, 2 × 4	1, 2, 4, 8
9	2	1 × 9, 3 × 3	1, 3, 9
10	2	1 × 10, 2 × 5	1, 2, 5, 10
11	1	1 × 11	1, 11
12	3	1 × 12, 2 × 6, 3 × 4	1, 2, 3, 4, 6, 12
13	1	1 × 13	1, 13
14	2	1 × 14, 2 × 7	1, 2, 7, 14
15	2	1 × 15, 3 × 5	1, 3, 5, 15
16	3	1 × 16, 2 × 8, 4 × 4	1, 2, 4, 8, 16
17	1	1 × 17	1, 17
18	3	1 × 18, 2 × 9, 3 × 6	1, 2, 3, 6, 9, 18
19	1	1 × 19	1, 19
20	3	1 × 20, 2 × 10, 4 × 5	1, 2, 4, 5, 10, 20
21	2	1 × 21, 3 × 7	1, 3, 7, 21
22	2	1 × 22, 2 × 11	1, 2, 11, 22
23	1	1 × 23	1, 23
24	4	1×24, 2×12, 3×8, 4×6	1, 2, 3, 4, 6, 8, 12, 24
25	2	1 × 25, 5 × 5	1, 5, 25
26	2	1 × 26, 2 × 13	1, 2, 13, 26
27	2	1 × 27, 3 × 9	1, 3, 9, 27
28	3	1 × 28, 2 × 14, 4 × 7	1, 2, 4, 7, 14, 28
29	1	1 × 29	1, 29
30	4	1×30, 2×15, 3×10, 5×6	1, 2, 3, 5, 6, 10, 15, 30

*In this exercise we view a rectangle a x b and b x a as the same rectangle.

Possible Rectangles Key (pg 2)

n	Number of Possible Rectangles	Possible Rectangle Dimensions	Possible Side Lengths (in increasing order)
31	1	1 x 31	1, 31
32	3	1 x 32, 2 x 16, 4 x 8	1, 2, 4, 8, 16, 32
33	2	1 x 33, 3 x 11	1, 3, 11, 33
34	2	1 x 34, 2 x 17	1, 2, 17, 34
35	2	1 x 35, 5 x 7	1, 5, 7, 35
36	5	1 x 36, 2 x 18, 3 x 12, 4 x 9, 6 x 6	1, 2, 3, 4, 6, 9, 12, 18, 36
37	1	1 x 37	1, 37
38	2	1 x 38, 2 x 19	1, 2, 19, 38
39	2	1 x 39, 3 x 13	1, 3, 13, 39
40	4	1 x 40, 2 x 20, 4 x 10, 5 x 8	1, 2, 4, 5, 8, 10, 20, 40
41	1	1 x 41	1, 41
42	4	1 x 42, 2 x 21, 3 x 14, 6 x 7	1, 2, 3, 6, 7, 14, 21, 42
43	1	1 x 43	1, 43
44	3	1 x 44, 2 x 22, 4 x 11,	1, 2, 4, 11, 22, 44
45	3	1 x 45, 3 x 15, 5 x 9	1, 3, 5, 9, 15, 45
46	2	1 x 46, 2 x 23	1, 2, 23, 46
47	1	1 x 47	1, 47
48	5	1 x 48, 2 x 24, 3 x 16, 4 x 12, 6 x 8	1, 2, 3, 4, 6, 8, 12, 16, 24, 48
49	2	1 x 49, 7 x 7	1, 7, 9
50	3	1 x 50, 2 x 25, 5 x 10	1, 2, 5, 10, 25, 50

Possible Rectangles Chart

n	Number of Possible Rectangles	Possible Rectangle Dimensions	Possible Side Lengths (in increasing order)
1			
2			
3			
4	2	1×4, 2×2	1, 2, 4
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			

Possible Rectangles Chart (pg 2)

n	Number of Possible Rectangles	Possible Rectangle Dimensions	Possible Side Lengths (in increasing order)
31			
32			
33			
34			
35			
36			
37			
38			
39			
40			
41			
42			
43			
44			
45			
46			
47			
48			
49			
50			

Shikaku- Rectangle

Objective: The goal of a Shikaku puzzle is to generate rectangles whose area is the same as the member encompassed in each rectangle.

Materials:
Copy of puzzle sheet. (to be found on CD)

Activity Instructions:

1. Each of the following Shikaku puzzles needs to be sectioned into rectangles (and squares) along the grid lines, so that the number in each rectangle refers to the area of that rectangle.

2. Only one number can appear in each rectangle.

3. Each square on the gird is used in exactly one rectangle (that is, no rectangles may overlap.

You can act this out on a number line, which connects to the linear model developed in Section 2.3. You can also use 3 groups of 4 marbles with 2 left over.

Teacher Tip: There are a lot of new words and phrases introduced in this section, make a chart to clarify the relationships between factor, multiple and divisible.

FACTORS AND MULTIPLES 3

SECTION 3.1 FACTORS, MULTIPLES, PRIMES, AND COMPOSITES

One of the most important concepts in mathematics is the idea of *divisibility*. Suppose you have 14 marbles, and you want to give the same number of marbles to each of three friends. Is it possible to give each friend the same number and have none left?

You can let each person have 4 marbles, but there are two left. This process is called division. You divide 14 by 3 to get the quotient 4 with remainder 2. This is equivalent to the calculation $14 = 3 \cdot 4 + 2$. In building the multiplication table for 3, your skip counting by 3, starting at 0, does not list 14.

On the other hand, if you have exactly 12 marbles, you can give each friend 4 marbles, and everybody has an equal number of marbles. This process corresponds to $12 = 3 \cdot 4$.

What does this have to do with divisibility? We know that 14 objects cannot be divided equally among 3 people. Another way to say this is

$$\begin{array}{r} 4 \\ 3\overline{)\,14} \\ \underline{-12} \\ 2 \end{array}$$

- 14 is not *divisible* by 3.
- 3 is not a *factor* of 14.
- 14 is not a *multiple* of 3.

On the other hand, we can divide 12 things equally among 3 people. Mathematically,

$$\begin{array}{r} 4 \\ 3\overline{)\,12} \\ \underline{-12} \\ 0 \end{array}$$

- *12* is *divisible* by 3.
- *3* is a *factor* of 12.
- *12* is a *multiple* of 3.

Although there is no integer that you can multiply by 3 to equal 14, there is the integer 4 that you can multiply by 3 to equal 12.

As a leadin to Exploration 1, review the idea of a rectangle. Point out that we are picking *n* to be the area and are trying to find the length and width that will give us this area of *n* square units. You may allow counting rectangles as unique n x m and m x n but for Exploration 1 we count only once a rectangle using the same factors of n and m..

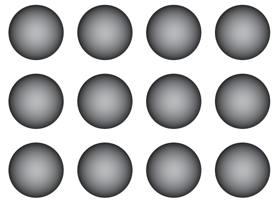
EXPLORATION 1:

Materials: teachers, make copies of the Possible Rectangle Chart found at the beginning of this section

1. In small groups, you can have different groups do different values of *n*. Students can draw rectangles on graph paper and post on the wall as the number n increases. They can also use tiles to physically make the rectangles in groups. Have your students share their results in small groups. After an initial period, have a class discussion about several possible values for *n*, like 6, 7, 8 and 9. Then continue either individually or in small groups through 20. For each *n*, give students who have discovered rectangles record their results on a bulletin grid paper in a prominent place. Record on Class Chart as groups report. Reflect with them what the different columns of the chart are telling us.

2. It might be good to point out that the values of the sides are factors of *n* and that they come in pairs. You might dramatize this by connecting the pairs with an arch. Make copies of the black line master of the table for students from the Teachers Supplement.
3. We are foreshadowing the idea of composites and primes. The definition of these are in the next section. Don't spoil students' attempts at articulating what the properties of these numbers are. They might notice (1) some *n* have only 1 possible rectangle because they are primes or (2) a few have an odd number of possible side lengths because they are perfect squares. If your students don't notice any patterns yet, the following steps will lead them to discover the patterns. If your class has already discovered the factors, go to Step 6.

For example, we know that 12 is divisible by 3 because $12 = 3 \cdot 4$. We can show this using our marble example:



Factor · Factor = Product

3 · 4 = 12

Notice that the twelve marbles can be arranged in a rectangular array. This suggests another way for us to see the divisors of a positive integer.

DEFINITION 3.1: DIVISIBILITY
Suppose that n and d are integers, and that d is not 0. The number n is **divisible** by d if there is an integer q such that $n = d \cdot q$. Equivalently, d is a **factor** of n, and n is a **multiple** of d.

We'll experiment with this in the following activity.

EXPLORATION 1: THE POSSIBLE RECTANGLE MODEL

Materials: You will need graph paper for this activity and the Possible Rectangles Chart.

1. For each positive integer n from 1 to 30, make as many rectangles with integer side lengths as you can that have area equal to n square units. Count a rectangle only once if the factors are the same. For example, $3 \cdot 4$ and $4 \cdot 3$ will be counted only once.

2. Organize the data in a table provided by your teacher. In the first column, write the positive integer n. In the second column, write the number of rectangles possible with area n. In the third column, list all the possible dimensions of the rectangles. In the fourth column, list all the possible lengths of sides of the rectangles, in increasing order. For example, we have filled in the results for the value of $n = 4$ on the table.
3. What do you notice in the table so far? **Answers will vary.**

Notice that some rectangle lists contain one square among the possible rectangles. Hence the numbers such as 16 are called square numbers.

6. Each is a list of all the factors of n.

7. Factors or divisors. Have students draw lines above the list in the last column connecting factor pairs. This forms a factor rainbow. The only exception is for perfect squares, where there is a factor in the middle with no pair, such as 1, 2, 4, 8, 16 for $n = 16$. This foreshadows the question in step 10, where the answer is perfect squares with the odd factor in the middle of the list.

8. This might be a good time to talk about how important and unique the number 1 is in multiplication. That is why mathematicians call it the multiplicative identity. Multiplying any number, n, by 1 gives you the starting number n. In the next exploration, your students will find that 1 is also the only number that is neither prime nor composite.

9. 3, 5, 7, 11, 13, 17, and 19 all generate only one rectangle. They are called prime numbers or primes. These numbers each have exactly 2 positive factors, the number itself and 1.

10. They are the perfect squares. You might ask your students why the number is odd. This could lead to the discussion of multiple factors. We say that 2 is a factor of 4 with multiplicity 2.

It is probably worth repeating that the number 1 is neither prime nor composite several times in the lesson.

4. Continue the table for n from 31 to 50.
5. Looking at the extended table, do the patterns continue?
6. Looking at a given number n, what do you notice about the numbers in the last column for this value of n?
7. What do we call the numbers in the last column in relation to n? For each rectangle, the dimensions form a factor pair, such as 3 and 6 for $n = 18$. If you put all the factors in the last column in order, such as 1, 2, 3, 4, 6, 12 for $n = 12$, how do the factor pairs line up?
8. What do you notice about the number 1? Find any other numbers that have this same property, if possible.
9. Circle the values of n that generate only one rectangle.
   How many factors does each of these have?
   How would you describe the circled numbers, excluding 1?
10. Use a different color pen or marker to box the values of n that have an odd number of positive divisors.

Notice that all of the factors in our chart are positive. Generally, when talking about *factors*, we just mean the positive factors.

The numbers that have only two positive factors play a special role in mathematics and have a special name.

DEFINITION 3.2: PRIME AND COMPOSITE
A **prime** number is an integer p greater than 1 with exactly two positive factors: 1 and p. A **composite** number is an integer greater than 1 that has more than two positive factors. The number 1 is the **multiplicative identity**; that is, for any number n, $n \cdot 1 = n$. The number 1 is neither a prime nor a composite number.

Your work with the Possible Rectangle Table (PR Table) allows you to see the relationships between a given number, n, the number of rectangles possible with n as its area, and the number of factors n has. In particular, you can see from

Allow students to work on Problem 1 and then discuss as a whole class different approaches students used. This is exploratory and in Example 1 we present one systematic approach.

your PR Table that the prime numbers between 1 and 40, in increasing order, are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37. These are the only integers greater than 1 and less than or equal to 40 that had exactly one rectangle possible and hence, exactly two factors. Let's consider a larger number and determine whether it is prime or not.

PROBLEM 1

Is 119 prime or composite?

Here is another approach that is not as geometric, but it is systematic.

EXAMPLE 1

Is the number 171 prime?

SOLUTION

To see if there are any other factors of 171 between 1 and 171, begin to divide 171 by numbers less than 171, in increasing order beginning with 2.

Divide by	Quotient	Remainder	Factor?
2	85	1	no
3	57	0	yes

Dividing 171 by 3 gives quotient 57 and remainder 0, showing that 3 is a factor of 171. This is enough information to conclude that 171 is composite because 171 has not just 1 and itself as factors, but it also has 3 as a factor.

The number 171 does not appear on the Sieve of Eratosthenes chart. Instead of using the possible rectangle model, we approached the problem by finding factors. Why was that smart? We could have shortened our search even more. Because 171 is odd, we know that 2 is not a factor of 171.

EXPLORATION 2:

a. composite b. prime c. composite d. composite e. prime

f. composite g. composite h. prime i. composite

Provide students opportunities for practice with determining factors of a number through division. This provides a good review of division, multiplication, and the use of the division algorithm to see how the quotient and the remainder are related. To be a factor the remainder must be zero.

Note the different ways a question can be asked. It is very important to make students comfortable with the language of mathematics.

PROBLEM 2

Is the number 127 prime? Explain why or why not.

EXPLORATION 2

In small groups or individually, determine whether the following numbers are prime or composite. Try to devise as many time-saving strategies as you can, so you don't have to check every integer between 1 and the target number.

a. 51	b. 67	c. 81
d. 99	e. 113	f. 123
g. 171	h. 131	i. 323

Let's explore some strategies for finding all the factors or divisors of a given positive integer n. In particular, if n is a positive integer and k is a positive integer, how do we determine whether k is a factor of n; or equivalently, whether n is a multiple of k? The method used in the Possible Rectangle Activity works well for small numbers but does not work as well for larger numbers. So, we want to find a method that can be used when we have to deal with large numbers.

EXAMPLE 2

Is 9 a factor of the number 112? Equivalently, is 112 a multiple of 9?

SOLUTION

Starting with the second question, we could skip count by 9 to determine whether 112 is a multiple of 9. However, this is not an efficient strategy. Instead, ask if there is a positive integer q such that $112 = 9 \cdot q$. You can answer this using long division, which results in a quotient of 12 and a remainder of 4. This means that $112 = 9 \cdot 12 + 4$. The goal is to find an integer q so that $112 = 9q$. If you skip counted by 9, you would not land on 112. That is, there is no integer q for which $112 = 9q$. Therefore, 9 is not a factor of 112.

EXAMPLE 3

Teacher will ask students to use a T-chart of factor pairs to solve this probem. Teacher will clarify the meaning and use of a T-chart. some students have seen this as a strategy since 5th grade.

144	
1	144
2	72
3	48
4	36
6	24
8	18
9	16
12	12

From the t-chart, the students can see that 8(18) = 144 and therefore 18 is a factor of 144.

PROBLEM 3: The remainder is not 0 in the non-divisible case and the remainder is 0 in the divisible case. We usually say that 15 divides "evenly into 105" rather than "into 105 with remainder 0."

Notice that $15 = 3 \cdot 5$. Every number that is divisible by 15 is also divisible by 3 and 5. One way for students to see this is numerically. If a number $n = 15 \cdot d$, then $n = 3 \cdot 5 \cdot d$. So $n = 3 \cdot (5d)$ or $n = 5 \cdot (3d)$ and n is divisible by 3 and 5. Another way is geometrically.

EXPLORATION 3: SIEVE OF ERATOSTHENES

A sieve is a meshed appliance that filters and separates particles usually by size, letting small particles pass through the mesh while capturing larger particles. The Sieve of Eratosthenes is not an appliance, but is a process that separates numbers, in our case removing certain numbers and keeping others. This is a way to separate and keep the prime numbers while removing the composite numbers. The students can save their work to use as a reference for later sections.

1. You can get this from the front of this section in the Teacher's Edition.
2. Do not refer to prime and composite yet because we have not defined them. Just say that we cross out the number "1" because it is special (the multiplicative identity).
3. Be sure to have your students use a different color for each step. For example, use red for all the multiples of 2, then blue for 3 and so on. Have students make a key of color to factor. Ask your students to look for visual patterns or short cuts as they go through this activity. At this point in the activity, half of the numbers should be marked out from the columns headed by the even numbers.
4. The first number is 3 because 3 is the second prime. Again, as always with mathematics, ask your students to be alert for patterns. Be sure the multiples of 3 are crossed out with a different color. The students should notice that some numbers are crossed out multiple times by different factors.

6. The circled numbers 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97 are called primes. Students may notice that each is not a multiple of any natural number that comes before it in the original list except for 1. Ask your students to notice any pattern that involved these same numbers in Possible Rectangle Activity. They were the numbers that had only one rectangle. Make sure your students know that these are the prime numbers. We will give a careful definition in Section 5.2.

EXAMPLE 3

Is 18 a factor of 144? Equivalently, is 144 a multiple of 18? Use a T-chart of factor pairs to solve this problem.

PROBLEM 3

a. Is the number 105 divisible by 15?

b. Is every number divisible by 15 also divisible by 3 and 5? Explain.

What distinction in the remainders did you notice between the examples of the divisible case and the not divisible case? Check with a few other examples to confirm that the distinction holds in those cases.

EXPLORATION 3: SIEVE OF ERATOSTHENES

This Exploration is based on an ancient method attributed to a famous Greek mathematician, Eratosthenes of Cyrene. The process involves letting a certain kind of number pass through the sieve leaving only another kind of number left in the sieve. Try the Exploration, and see for yourself.

1. Use the grid of the first 100 natural numbers in the rows of ten handout.
2. Mark out the number 1. We will see why in the next section.
3. Using a colored pencil or marker, circle the number 2, and then mark out every remaining multiple of 2 until you have gone through the whole list. What is a mathematical term for the marked out numbers?
4. From the beginning, with a different colored pencil or marker, circle the first number that is not marked out and not circled. Then, mark out all remaining multiples of that number. Notice that some numbers are crossed out by two different colored pencils.
5. Repeat this process until you have gone all the way through the list.
6. Make a new ordered list of all the circled numbers. What do these numbers have in common? How is this list of numbers related to patterns from the possible rectangle activity?

7. For students who have a difficult time understanding what the question is asking, a teacher might ask, “are these numbers also a multiple of another number?”
   6, 12, 18, …. These numbers are also multiples of 2, hence why they were already marked out. Since they are multiples of 2 and multiples of 3, they are multiples of 6.

The last question is really the key to determining whether a number n is prime or composite; one need check only if $p|n$, prime numbers, p, with $n \leq p^2$, to determine if n is or is not prime.

EXPLORATION 4:

1. Divisibility by 2: all numbers ending in 0, 2, 4, 6, 8 are divisible by 2.
2. Divisibility by 5: all numbers ending in 0 and 5 are divisible by 5.
3. Divisibility by 10: all numbers ending in 0 are divisible by 10.
4. Divisibility by 3: If the sum of the digits is divisible by three, then the number is divisible by 3.
5. Divisibility by 9: If the sum of the digits is divisible by nine, then the number is divisible by 9.
6. Divisibility by 6: If the number is divisible by 2 and divisible by 3, then the number is also divisible by 6.

Note that the different color pencils used on the Sieve of Eratosthenes will help bring out the pattern.

7. You might have noticed that in the third round, some of the multiples of 3 were already crossed out in the second round. Find 3 such numbers. Why did this happen?

You might have noticed that your Investigation involving the Sieve of Eratosthenes allowed you to find the prime numbers less than 100 quickly. Can you explain how the process worked? At what point is it possible to know that all the numbers left are prime?

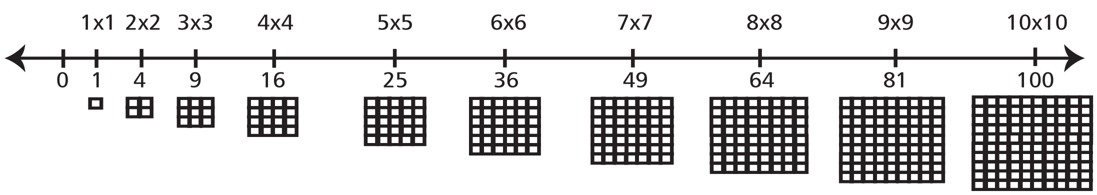
EXPLORATION 4: DIVISIBILITY RULES

Use the Sieve of Eratosthenes to explore the following:

1. What pattern do you notice about numbers that are multiples of 2? Make a conjecture for a rule to determine whether or not a number is divisible by 2.
2. What pattern do you notice about numbers that are multiples of 5? Make a conjecture for a rule to determine whether or not a number is divisible by 5.
3. What pattern do you notice about numbers that are multiples of 10? Make a conjecture for a rule to determine whether or not a number is divisible by 10.
4. What pattern do you notice about numbers that are multiples of 3? Make a conjecture for a rule to determine whether or not a number is divisible by 3.
5. What pattern do you notice about numbers that are multiples of 9? Make a conjecture for a rule to determine whether or not a number is divisible by 9.
6. What pattern do you notice about numbers that are multiples of 6? Make a conjecture for a rule to determine whether or not a number is divisible by 6.

EXPLORATION 5

1. No, 36 has an odd number of factors ({1, 2, 3, 4, 6, 9, 12, 18, 36})
2. 4, 9, 16, 36, 49, 64, 81, 100
3.



EXERCISES

1. a. 1, 2, 3, 6, 9, 18
   b. 1, 2, 3, 4, 6, 8, 12, 24
   c. 1, 2, 3, 4, 6, 8, 12, 16, 24, 48
   d. 1, 2, 4, 7, 8, 14, 28, 56
   e. 1, 2, 3, 6,
   f. 1, 2, 4, 8, 16, 32
   g. 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90
   h. 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120
   i. 1, 2, 3, 5, 6, 10, 15, 30
   j. 1, 17
   k. 1, 3, 17, 51
   l. 1, 3, 5, 9, 15, 45
   m. 1, 7, 49
   n. 1, 71
   p. 1, 43

2. Composites: 18, 24, 48, 56, 6, 32, 90, 120, 30, 51, 45, 49
   Primes: 17, 71, 43

Composite numbers have more than two factors while prime numbers have exactly two factors.

We summarize the divisibility rules that can be useful for determining whether any given number is divisible by numbers 2, 3, 5, 6, 9, 10.

If this is true about a number:	Then the number is divisible by:
One's digit is even,	2
The sum of the digits is divisible by 3,	3
The last digit is 0 or 5,	5
The number is divisible by 2 and 3,	6
The sum of the digits is divisible by 9,	9
The number ends in 0,	10

EXPLORATION 5: SQUARE NUMBERS

1. Is it possible to pair up all the positive factors of 36? Explain.
2. Find all the other numbers between 1 and 100 with an odd number of factors. What might these numbers be called?
3. Draw a number line, and locate the square numbers below the number with the associated square shapes.

EXERCISES

1. Find all the factors of the following numbers. You may wish to use a t-chart or table.

a. 18	b. 24	c. 48	d. 56	e. 6
f. 32	g. 90	h. 120	i. 30	j. 17
k. 51	l. 45	m. 49	n. 71	p. 43

2. Classify the numbers from Exercise 1 as prime or composite. Then, write a sentence explaining how you know numbers are prime or composite.
3. In each of the following problems, values of n and d are given. Determine whether d is a factor of n. If d is a factor of the product n, find an integer q such that $n = dq$. Explain how you use any patterns or prior knowledge that help you answer the questions. Use long division only on the starred items.

	d (factor)	n	Is d a factor of n? Explain.
a.	5	26	**no**
*b.	4	52	**yes, 13**
c.	1	38	**yes, 38**
*d.	8	77	**no**
e.	35	0	**yes, 0**
f.	4	81	**no**
g.	13	195	**yes, 15**
h.	6	78	**yes, 13**
*i.	14	159	**no**
j.	9	135	**yes, 15**

4. a. 101, 103, 107, 109
   b. 113, 127, 131, 137, 139, 149
   c. 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199

5. a. 3, 6, 9, 12, 15, 18, 21, 24, 27, 30
   b. 4, 8, 12, 16, 20, 24, 28, 32, 36, 40
   c. 7, 14, 21, 28, 35, 42, 49, 56, 63, 70
   d. 8, 16 ,24, 32, 40, 48, 56, 64, 72, 80
   e. 9, 18, 27, 36, 45, 54, 63, 72, 81, 90
   f. 11, 22, 33, 44, 55, 66, 77, 88, 99, 110
   g. 12, 24, 36, 48, 60, 72, 84, 96, 108, 120
   h. 13, 26, 39, 52, 65, 78, 91, 104, 117, 130
   i. 15, 30, 45, 60, 75, 90, 105, 120, 135, 150
   j. 24, 48, 72, 96, 120, 144, 168, 192, 216, 240
   k. 25, 50, 75, 100, 125, 150, 175, 200, 225, 250

6. a. 48
   b. 56

7. a. 144
   b. 135

	d (factor)	n	Is d a factor of n? Explain.
a.	5	26	
*b.	4	52	
c.	1	38	
*d.	8	77	
e.	35	0	
f.	4	81	
g.	13	195	
h.	6	78	
*i.	14	159	
j.	9	135	

4. Find all the prime numbers between the given pairs. Verify that your choice is a prime. Example: Find the prime numbers between 70 and 80. List: ~~70~~, 71, ~~72~~, 73, ~~74~~, ~~75~~, ~~76~~, ~~77~~, ~~78~~, 79, ~~80~~. So, only the numbers not crossed are prime numbers.

a. 100 and 110 b. 110 and 130 c. 130 and 150

5. Find the first 10 multiples of the following numbers:

a. 3 b. 4 c. 7 d. 8 e. 9

f. 11 g. 12 h. 13 i. 15 j. 24

k. 25

6. a. What is the least multiple of 6 that is greater than 43?
   b. What is the least multiple of 7 that is greater than 50?

7. a. What is the least multiple of 12 that is greater than 132?
   b. What is the least multiple of 15 that is greater than 120?

8. a. Yes, because 85 is a multiple of 5 (85 = 15(5) or 0.85 = 17(0.05)), so she can pay with 17 nickels.
   b. No, since 10 is not a factor of 85, dimes cannot be used to get $0.85 evenly.

9.

Length of each ribbon	1	2	3	4	5	6	10	12	15	20	30	60
Number of ribbon pieces	60	30	20	15	12	10	6	5	4	3	2	1

10.

Columns	1	2	3	4	5	6	8
Number of Names in each column	120	60	40	30	24	20	15

11. a. The list of factors of 30 is 1, 2, 3, 5, 6, 10,15, 30.
    b. The pairs are (1,30), (2,15), (3,10) and (5,6). Ask your students if they discovered a technique they can use so that they do not skip any factor. If they haven't, lead them to the technique. Have students show how they found these pairs, such as a T-chart or connecting the pairs with an arching line above the list to create a rainbow effect. 1 connects to 30, 2 connects to 15, 3 connects to 10, and 5 connects to 6.

12. The idea is to use factor pairs. For each of the factors given in the list, find the other member of its pair. The pairs are (1, 140), (2, 70), (4, 35), (7, 20), (10, 14). Since 11, 12, and 13 are not factors and 14 is, this means we have all of the factors because any factor greater than 14 would be in a pair with a factor less than 14. So if the list of factors we started with is all of the factors from 1 to 10, then we now have them all.

14. **Spiral Review** (6.1C)
    13ºC

8. a. Sara has a large number of nickels in her purse. Can she pay for a pack of gum that costs \$0.85? If so, how many nickels will she need? Explain your answer in terms of either factors or multiples. (In other words, write a sentence explaining your answer using the words factor or multiple.)

   b. If Sara only had dimes in her purse, could she pay for the same pack of gum with exact change? If so, how many dimes will she need? Explain your answer in terms of either factors or multiples.

9. Ms. Ellis wants to make a wall decoration from the 60 feet of colorful ribbon she bought. She wants to use all the ribbon and wants each piece she cuts to be of equal length. List all the possible number of ribbons she can cut using only whole number and the length of each.

10. The local newspaper will be printing the names of students who made the honor roll for the entire year. They will split the names on the list into equal columns. There are 120 students on the list, and the newspaper wants to have fewer than 10 columns. How many different columns can they use, and how many names will be in each column?

11. a. Using your data from the Possible Rectangle Exploration, write all the factors of 30 in order from least to greatest.

    b. There is a natural way to pair up the positive factors of 30. What do you notice from the rectangle model that can help you pair up the factors? What factor pairs did you get?

12. The numbers 1, 2, 4, 5, 7, and 10 are six factors of 140 in numerical order. How can you use this information to find larger factors of 140? Find all those factors.

13. Numbers like 536 and 712 are divisible by 4, while 378 is not. Make a conjecture for a rule to determine whether a number is divisible by 4.

Spiral Review:

14. At 6 A.M. the temperature was -2° C. At noon, the temperature had risen to 11 °C. By how many degrees did the temperature increase?

15. **Spiral Review** (6.12 A)
    B

16. **Ingenuity**
    a. **Yes .**Students can do this by dividing the number 60 by each of the numbers 2, 3, 4, 5, and 6, and seeing that the remainder is 0 in each case.
    b. **Yes .**We know that 60 is divisible by 3, so 60 + 3 is divisible by 3. We can repeat this reasoning for 60 + 4, 60 + 5, and 60 + 6. The same reasoning holds if we know that N is divisible by 2, 3, 4, 5, and 6
    c. If we can find a positive integer N that is divisible by 2, 3, 4, 5, 6, 7, 8, 9, and 10, then we know that the nine integers N + 2, N + 3, N + 4, ..., N + 10 will all be composite. To get an integer that is divisible by all the numbers from 2 to 10, we can multiply 2 x 3 x 4 x...x 10 to get 3628800. Another integer that is divisible by all of the whole numbers from 2 to 10 is 8 x 9 x 5 x 7 = 2520. If we use the number 2520, then we find that 2522, 2523, 2524, ..., 2530 are all composite. (Note that there are many other sets of nine consecutive composite numbers; the important thing is to see the student's reasoning.)
    d. If we could find a positive integer N that is divisible by all of the positive integers from 2 to 1001, then the integers from N + 2 through N + 1001 will all be composite. One possible value of N is 1001!, the product of all the positive numbers from 2 to 1001. (This number is around 2500 digits long - don't expect students to compute it without a very sophisticated calculator!)

Investigation

17. a. Yes; Anne could write 2 x 2 x 4 x 5. She could also write 2 x 2 x 2 x 2 x 5.

15. Four friends attended a concert and agreed to share the cost evenly. The total of the tickets was \$120, the rental car was \$65 and snacks and drinks were \$50. Which expression can be used to represent the amount, C, each friend should have paid?

    a. C = (120 + 65 + 50)(4)

    b. C = (120 + 65 + 50) ÷ 4

    c. C = 120 + 65 + 50 + 4

    d. C = 120 + 65 + 50 - 4

16. **Ingenuity:**

    In this problem, we will discover how to find long sequences of consecutive composite numbers.

    a. Verify that the number 60 is divisible by 2, 3, 4, 5, and 6
    b. Explain how we know that the numbers 60 + 2, 60 + 3, 60 + 4, 60 + 5, and 60 + 6 are all composite. More generally, if N is a positive integer that is divisible by 2, 3, 4, 5, and 6, explain how we know that N + 2, N + 3, N + 4, N + 5, and N + 6 are all composite.
    c. Based on your work in part (b), you now have a sequence of five consecutive composite numbers: 62, 63, 64, 65, and 66. Now, find a sequence of nine consecutive composite numbers.
    d. Suppose you had a powerful calculator that could compute products as large as you wanted to compute. Explain how you would find a sequence of one thousand consecutive composite numbers.

17. **Investigation:**

    Anne and Sam are playing a number game. One of them writes down a number, and the two then write the number as a product of integers greater than 1. The winner is the player who writes the number as the product of the most integers. For example, if the number is 60, and Sam writes the number as 5 x 12, while Anne writes the number as 3 x 2 x 10, then Anne is the winner, since she used three integers and Sam used only two.

    a. Suppose the number is 80, and Sam writes 80 = 4 x 4 x 5. Is there a product that Anne could write down that would beat Sam's product?

b. There is no product that Sam could write down that would beat this product.
c. One "best possible" product for 84 is 84 = 2 x 2 x 3 x 7. There are other ways to write this product, such as 84 = 3 x 2 x 7 x 2. However, any product that contains four factors (all of which are greater than 1) will contain these four factors: two 2's, a 3, and a 7.
d. One "best possible" product is 600 = 2 x 2 x 2 x 3 x 5 x 5.

Challenges:

18. 193

b. Suppose the number is 125, and Anne writes 125 = 5 x 5 x 5. Is there a product that Sam could write down that would beat Anne's product?

c. Write down a product, using as many factors as possible, for the number 84. Is there more than one product that has the largest possible number of factors? If so, do these products have different factors?

d. Write down a product, using as many factors as possible, for the number 600.

18. **Challenge:**

Find the smallest prime number p such that $x^2 - y^2 + 1 = p$ where x and y are both multiples of 8 and x is larger than y.

Section 3.2 - Exponents and Order of Operations

Big Idea:

Order of operations and exponents

Key Objectives:

- Discover patterns for simplifying operations with exponents.
- Use of Order of Operations

Materials:

Graphing calculators, Colored paper

Pedagogical/Orchestration:

- Students need to work problems on paper first, then check work with calculator.

Activities:

"PEMDAS" and "Order of Operations"

Explorations 1 and 2 offer opportunities for group work and good class discussion.

Exercises:

The exercise problems are very computational. Using a calculator may be stressed.

Foreshadowing square numbers and square roots with numbers squared.

Internet Resources:

Rags to riches review of order of operations: http://www.quia.com/rr/116044.html

Vocabulary:

exponential notation, power, base, exponent, order of operations

TEKS:

6.2(E); 6.3(E); New: 6.7(A)

Launch for Section 3.2

Ask the students, "What would you do if you had this decision to make? Your uncle is asking you what you would prefer for your birthday gifts and gives you the choice of $50 every year on your birthday, or start with $1.00 for this birthday and double the amount every year thereafter." Allow students some time to discuss their preference, but do not give an answer or explanation at this time. After students discuss their ideas tell them, "Pay attention to the examples we do in class today and see if any of them will help you decide what the best option would be. Our lesson is about exponents so see if you can relate today's lesson to the birthday present decision." After the lesson, remember to relate back to this launch and ask students if they have made a decision on which present they would choose. It is interesting to note that if the student chose the $50 option, on the 21st year he would have received $50 for a twenty-one year total of $1050, whereas on the doubling dollar option, on the 21st year alone, the student would receive over one million dollars. Hopefully by the end of today's lesson the student will have begun to appreciate the power of the exponent.

PEMDAS

Objective: Students will be able to solve number expressions that may include the four basic operations (+, –, •, ÷), parentheses, and/or exponents.

Materials:
PEMDAS Chart
Index cards

Activity Instructions:

1. Teacher will have students work in groups to create number expressions with numbers 0-9 written in index cards, including each of the basic operations (+, –, •, ÷), parentheses, and exponents 2 and 3.
2. Each group will solve number expressions created by each student. As each expression is solved, students in the groups check each other's solutions.
3. Students will refer back to the PEMDAS (Please Excuse My Dear Aunt Sally) chart. Students need to remember that multiplication and division are at equal level. Addition and subtraction are also in the same level. The order is decided by which occurs first from left to right.
4. After you have done parentheses and exponents, work from left to right and multiply or divide whichever comes first. Do the same if there is an addition and a subtraction.

PEMDAS Chart

P	Parentheses first
E	Exponents (ie Powers and Square Roots, etc.)
MD	Multiplication and Division (left-to-right)
AS	Addition and Subtraction (left-to-right)

Order of Operations

Activity Instructions: Students will create a PEMDAS flip chart in the following manner:

1. Take a single sheet of card stock or other similar paper.
2. Fold the page in half lengthwise. The top will be the folded side and the bottom is the open side.
3. Measure a 1 inch section at the top (near the fold) to create the longest rectangle. This will be the title section.
4. Divide the remaining section into 4 windows. Remember to cut only the top layer.

Order of Operations **$18 + 7^2 \times (8 - 2) \div 3 + 8$** (This will be written in the title section)

Window 1	Window 2	Window 3	Window 4

Write the following information on the cover and inside each window in order from left to right.

Window:	**Outside:**	**Inside:**
1	P Simplify (solve) grouping symbols like Parenthesis	$18 + 7^2 \times (8 - 2) \div 3 + 8$ $18 + 7^2 \times 6 \div 3 + 8$
2	E Find the value of powers (exponents)	$18 + 7^2 \times (8 - 2) \div 3 + 8$ $18 + 7^2 \times 6 \div 3 + 8$ $18 + 7 \times 7 \times 6 \div 3 + 8$ $18 + 49 \times 6 \div 3 + 8$
3	MD Multiply and/or divide in order from left to right	$18 + 7^2 \times (8 - 2) \div 3 + 8$ $18 + 7^2 \times 6 \div 3 + 8$ $18 + 7 \times 7 \times 6 \div 3 + 8$ $18 + 49 \times 6 \div 3 + 8$ $18 + 294 \div 3 + 8$ $18 + 98 + 8$
4	AS Add and/or subtract in order from left to right	$18 + 7^2 \times (8 - 2) \div 3 + 8$ $18 + 7^2 \times 6 \div 3 + 8$ $18 + 7 \times 7 \times 6 \div 3 + 8$ $18 + 49 \times 6 \div 3 + 8$ $18 + 294 \div 3 + 8$ $18 + 98 + 8$ $116 + 8$ 124

1. **1024**
2. **between the 13th and 14th hour**

   This is a perfect time to use the graphing calculator as an exploratory tool. First, on the home screen, enter a 2. Mentally count that as 1. Then press x 2 and enter. The answer should be 4. Mentally count that as 2. Continue to press enter as you count. For instance, the next time you press enter, your should count 3 and see 8. What you are seeing is the powers of 2 you are counting. Continue to press enter, counting and looking for a number that is greater than 10,000. The 13th time you press enter, you should see 8192 on the screen. The 14th time you press enter, you should see 16384, which is larger than 10,000. You can show them how to use the "^" in computing powers of 2, such as 2^5 = 32 and 2 ^ 7 = 128.

SECTION 3.2 EXPONENTS AND ORDER OF OPERATIONS

In Section 2.1, we modeled multiplication by repeatedly adding an integer to itself. There are also situations in which it is useful to multiply a number repeatedly by itself.

EXAMPLE 1

Escherichia coli bacteria are more commonly known as E. coli. A scientist places one of the living bacteria in a petri dish. The number of bacteria in the dish doubles each hour. How many bacteria are living in the dish after 1 hour? 2 hours? 3 hours? 5 hours? n hours?

SOLUTION

Because the number of bacteria doubles each hour, after 1 hour there will be 2 bacteria. After 2 hours, there will be $2 \cdot 2 = 4$ bacteria. We can write this information using **exponential notation** as

Number of bacteria after the 1st hour $= 2 = 2^1$
Number of bacteria after the 2nd hour $= 2 \cdot 2 = 2^2 = 4$
Number of bacteria after the 3rd hour $= 2 \cdot 2 \cdot 2 = 2^3 = 8$

Continuing this pattern, the number of bacteria living in the petri dish after 5 hours is equal to $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$. If we let n = number of hours, there are 2^n bacteria after n hours.

Use a calculator to answer the following questions about Example 1:

1. How many bacteria will there be after 10 hours? **1024**
2. We need at least 10,000 bacteria for an experiment. When can we harvest this many bacteria?

DEFINITION 3.3: EXPONENTS AND POWERS
Suppose that n is a whole number. Then, for any number x, the n^{th} **power** of x, or x to the n^{th} power, is the product of n factors of the number x. This number is usually written $\mathbf{x^n}$. The number x is usually called the **base** of the expression x^n, and n is called the **exponent**.

EXAMPLE 2

Rewrite the following repeated multiplication in exponential notation. Identify the base and exponent for each expression.

a. $2 \cdot 2 \cdot 2 \cdot 2$ b. $5 \cdot 5 \cdot 5$ c. $7 \cdot 7$ d. $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

e. $n \cdot n \cdot n \cdot n$

Solution:

a. 2^4 The base is 2, exponent 4
b. 5^3 The base is 5, exponent 3
c. 7^2 The base is 7, exponent 2
d. 10^5 The base is 10, exponent 5
e. n^4 The base is n, exponent 4

PROBLEM 1

Rewrite the following expressions using repeated multiplication. Find the values for parts a through f.

a. 4^3
b. 3^4
c. $2^3 \cdot 3^2$
d. $5^2 \cdot 7^2$
e. m^3
f. $p^2 \cdot q^5$

Exploration 1 and **2** might be a good time to use the power of the calculator. Even though it is tempting to express numbers as powers only, it is a good exercise in the power of exponents to see how quickly the numbers grows, within reason. Have students first work out the calculations in both Explorations and then check with a calculator.

Exploration 1

The students may observe a pattern for $x^n \cdot x^m = x^{(n+m)}$. The objective here is not to get to this rule of exponents, but you may find the students will naturally discover this rule.

a. $3^2 \cdot 3^3$ **3^5 or 243** c. $3^3 \cdot 3^2$ **3^5 or 243**

b. $2^2 \cdot 2^3$ **2^5 or 32** d. $10^3 \cdot 10^5$ **10^8 or 100,000,000**

What pattern do you observe when multiplying numbers in exponential form with the same base? Explain.**Answers will vary, but we hope yes.**

PROBLEM 2

$2^5 \cdot 2^4 =$ **2^9**

PROBLEM 3

a. $4^8 = 65{,}536$ c. $10^7 = 10{,}000{,}000$

b. $2^{10} = 1{,}024$ d. $3^2 \cdot 4^3 = 576$

We would encourage the students to note that $x \cdot 1 = x$ and $x = x$, but the 1 is used in different ways.

EXPLORATION 1

By using the definition of exponential notation and multiplication, we see that:

$$3^4 \cdot 3^6 = (3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^{10} = 3^{4+6}.$$

Compute the following products, showing all your work.

a. $3^2 \cdot 3^3$ **3^5 or 243**

b. $2^2 \cdot 2^3$ **2^5 or 32**

c. $3^3 \cdot 3^2$ **3^5 or 243**

d. $10^3 \cdot 10^5$ **10^8 or**

What pattern do you observe when multiplying numbers in exponential form with the same base? Explain.**Answers will vary, but we hope yes.**

PROBLEM 2

Compute the product: $2^5 \cdot 2^4$. **2^9**

The pattern leads to the multiplication property for exponents.

PROPERTY 3.1: MULTIPLICATION OF POWERS
Suppose that x is a number and a and b are whole numbers. Then, $$x^a \cdot x^b = x^{a+b}$$

Use your observation to determine the product in the problem below.

PROBLEM 3

Use these properties to rewrite the following products and then compute them. Use a calculator to compute both equivalent forms.

a. $4^3 \cdot 4^5$

b. $2^5 \cdot 2^5$

c. $10^3 \cdot 10^4$

d. $3^2 \cdot 4^3$

Special Cases: What do 4^1 and 4^0 equal?

We note that $4 \cdot 4 = 4^2 = 4^{1+1} = 4^1 \cdot 4^1$, so 4^1 must be the same as 4. We can use the same process for any number x: $x \cdot x = x^2 = x^{1+1} = x^1 \cdot x^1$, so $x^1 = x$.

Special Cases:

You may also have student look for patterns of powers

$2^5 = 32$, $2^4 = 16$, $2^3 = 8$, $2^2 = 4$. Have students conjecture 2^1 and 2^0 by the pattern observed.

$4^4 = 256$ (divide by 4)
$4^3 = 64$ (divide by 4)
$4^2 = 16$ (divide by 4)
$4^1 = 4$ (divide by 4)
$4^0 = 1$.

Include pattern justification for 4 raised to the 0. Ask students how they would reach conclusion that 4 to the zero power is 1. Then ask if this works for every number? You might want to talk about the problem with 0^0, depending on your class. 0^0 is undefined. There are a couple of ways to convince students of this. First, take any number n to the 0th power. Then $n^0 = n^{x-x} = n^x/n^x$. But no one can divide by 0, so n cannot be 0. Or ask what your students think 0^0 is. There are two logical answers: $0^0 = 1$ because any number to the 0^{th} power = 1 OR $0^0 = 0$ because 0 to any power is 0. No number can have two values, so 0^0 is undefined.

EXPLORATION 2

Use the problem in Exploration 2 in the following way.

1. Pose the problem to students to solve using prior knowledge.
2. List their answer choices
3. Ask "Why are there several answers to the same problem?"
4. Lead students to the conclusion that order of operations is important.
5. Explain that the conventional process for order of operations will be examined in the following activity and it will lead to the correct answer to this problem. Then guide students through the Order of Operations Activity found at the beginning of this section in the Teacher's Edition.

You may wish to use simpler computations that highlight the difference in the obtained values depending on the order of operations. For example, consider the sequence: $3 + 4 \times 5$, $3 \times 4 + 5$, 3×2^2, $(3 \times 2)^2$ and discuss multiplication before addition, exponents before multiplication, parenthesis before exponents and other rules in the order of operations.

We are using PEMDAS to compute $20 - 10 \div 2 + 3^3 - 9 = 20 - 10 \div 2 + 27 - 9 = 20 - 5 + 27 - 9 = 33$
What we are looking for here is the idea of order of operations. That is, the calculator does not simply multiply or add from left to right, as we read.

Parentheses can be used when necessary to help us keep the order of operations straight in our minds. For example, $(7 \cdot 8) - (6 \div 2)$ is the same as $7 \cdot 8 - 6 \div 2$.

What does 4^0 equal? Because $4 \cdot 4^0 = 4^1 \cdot 4^0 = 4^{1+0} = 4^1 = 4 = 4 \cdot 1$, we see that multiplying by 4^0 is the same as multiplying by the number 1. We, therefore, assume that for any positive integer n, $n^0 = 1$.

Consider the number 4638, which we read as four thousand six hundred thirty eight. Using place value and our notation of exponents, we can rewrite 4638 using expanded notation in the following way:

$$4 \cdot 1000 + 6 \cdot 100 + 3 \cdot 10 + 8 \cdot 1 = 4 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$$

Start with the expression $4 \cdot 10^3 + 6 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0$, or in calculator notation, $4 \times 10^3 + 6 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$. In what order can we perform the calculations in this expression so the sum equals 4638?

EXPLORATION 2

Compute the following, showing all your work.

$$20 - 10 \div 2 + 3^3 - 9$$

ORDER OF OPERATIONS

To compute 2^4 on a calculator, we enter 2^4. To multiply 3 by 5, we enter 3×5. How would we enter the following into a calculator:

a. $1 + 2^3$
b. $3 + 2 \cdot 5$
c. $2 \cdot 4^3$
d. $5 - 2 + 4$

What will the results be? Can you explain what the calculator is doing? You might wonder why the calculator does not always perform these calculations from left to right, as we read them. We can see that the order the calculator uses, which is called the **order of operations**, is natural by examining our place value system.

For example, to compute $7 \cdot 8 - 6 \div 2$, the rule tells us that we perform any multiplication and division before any addition or subtraction, from left to right. So in the above example, we compute $56 - 3$ or 53 as our answer; the parentheses $(7 \cdot 8) - (6 \div 2)$ are understood. However, if there were any parentheses in the problem, we would compute them first. For example, for $7 \cdot (8 - 6) \div 2$ the order of operations is $7 \cdot 2 \div 2$ or $14 \div 2$ or 7. The operation in the parentheses is performed before any of the other operations.

PROBLEM 4

$4 + 23 \cdot 3 - (17 - 5) \cdot 2 + (17 - 5) \div 2$

$= 4 + 8 \cdot 3 - (12) \cdot 2 + (12) \div 2$

$= 4 + 24 - 24 + 6$

$= 4 + 6$

$= 10$

EXERCISES

1. a. 2^5 d. $2^4 3^3$
   b. 3^4 e. n^5
   c. $4^3 5^2$ f. $a^4 b^3$

2. a. $4^3 = 64$, $3^4 = 81$, $3^4 > 4^3$ c. $5^3 = 125$, $10^2 = 100$, $5^3 > 10^2$
   b. $6^3 = 216$, $3^6 = 729$, $3^6 > 6^3$ d. $4^2 = 16$, $2^4 = 16$, $2^4 = 4^2$

3. a. 12
   b. 17
   c. 27
   d. 72

We summarize below the order in which mathematical operations are performed:

Order of Operations

- Compute the numbers inside any **p**arentheses.
- Compute any **e**xponential expressions.
- **M**ultiply or **d**ivide as they occur from left to right.
- **A**dd or **s**ubtract as they occur from left to right.

Why do these two problems have different solutions?

a. $7 \cdot 8 - 6 \div 2$ b. $7 \cdot (8 - 6) \div 2$

EXAMPLE 3

Generate equivalent numerical expressions to simplify the following expressions:

a. $3 - 5 + 4$
b. $23 \div 4 \cdot 2$
c. $(7 - 2 + 3) + 2 \cdot 32$

SOLUTION

a. This problem only has addition and subtraction, so we do the operations from left to right, first subtracting, then adding.
$3 - 5 + 4 = (3 - 5) + 4 = -2 + 4 = 2$

b. First we simplify the exponent.
$2^3 \div 4 \cdot 2 = 8 \div 4 \cdot 2$
Then we do the multiplication and divisions, from left to right.
Multiplication and divisions are done left to right in order, just like addition and subtraction are done left to right in order.
$8 \div 4 \cdot 2 = (8 \div 4) \cdot 2 = 2 \cdot 2 = 4$

c. Step 1: Simplify inside the parenthesis.
We first subtract, then add, from left to right
$(7 - 2 + 3) + 2 \cdot 3^2 = (5 + 3) + 2 \cdot 3^2 = 8 + 2 \cdot 3^2$
Step 2: Next we simplify the exponent, $3^2 = 9$, so the above becomes: $=8+2 \cdot 9$
Step 3: We then multiply, since we always multiply or divide before adding or subtracting: $= 8 + 18$.
Step 4: Add or Subtract, left to right: $= 26$

4. a. 19
   b. 7

5. Go over the first step in the process to make sure the students understand the tripling process resulting from the cutting and the stacking.
   243 units

6. a. 10
   b. 100
   c. 1,000
   d. 100,000,000
   e. Write a 1 followed by 200 zeros

7. a. 719
   b. 10,006

8. Notice that each student must eat a number of Skittles that is a power of 4: the first student must eat 4^0, the second 4^1, the third 4^2, and so on. The sixth must eat $4^5 = 1024$

9. a. 0
   b. 27

10.
   a. $(5)(5) = 25$
   b. $(8)(8)(8) = 512$
   c. $(7)(7)(7) = 343$
   d. $(6)(6) = 36$
   e. $(15)(15)(15) = 3375$
   f. $(12)(12)(12) = 1728$

PROBLEM 4

Compute the following showing all of your work:

$$4 + 2^3 \cdot 3 - (17 - 5) \cdot 2 + (17 - 5) \div 2$$

EXERCISES

1. Rewrite each of the following multiplication expressions into expressions using exponents.

   a. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

   b. $3 \cdot 3 \cdot 3 \cdot 3$

   c. $4 \cdot 4 \cdot 4 \cdot 5 \cdot 5$

   d. $(2 \cdot 2 \cdot 2 \cdot 2) + (3 \cdot 3 \cdot 3)$

   e. $n \cdot n \cdot n \cdot n \cdot n$

   f. $a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b$

2. Expand and compute the answer of the following. Tell which expression is greater or if they are equal.

   a. 4^3 or 3^4

   b. 3^6 or 6^3

   c. 5^3 or 10^2

   d. 4^2 or 2^4

3. Evaluate the following expressions:

   a. $3^2 + 3$ **10** b. $2^3 + 3^2$ **25** c. $3^2 \cdot 3$ d. $2^3 \cdot 3^2$ **16**

4. Evaluate the following numerical expressions using Order of Operations:

   a. $22 - 3^2 + 6$ **14** b. $22 - (3^2 + 6)$ **3**

5. Rhonda has one sheet of paper. She cuts it into thirds and stacks the three sheets. If she completes this process a total of 5 times, how many sheets thick will the resulting stack be? By the way, she only has to complete the process 27 times before the stack reaches the moon. **243 units**

11. Spiral Review (6.1C)
    Answer: C

12. Spiral Review (7.7A)
    Answer: (3,-1)

13. even, divisible by 2, 3, 6

6. Calculate the following:
   a. 10^1 **10**
   c. 10^3 **1,000**
   b. 10^2 **100**
   d. 10^8**100,000,000**
   e. Explain how you would calculate 10^{200}.**rite 1 followed by 200 zeros.**

7. Evaluate the following numerical expressions using Order of Operations:
   a. $7 \times 10^2 - 53 + (8 \times 3^2)$ **8096**
   b. $7016 + (3 \times 10^2) \times 10 - 10^2 \div 10$ **12024**

8. Six students are having a Skittles eating contest. The first student eats 1 Skittle. Each student after that must eat 4 times as many Skittles as the previous student. How many Skittles does the sixth student eat?**See TE.**

9. Evaluate the following numerical expressions using Order of Operations:
   a. $4 \times 2 + 48 \div 6 - 2^2 \times 4$ **0**
   b. $5^3 - 100 + 4(7 - 4) \div 6$ **170**

10. Compute the following:
    a. $(2+3)^2$
    d. $(2 \cdot 3)^2$
    b. $(3+5)^3$
    e. $(3 \cdot 5)^3$
    c. $(3+4)^3$
    f. $(3 \cdot 4)^3$

Spiral Review:

11. Rocky, the squirrel, is busy storing pecans for the winter. Sadie, the golden retriever, is busy finding Rocky's hidden pecans. The table below shows the number of pecans stored by Rocky and the number of pecans eaten by Sadie over the last 4 days. Which expression best describes the number of pecans still stored for the winter after the 4th day?

Day	Number Buried	Number Eaten
Monday	22	5
Tuesday	24	8
Wednesday	21	11
Thursday	15	4

    a. -22 + 5 - 24 +8 - 21 + 11 - 15 + 4

14. **Ingenuity**

It is a good idea to put parentheses around the expression 5 + 3, so that we will multiply the sum 8 by 7 rather than just multiplying 3 by 7. For the same reason, we will put parentheses around the expression 7 + 6, so we multiply the sum 5 + 3 = 8 by 7 + 6 rather than just 7. So far we have

$$(5 + 3) \times (7 + 6) = 8 \times 13 = 104.$$

We could include the "minus 4" in the parentheses with the 7 + 6 and have (5 + 3) x (7 + 6 – 4), but then we would be forced to subtract 8 x 4 from our expression rather than just subtracting 4. Therefore, we will leave the 4 out of the parentheses. We also will choose not to put the expression 2 + 6 in parentheses; if we do, we will be forced to subtract 4 x (2 + 6) = 32 rather than just subtracting 4 x 2. So the best possible expression is

$$(5 + 3) \times (7 + 6) - 4 \times 2 + 6 = 102.$$

Note that we can put parentheses around the 4 x 2, but it is not necessary to do so because the order of operations prioritizes multiplication over addition.

15. **Investigation**:

This Investigation foreshadows the Rule of Product, which is introduced in Chapter 10. One mistake that students frequently make with this type of problem is to add the numbers given in the problem (in this case, the threes) rather than taking the product. This is typically the result of the student not having a clear picture of all the possibilities that can occur when Donny makes several choices in succession. Therefore, the tree diagram introduced below is a powerful tool to help students understand this idea.

a. Let's suppose the colors of Donny's shirts are red, white, and blue, and the colors of his pants are blue, grey, and tan. Then the outfits Donny can make are

Red shirt, blue pants

Red shirt, grey pants

Red shirt, tan pants

White shirt, blue pants

White shirt, grey pants

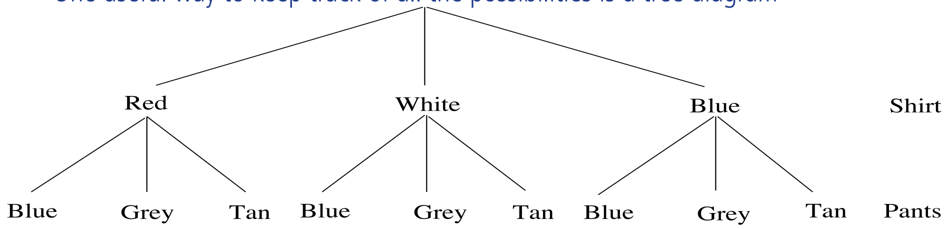
White shirt, tan pants

Blue shirt, blue pants

Blue shirt, grey pants

Blue shirt, tan pants

One useful way to keep track of all the possibilities is a tree diagram

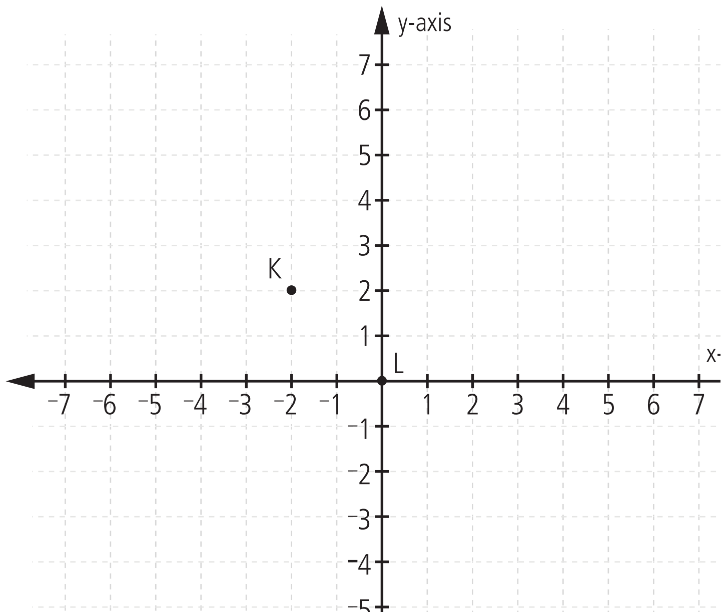


b. 22 + 5 + 24 + 8 + 21 + 11 + 15 + 4

c. 22 + 24 + 21 + 15 - 5 - 8 - 11 - 4

d. 22 + 24 + 21 + 15 - 5 + 8 + 11 + 4

12. If point K is translated 5 units to the right and 3 down, what will point K’s new coordinates be?



13. What do the numbers 30, 12, 90, 60, and 24 have in common? List all common traits.

14. Ingenuity:

Insert parentheses in the following expression so that the value of the expression is as large as possible.

5 + 3 x 7 + 6 - 4 x 2 + 6

For example, one way to insert parentheses would be

(5 + 3) x 7 + 6 -4 x (2 + 6) = 30.

15. Investigation

Suppose that Donny has three shirts and three pairs of pants that he can wear to school. Donny has poor fashion sense and is willing to wear any shirt with any pair of pants, even if the colors do not go well together.

We see that there are 9 possible outfits.

b. There are 9 possible ways for Donny to choose a shirt and pants. For each of these configurations, there are 3 caps that Donny can choose. Therefore, there are 9 x 3 = 27 possible outfits if Donny chooses a shirt, a pair of pants and a cap. If we use our tree diagram to see this, we see that each of the 9 branches at the "Pants" stage will split into three more branches at the "Cap" stage, for a total of 9 x 3 = 27 branches.

c. There are 27 ways for Donny to choose a shirt, pants, and a cap. For eacho f these possibilities, there are 3 jackets Donny can choose, so the number of ways to choose a shirt, pants, a cap, and a jacket is 27 x 3 = 81.

d. We find that if Donny has to make n successive choices in order to make his outfit, and he has 3 options each time, then the number of possible outfits he can make is 3^n. More generally, if Donny has to make n choices in succession, and he has k options each time, then the total number of possible outcomes is k^n.

a. Come up with three colors for Donny's shirts and three colors for his pairs of pants (use your imagination). Make a list or chart showing all of the combinations of a shirt and a pair of pants that Donny can wear to school. How many possible combinations are there?

b. Suppose that, in addition to having three shirts and three pairs of pants, Donny also has three baseball caps that he can wear to school. How many ways are there for Donny to choose a shirt, a pair of pants, and a baseball cap to wear to school?

c. Suppose that Donny also has three jackets that he can wear. How many ways are there for Donny to choose a shirt, a pair of pants, a baseball cap, and a jacket to wear?

d. What pattern do you notice in (a) through (c)? Write a sentence or two explaining your findings.

Section 3.3 - Unique Prime Factorization

Big Idea:

Developing the understanding that any integer greater than 1 is either prime or can be written as a unique product of prime numbers. (Unique Prime Factorization)

Key Objectives:

- Use a factor tree to factor an integer.
- Use exponents to shorten prime factorizations.
- Understand that multiplication is commutative.
- Understand that the number 1 is neither prime nor composite.

Materials:

Chart from Sieve of Eratosthenes, Calculator

Pedagogical/Orchestration:

- Introduce the lesson with the Factor Chain activity. While the Fundamental Theorem of Arithmetic may seem complicated to middle school students, it provides the reason why prime factorizations are written in order from least to greatest.
- This section relies on students' number sense and can help develop their appreciation and facility for it. There are some students and teachers who remain confused about the fact that 1 is not a prime number. The Fundamental Theorem of Arithmetic can demonstrate the reason 1 is not a prime. If 1 were a prime, there would never be a unique prime factorization, because you could have any number of factors of 1 and the Fundamental Theorem of Arithmetic would not hold.

Activity:

"Doughnut Races"; Matching game to review prime factorization: http://www.quia.com/cm/26221.html

Exercises:

Exercises 2 and 3 build new vocabulary words: perfect squares and cubes. Exercises 4 and 5 can be challenging for students with weak number sense ability or poor recall of multiplication facts.

Vocabulary:

unique prime factorization, prime factorization, factor tree, perfect square, perfect cube, factor tree

TEKS:

6.1(D); New: 6.7(A); 5.4(A)

Launch for Section 3.3:

Factor Chains

In groups, instruct students to represent the number 360 as the longest chain of factors they can find without using 1 as a factor. Give students about 5 minutes to come up with a response.

Have a member from each group come to the board to report the longest "chain" they found for the number.

Once all groups have reported, conduct a class discussion to observe patterns found. Some groups will have only factor pairs. Others will have longer "factor chains." Discuss what those groups did to expand the "chain."

Example: $360 = 10 \cdot 36$ (group 1)

$360 = 2 \cdot 5 \cdot 36$ (group 2)

$360 = 2 \cdot 5 \cdot 4 \cdot 9$ (group 3)

In many cases, groups will have the same factors but in different orders. This is an example of multiplication being commutative.

Focus on the longest "factor chain" and ask students to explain how it is related to shorter chains.

Ask students to explain why the original directions stated that you cannot use a 1.

Doughnut Races

Objective: Students will practice and review multiplication, prime /composites, factors, multiples, prime factorization, fractions, decimals, percents

Materials:
Doughnut Circles (for whole group game); or Doughnut worksheet (for individual or small group game)
Pencils if using student copy (for individual or small group play)
Chalk or Dry erase markers (if using board for whole group play)
Two Dice

Activity Instructions:
Teacher calls out skill in random order:

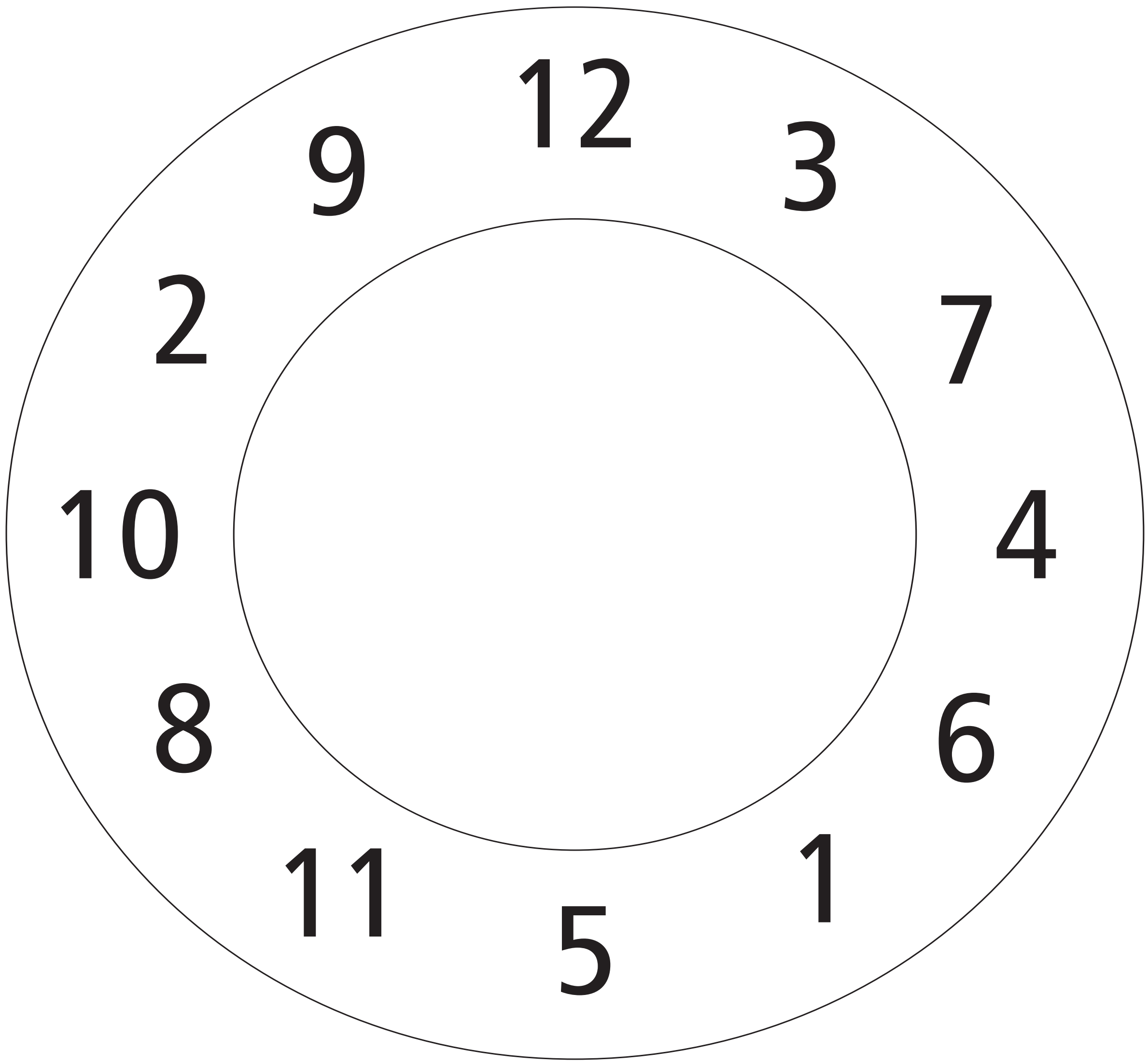
- *Multiplication* (teacher rolls the dice, calls out the sum of the two dice and students multiply by each number around the "doughnut" as fast as they can. First player to correctly complete their doughnut wins the round.
- *Prime/composites* (students write P or C around the donut)
- *Factors* (students list the factors of each number)
- *Multiples* (students list the first 4 multiples of each number)
- *Prime factorization* (students write the prime factors of each number and use exponential notation if needed.)
- *Fractions* (teacher rolls a die and calls out the number. The number rolled is the denominator and the numbers on the "doughnut" are the numerators. Students are to write each fraction and simplify them if needed. (reduce & convert)
- *Fractions decimals* (repeat the above except convert fractions to decimals and/or to percents.)
- *Fractions, decimals, percents* (on the backside of the donut use the decimal forms to convert to fractions and/or to percents.)

Students work out the skill and the first person to complete their "doughnut" correctly earns a point. Team to earn 10 points wins the game.

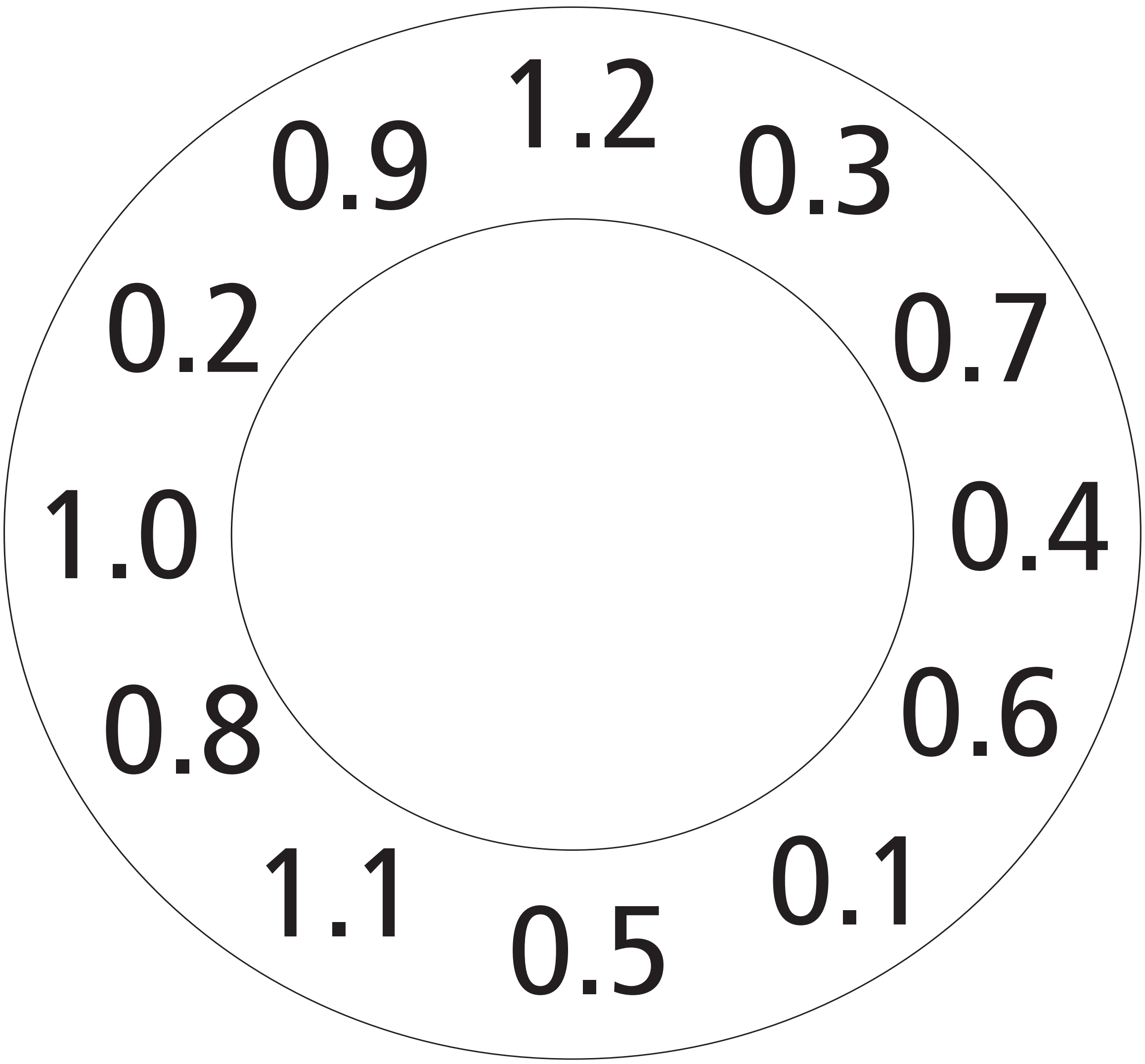
Variation 1:
Students can play in 2 teams, using the giant doughnuts (made out of poster board, then laminated and taped on the chalkboard) Two players compete at a time and earn points for their team then rotate players. Winning team gets to eat doughnuts, other team gets doughnut holes. We play this game on the last Friday of the 6 weeks and 2 students volunteer to bring doughnuts and d-holes for the prize.

Variation 2:
Use the multiple doughnuts worksheet to play many rounds at a time individually or in small groups. This allows for more participation and success.

Doughnut Races



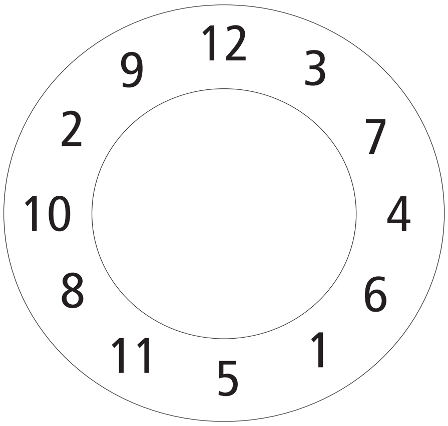
Decimal Doughnut Races



Name____Date____________Period__________

Doughnut Races - Student Copy

1.



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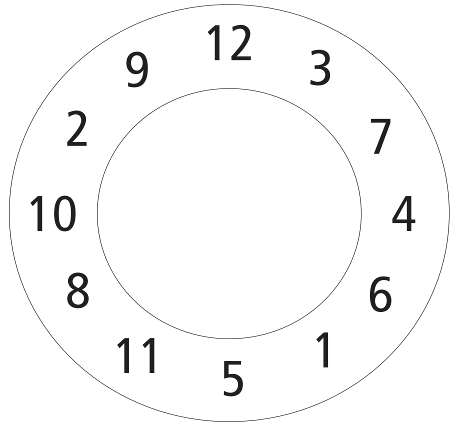

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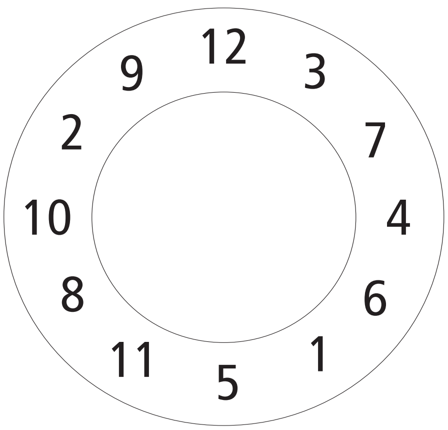

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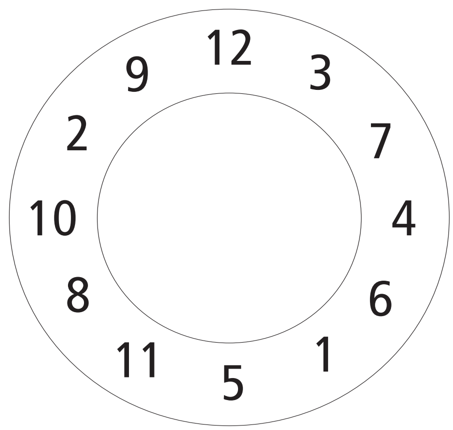

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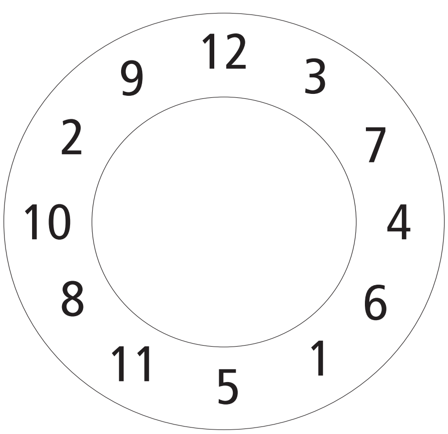

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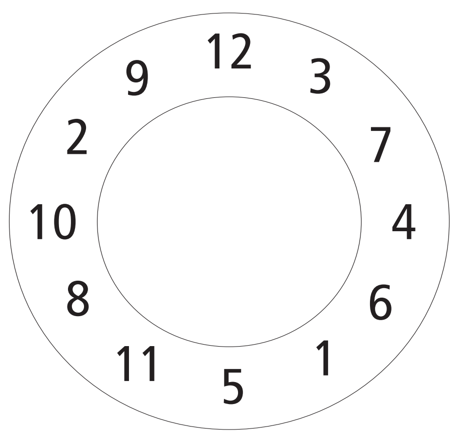

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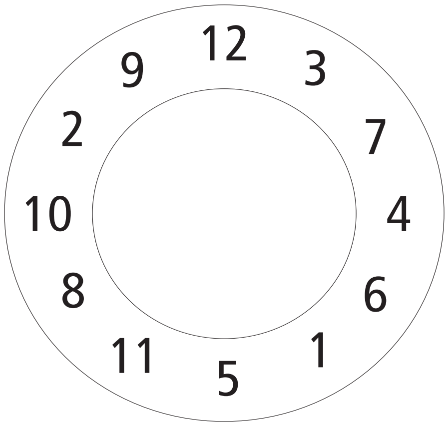

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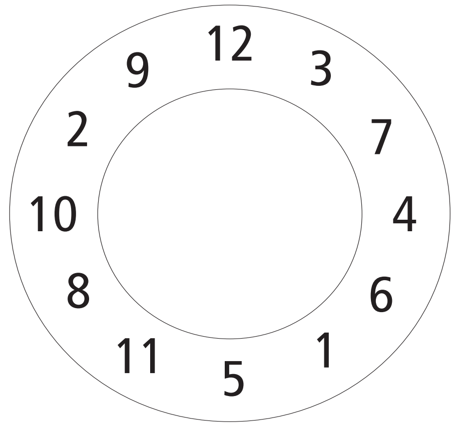

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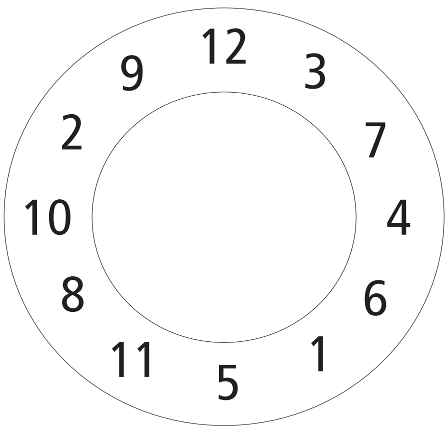

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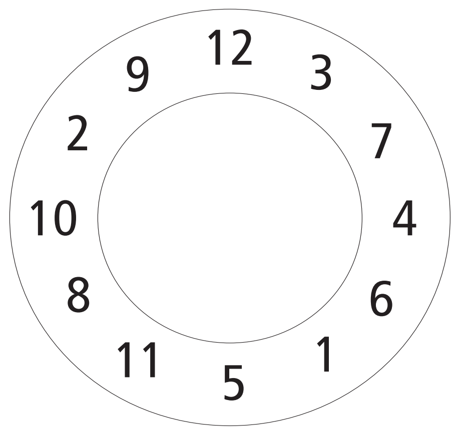

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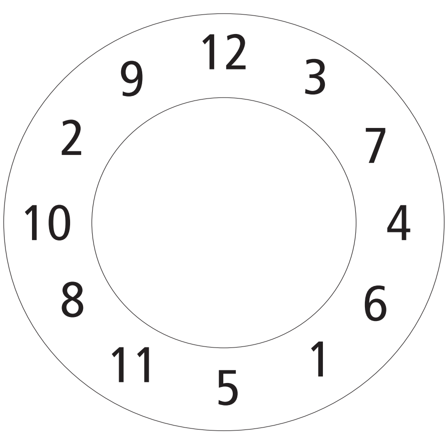

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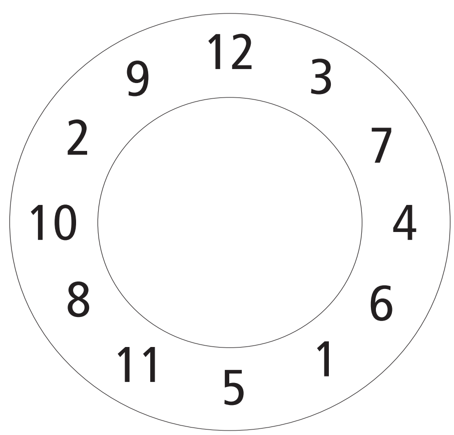


SECTION 3.3 UNIQUE PRIME FACTORIZATION

One reason we are so interested in prime numbers is that they are the building blocks of the integers. In the previous section, we learned that a prime number is a positive integer greater than 1 that can be written as a product of two positive integers in only one way ignoring the order of the factors. For example,

$$13 = 13 \cdot 1 = 1 \cdot 13.$$

We cannot write 13 as a product of two positive integers without using the number 13 itself. In this way, the number 13 cannot be divided into smaller equal whole parts. We can, however, use the number 13, together with other prime numbers, to form many other numbers:

$$\begin{aligned} 13 \cdot 2 &= 26 \\ 13 \cdot 3 &= 39 \\ 13 \cdot 5 &= 65 \\ 13 \cdot 7 &= 91 \end{aligned}$$

Primes are combined in various ways to form different positive integers. In some cases, you might use a certain prime factor more than once when building a number:

$$\begin{aligned} 4 &= 2 \cdot 2 &&= 2^2 \\ 8 &= 2 \cdot 2 \cdot 2 &&= 2^3 \\ 12 &= 2 \cdot 2 \cdot 3 &&= 2^2 \cdot 3 \\ 36 &= 2 \cdot 2 \cdot 3 \cdot 3 &&= 2^2 \cdot 3^2 \end{aligned}$$

In a similar way, every positive integer greater than 1 that is not prime can be written as the product of prime factors. In other words, each positive integer can be identified by its prime factors and the number of times each of these factors occurs. For example, n is a positive integer that is composed of 3 factors of 2, 1 factor of 3, and 2 factors of 5. What is the exact value of n? Does it matter if n is $2 \cdot 3 \cdot 5 \cdot 2 \cdot 5 \cdot 2$ or $2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$? Is there an accepted way to organize these factors as a product of n?

The notation of Pi and the precise language used here may be new to the students. The teacher may want to have students state in their own words their understanding of this theorem.

Make sure students notice that the even numbers always have at least one factor that is 2 in the first two steps.

We can answer these questions with the following:

THEOREM 3.1: FUNDAMENTAL THEOREM OF ARITHMETIC
If n is a positive integer, $n > 1$, then n is either prime or can be written as a product of primes $$n = p_1 \cdot p_2 \cdot \; \cdots \; \cdot p_k,$$ for some prime numbers p_1, p_2, …, p_k such that $p_1 \leq p_2 \leq \cdots \leq p_k$, where k is a natural number. In fact, there is only one way to write n in this form.

Positive integers greater than 1 are either prime or product of primes. Note that some of these primes might be repeated, as in the examples above. But the Fundamental Theorem of Arithmetic, or FTA, also tells us that there is only one way to decompose a given integer into prime factors with the factors written in order. The Fundamental Theorem of Arithmetic is also referred to as the Unique Prime Factorization Theorem. That is, if two different people correctly write a positive integer as a product of prime factors, their products always contain exactly the same prime factors, whether the order is the same or not. Using the FTA, however, the prime factors should be in increasing order.

The prime factorization of an integer gives us useful information about its factors. Factoring a number into primes usually takes some trial and error, but there is a technique that makes the process easier. Let's look at an example.

EXAMPLE 1

Find the prime factorization of 60.

SOLUTION

When looking for prime factors of a positive integer, it is useful to have a list of prime numbers. Look at the Sieve of Eratosthenes from the activity in Section 3.1 to confirm that the first few primes are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$$

Work your way through these primes to see if any of them are factors of 60. Divide 60 by 2 to get a quotient of 30 and a remainder of 0. So, 2 is a factor of 60, with

$$60 = 2 \cdot 30.$$

Reinforce the multiplication facts and the fact that any number ending in 5 has a factor of 5.

It is tempting to go on to the next prime in our list, but remember that a prime might appear more than once in a prime factorization. So before you continue, notice that 30 is even and realize that 2 is a factor of 30. Dividing, you will find that $30 = 2 \cdot 15$. So,

$$60 = 2 \cdot 2 \cdot 15.$$

Using the multiplication facts, divide 15 by 3 to find that $15 = 3 \cdot 5$. So,

$$60 = 2 \cdot 2 \cdot 3 \cdot 5.$$

Because each of the factors 2, 2, 3, and 5 is prime, you are through. You have written 60 as a product of prime factors. A useful way to write your result is to use exponents:

$$60 = 2^2 \cdot 3 \cdot 5$$

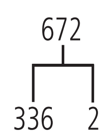
The process of finding the prime factors of a number is called **prime factorization**. We also use the same term to describe the result of the process. For example, the prime factorization of 60 is $2^2 \cdot 3 \cdot 5$.

EXAMPLE 2

Find the prime factors of 672.

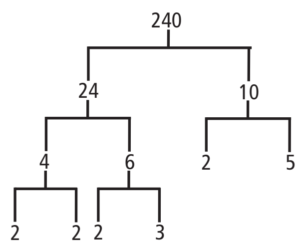
SOLUTION

If you want, use the same step-by-step process you used in the previous example. But there is a faster way to write the information. You can track the prime factors using a *tree diagram or factor tree*. You know that 2 is a prime factor of 672 because 672 is even, so start by dividing 672 by 2:

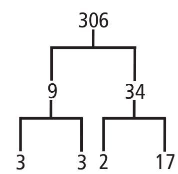


PROBLEM

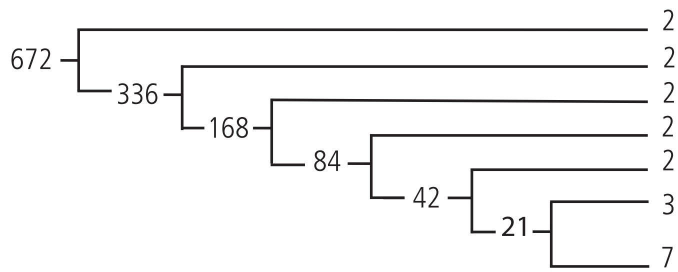
a.



b.



Continue factoring by 2 until the remaining number is odd:



Remember that $7 \cdot 3 = 21$, so you know that the last 2 factors in the process are 7 and 3. The prime factorization of 672 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^5 \cdot 3 \cdot 7$.

Notice that the processes used in Examples 1 and 2 are related. The second process is the same as the first, except that it organizes the factoring process more visually using a factor tree, which can be helpful when working with larger numbers.

PROBLEM

Find the prime factorization of each number below and write the answer using exponents.

a. 240 b. 306

We can also use prime factorization to generate equivalent numerical expressions.

EXAMPLE 3

Use prime factorization to simplify the following expression and leave your answer in factored form:

a. $15^2 \cdot 3^2$ b. $10^4 \div 2^3$

SOLUTION

a. The most straightforward way to simplify this is to use order of operations:

$15^2 \cdot 3^2 = 225 \cdot 9 = 2025.$

However, there is another way using prime factorization. We factor $15 = 3 \cdot 5$.

EXERCISES

1. a. $10 = 2 \cdot 5$
   b. $8 = 2^3$
   c. $25 = 5^2$
   d. 31 prime
   e. $28 = 2^2 \cdot 7$
   f. $60 = 2^2 \cdot 3 \cdot 5$
   g. $64 = 2^6$
   h. $72 = 2^3 \cdot 3^2$
   i. $48 = 2^4 \cdot 3$
   j. $51 = 3 \cdot 17$
   k. $77 = 7 \cdot 11$
   l. $84 = 2^2 \cdot 3 \cdot 7$
   m. $169 = 13^2$
   n. $360 = 2^3 \cdot 3^2 \cdot 5$
   o. $1225 = 5^2 \cdot 7^2$
2. Yes, 25, 64, 169, 1225. Perfect squares have prime factors with even powers. $25 = 5^2$; $64 = 8^2$; $169 = 13^2$; $1225 = 35^2$
3. The powers of every prime factor must be multiples of 3.
4. 30

So, $15^2 \cdot 3^2 = (3 \cdot 5)^2 \cdot 3^2$. Now we use properties of exponents,

$(3 \cdot 5)^2 = (3 \cdot 5) \cdot (3 \cdot 5) = 3^2 \cdot 5^2$

So, $(3 \cdot 5)^2 \cdot 3^2 = 3^2 \cdot 5^2 \cdot 3^2 = 3^2 \cdot 3^2 \cdot 5^2 = 3^4 \cdot 5^2 = 81 \cdot 25 = 2025$.

Either way gives you the same answer!

b. Again, we could use order of operations, $10^4 \div 2^3 = 10000 \div 8 = 1250$. A different way is to use prime factorization,

$$10^4 \div 2^3 = \frac{(2\bullet 5)^4}{2^3} = \frac{(2\bullet 5)\bullet(2\bullet 5)\bullet(2\bullet 5)\bullet(2\bullet 5)}{2\bullet 2\bullet 2} = \frac{(2\bullet 2\bullet 2)\bullet 2\bullet(5\bullet 5\bullet 5\bullet 5)}{(2\bullet 2\bullet 2)} = \frac{(2\bullet 2\bullet 2)}{(2\bullet 2\bullet 2)}\bullet\frac{2\bullet(5\bullet 5\bullet 5\bullet 5)}{1} = 2\bullet(5\bullet 5\bullet 5\bullet 5)$$

While order of operations at first may seem much simpler, when you have large exponents, using prime factorization can sometimes make the problem easier. In the exercises, try using both ways, and see which you like better

EXERCISES

1. Factor each of the following integers into primes. Write the integer as a product of its prime factors, using exponents when there are repeated prime factors. **See TE.**

a. 10	d. 31	g. 64	j. 51	m. 169
b. 8	e. 28	h. 72	k. 77	n. 360
c. 25	f. 60	i. 48	l. 84	o. 1225

2. The prime factorization of a number can be used to find a **perfect square**, which is the square of an integer. Look at the numbers given in Exercise 1. Are any of these perfect squares? If so, which ones? How can you use their prime factorization to determine whether each is a perfect square? **See TE.**

3. A **perfect cube** is an integer n that can be written in the form $n = k^3$, where k is an integer. Some examples of perfect cubes are

   $$0^3 = 0,\ 1^3 = 1,\ 2^3 = 8,\ 3^3 = 27,\ 4^3 = 64,\ \ldots.$$

   How can you use the prime factors of a number to determine whether it is a perfect cube? **The powers of every prime factor must be multiples 3.**

4. What is the smallest positive integer that has three different prime factors?

5. The prime factorization of 125 is 5^3 and 144 is $2^4 3^2$. These two numbers, although they don't have any factors in common, 125 is a perfect cube and 144 is a perfect square.
6. Answers will vary. But students should note that numbers divisible by 5 have either a 0 or 5 in the units digit
7. Numbers divisible by 10 end in 0.
9. a. 105 is composite; $5 \cdot 21$
   b. 113 is prime
   c. 117 is composite; $9 \cdot 13$
   d. 153 is composite; $3 \cdot 51$.
   e. 213 is composite; $3 \cdot 71$
   f. 239 is prime.
10. $2 = 2$; $4 = 2^2$; $8 = 2^3$; $16 = 2^4$; $32 = 2^5$. $64 = 2^6$ and $128 = 2^7$.
11. **Spiral Review** (7.1A)
    -39, -16, -5, 15, 39, 42
12. **Spiral Review** (7.2E) 102

13 **Ingenuity**

Students may be able to find all of the divisors for all of the integers from 80 to 90 and use the lists to find out which numbers are balanced, but the problem is easier if we first try to get some general insight about which numbers are balanced.

We know that odd numbers cannot be balanced, because all of their divisors are odd.

So suppose that we have an even number, such as 72. Suppose we find all of the factor pairs of 72:

72 = 1 x 72
72 = 2 x 36
72 = 3 x 24
72 = 4 x 18
72 = 6 x 12
72 = 8 x 9

Notice that each pair has at least one even factor. This is necessary because 72, an even number, cannot be written as a product of two odd factors. However, some pairs (such as 4 x 18) have two even factors. Therefore, in this case, the even factors outnumber the odd factors, and thus 72 is not balanced. In order for an even number to be balanced, every factor pair must contain one even factor and one odd factor. In particular, for an even number N to be balanced, the factor pair N = 2 x k must contain one even factor (2) and one odd factor (k). Thus N must be the product of 2 and an odd number in order to be balanced.

Thus the only integers between 80 and 90, inclusive, that can be balanced are 82 = 2 x 41, 86 = 2 x 43, and 90 = 2 x 45. A quick check of the divisors of these numbers show that all three are balanced. Therefore, 3 of the given integers are balanced.

5. Write the prime factorizations of 125 and 144. Looking at your factorizations, explain what the answers have in common and what their main difference is in terms of what you learned in problems 2 and 3.

6. Numbers like 25 and 120 are divisible by 5. Make a conjecture (educated guess) about what numbers divisible by 5 look like.

7. Make a conjecture about what numbers divisible by 10 look like.

8. Determine as efficiently as possible whether each of the following numbers is prime or composite. Prove your answer with a factor pair if you believe the number is composite.

   a. 105 b. 113 c. 117 d. 153 e. 213 f. 239

9. Find all the prime numbers of each of the following numbers: 2, 4, 8, 16, and 32. Write your prime factorization as a list of primes using exponents. Using the pattern you observed, find the prime factorizations for 64 and 128.

10. Use prime factorization to simplify the following expressions. Leave your answer in prime factored form.**e TE.**

    a. $2^{10} \div 2^2$ b. $6^4 \cdot 3^2$ c. $(3^4)^3$ d. $35^4 \div 5^2$
    e. $10^3 \cdot 12^2$

Spiral Review:

11. List 42, 39, -5, -39, 15, and -16 in order from least to greatest.

12. What is the value of the expression below?

    $6 + 6(12 \div 3)^2$**See TE.**

13. **Ingenuity:**

    For purposes of this problem, we say that a positive integer is *balanced*. if it has exactly as many odd divisors as it has even divisors. How many of the integers from 80 to 90 (inclusive) are balanced?**.**

14. **Investigation:**

    In this Investigation, we'll explore a useful technique for counting the divisors of a positive integer. Let us begin with the number 72 as an example. We know that the prime factorization of 72 is $72 = 2^3 \times 3^2$.

14. **Investigation**

This investigation is an extension of the Product Rule, which was introduced in the Investigation for Section 3.2

a. The divisors are as follows:

$1 = 2^0 \times 3^0$
$3 = 2^0 \times 3^1$
$9 = 2^0 \times 3^2$
$2 = 2^1 \times 3^0$
$6 = 2^1 \times 3^1$
$18 = 2^1 \times 3^2$
$4 = 2^2 \times 3^0$
$12 = 2^2 \times 3^1$
$36 = 2^2 \times 3^2$
$8 = 2^3 \times 3^0$
$24 = 2^3 \times 3^1$
$72 = 2^3 \times 3^2$

c. Each divisor of 72 has a prime factorization of the form $2^m \times 3^n$, where $0 \leq m \leq 3$ and $0 \leq n \leq 2$. Furthermore, every combination of a value of m from the set {0, 1, 2, 3} and a value of n from the set {0, 1, 2} is is represented in our table. There are 12 divisors in all, and there are 4 x 3 = 12 ways to choose a value of m and a value of n.

d. We have $100 = 2^2 \times 5^2$. So the divisors of 100 will have prime factorizations of the form $2^m \times 5^n$, where m comes from the set {0, 1, 2} and n comes from the set { 0, 1, 2}. There are 3 choices for m and 3 choices for n, so there are 3 x 3 = 9 divisors in all. (If students do not yet see the method explained here, encourage them to make a table of divisors of 100, the same way they did with 72.)

We have $112 = 2^4 \times 7^1$. So the divisors of 112 will be of the form $2^m \times 7^n$, where m is the set {0, 1, 2, 3, 4} and n is in the set {0, 1}. There are 5 choices for m and 2 choices for n, so there are 5 x 2 = 10 divisors in all.

e. We have $180 = 2^2 \times 3^2 \times 5^1$. So the divisors of 180 are of the form $2^k \times 3^m \times 5^n$, where k is in the set {0, 1, 2}, m is in the set { 0, 1, 2}, and n is in the set {0, 1} There are 3 choices for k, 3 choices for m, and 2 choices for n, so there are 3 x 3 x 2 = 18 divisors in all.

a. Make a table of the divisors of the number 72, as shown below. For each divisor , find the prime factorization, and write it in the form 2^m x 3^n , where m and n are whole numbers. For this exercise, if a prime p is not part of a divisor's prime factorization, write it as p^0. For example, since 2 is not a factor of 9, we will write 9 = 2^0 x 3^2 in the table. Be sure to include the divisors 1 and 72 in your table. Your table may need to be bigger than the one shown.

Divisor	Prime Factorization
9	$2^0 \times 3^2$
6	$2^1 \times 3^1$
24	$2^3 \times 3^1$

b. Now that you have recorded all of the divisors of the number 72 in the table, see if you can find a nice way to order the rows the table according to the prime factorizations of the divisors. (Your ordering will not necessarily have the divisors in increasing order.)

c. What do you notice about the prime factorizations of the divisors of 72? How many divisors are there? How could you have predicted this by looking at the prime factorization of 72?

d. Based on your work in (a) through (c), can you predict how many divisors the number 100 will have? How about the number 112?

e. How many divisors does the number 180 have?

15. **Challenge:**

Find the smallest positive integer whose prime factorization uses each odd digit (1, 3, 5, 7, 9) exactly once.

Section 3.4 - Common Factors and the GCF

Big Idea:

Finding common factors and greatest common factor (GCF)

Key Objectives:

- Use prime factorization to find common factors, including GCF.
- Use Venn diagrams to find common factors.
- Understand definition of "relatively prime".

Materials:

No extra materials needed.

Pedagogical/Orchestration:

To help students better understand Example 3 and the prime factorization method, give other examples where the numbers are smaller such as 14 and 18.

Activity:

"Factoring with Venn Diagrams" (Warning, this activity also includes LCM), "Math Bingo"

Exercises:

Be sure to emphasize what happens when two numbers have no common prime factors before the students attempt Exercise 2. This situation will come up when they get to Exercise 2e, and the students need to be aware that in these situations the GCF is 1.

Your students may find that Exercise 10 is time consuming, but it is a very important problem. Allow them to work together to find all of the values of *d*, and encourage them to work strategically to be sure that they have found all of the possible values.

The Investigation, Exercise 11, foreshadows LCM.

Vocabulary:

common factor, greatest common factor, relatively prime

TEKS:

6.1(B,C,E); 6.12(A); 6.13(B)

Launch for Section 3.4:

Tell students, "We are going to figure out some patterns, and see who can tell me in which column a pair of numbers belongs." Write the words "Column A" and "Column B" on the board. Tell students once they figure out the pattern they are not to tell their friends. Under Column A write the numbers 6 and 10. For Column B write the numbers 2 and 9. Next show students the following pair of numbers, but do not place them under a column yet: 15 and 3. Once students have thought about it for a few seconds and made their guesses, then put the numbers under Column A. Continue with this process with the following numbers always giving time for students to guess where the numbers belong before placing them in the correct column: 6 and 18 (Column A), 16 and 5 (Column B), 8 and 21 (Column B), 4 and 18 (Column A), 9 and 27 (Column A), 7 and 10 (Column B). Students will probably intuitively be able to figure out which column the numbers belong in before they are able to verbalize the rule that places them there. Allow students to try to explain to their classmates what they see. The students may eventually notice that the pairs of numbers under Column A have common factors greater than 1 whereas the pairs of numbers under Column B do not; however, do not tell them because that will be the point of the lesson. To help students visualize what is happening without telling them, draw a number line on the board that ranges from 0 to 27. For each pair of numbers, have students skip count on the number line by 1's, 2's, 3's, etc. to test which type of skip counting will cause them to land on both numbers in the pair. If they can skip count by greater than 1 and land on both numbers in the pair, then that pair of numbers belongs in Column A. Let students know that this is a preview of today's lesson and that there is actually a name for the types of number pairs under Column B. Tell students, "Listen carefully to today's lesson and you will learn what the Column B number pairs are called, plus various methods for finding common factors." (During the lesson, once the definition of "relatively prime" is covered, make sure you write that term above the Column B numbers. Hopefully the students will be the ones to notice that the term describes Column B.)

Math Bingo

Objective: Distinguish between prime and composite numbers, generate factors and multiples of a number, use divisibility rules, and find least common multiple and greatest common factor.

Materials Needed:
Math Bingo Cards (can be printed from http://print-bingo.com or made by teacher)
Chips, beans or small cubes (for students to cover their numbers)

Activity Instructions:

1. Give each student a Math Bingo card. (for free bingo cards visit: http://print-bingo.com)
2. Use chips, beans, or small cubes to cover numbers.
3. Call out a "Bingo" problem. Students can decide where to place their chip based on the problem description called.
4. Record each problem on the board to check answers once a "BINGO!" is called and so student answers can be checked.
5. A player wins by getting a vertical, horizontal, or diagonal lined filled. Student must justify their answers according to the problems written on the board: (example: "7 because it is prime; 12 because it is a multiple of 3; 10 because is is the LCM of 2 and 5, etc.)
6. Have FUN… and continue adding problems to this list!

Bingo Sample Problems:

A. Prime / Composite:
Randomly call out: "Prime" or "Composite"
Student can pick any prime or composite number on their bingo card for a strategy move.

B. Divisibility Rules: A number divisible by 2

A number divisible by 5	A number divisible by 6
A number divisible by 9	A number divisible by 10
A number divisible by 3	

C. Factors/ Multiples:

A factor of 8	A factor of 100	A factor of 84
A multiple of 9	A multiple of 4	A multiple of 12
A multiple of 30	A multiple of 24	A multiple of 11
A multiple of 16	A multiple of 21	A multiple of 7

E. GCF/LCM: (answer is written in parentheses)

GCF of 18, 30	(6)	GCF of 18,30	(5)
GCF of 44, 66	(22)	GCF of 15,30	(15)
GCF of 36,72	(36)	GCF of 55, 77	(11)
GCF of 6, 12	(6)	GCF of 16, 24	(8)
GCF of 25 , 100	(25)	LCM of 8, 12	(24)
LCM of 12, 10	(60)	LCM of 6,16	(48)
LCM of 21, 63	(63)	LCM of 5,9	(45)
LCM of 4, 14	(28)	LCM of 7, 21	(21)
LCM of 12, 18	(36)	LCM of 6, 14	(42)

Factoring with Venn Diagrams

Objective: The students will practice the skills learned in sections 3.1, 3.2, and 3.5 to find the prime factorization of numbers, categorize these factors in a Venn diagram and use this Venn diagram to find LCMs and GCFs.

Materials:
Computer with internet
Notebook paper

Activity Instructions:
Direct students to the following website and have them play several rounds of the activity. This activity will work best if you can provide one computer per student. If, however, you don't have enough computers for all of your students, you can do this as a class activity if you have a way to project your computer screen to the whole class. If you are doing the whole class activity, you can break the students into groups and have one group at a time come up and work a problem. If they get it correct, that team gets a point. It is important, however, that all students get a chance to work at least one problem on their own.

nlvm.usu.edu/en/nav/frames_asid_202_g_3_t_1.html

Once your students are at the correct website, direct them to the bottom of the page and ask them to click on "two" for the number of trees and "computer" for the problems. One problem will be in blue and the other will be in yellow. They find prime factors for each of these numbers until the factor trees are complete. Once both trees are complete and correct, the program will automatically display a Venn diagram. The students then click and drag the prime factors from the trees into the appropriate areas of the Venn diagram.

After the Venn diagram is complete and correct, the program prompts the students to fill in the LCM and GCF of the two numbers. Once they think they have found them, they should click the button to check. The computer lets them know if they are correct. If the student is incorrect, the computer lets them know where to look for their mistake.

It would be best if the students kept a written log of the original numbers and their final answers. Decide on a set number of problems for them to complete in a certain amount of time, and the students can turn in their written work to show you what they have accomplished.

EXPLORATION 1

Remind the students that "so on" refers to going on sequentially.

1. Have students draw a number line from 0 to 60. Use it to identify the frogs that land on both 24 and 36
. Solution: 2, 3, 4, 6, and 12 frogs.
2. The longest jumping frog that will land on both 24 and 36 is the 12-frog. This is because 12 is the greatest common factor of 24 and 36.
3. 4-frog
4. 1-frog

There is a problem in mathematics when it enters the symbolic stage. There are not enough unique symbols, in this case grouping symbols, to have unambiguous expressions. For instance, in function notation $f(x)$ means "the function of x" but in algebraic multiplication it means "f times x." This is always a source of confusion and frustration for some students. Call your students' attention to the fact that both points "(x,y)" and "GCF(m,n)" use parentheses with an ordered pair. The careful reader can, however, distinguish now the parentheses are being used and avoid confusion. Many students, also, have the same name but are very different in spite of that similarity.

SECTION 3.4 COMMON FACTORS AND THE GCF

We know that when we multiply 3 and 8 to obtain the product 24, the numbers 3 and 8 are factors of 24. Notice that $2 \cdot 12$ also equals 24, and 2 and 12 are factors of 24. You have discovered several other numbers that are factors of 24. In this section, we examine how we can use what we know about factors to determine factors common to two or more numbers. When is it necessary to find common factors? Let's examine the following situation to see.

EXPLORATION 1

To prepare for a frog-jumping contest, Fernando decided to train a group of his fellow frogs. Each frog was trained to jump a certain length along a number line starting at 0. He trained a 1-frog to jump a distance of 1 unit in each hop. He also trained a 2-frog to jump 2 units, a 3-frog to jump 3 units and so on. The frogs always start at the zero point on the number line. Now Fernando wants to know which frogs will land on certain locations on the number line.

1. Which of his frogs will land on both the locations 24 and 36?
2. Which is the longest jumping frog that will land on both 24 and 36? Explain why this answer makes sense.
3. What is the longest jumping frog that will land on both 20 and 32?
4. What is the longest jumping frog that will land on both 24 and 25?

If a frog lands on 24, then the length of its jump is a factor of 24. So, if a frog lands on *both* 24 and 36, then the length of its jump is a factor of 24 *and* 36.

DEFINITION 3.4: COMMON FACTOR AND GCF

Suppose m and n are positive integers. An integer d is a **common factor** of m and n if d is a factor of both m and n. The **greatest common factor**, or **GCF,** of m and n is the greatest positive integer that is a factor of both m and n. We write the GCF of m and n as $\text{GCF}(m, n)$.

EXPLORATION 2

In this exploration, each piece must be cut into a positive integer length with no fractions and no string left.

All the possible integer lengths both the strings can be cut are 1, 2, 3, 4, 6, 12, and 24 inches. The longest length possible is 24 inches.

If you revisit Exploration 1, you will see that in question 2, the GCF of 24 and 36 is 12. From question 3, the GCF of 20 and 32 is 4, and the GCF of 24 and 25 is 1.

EXPLORATION 2

Suppose you have two types of string with different lengths. The cotton string is 120 inches long, and the nylon string is 72 inches long. Determine every possible integer length both of the strings can be cut so that each piece is the same length and there are no string pieces leftover. Make a list of all the possible common lengths. What is the longest common length possible? In this Exploration, each piece must be cut into a positive integer length with no fractions and no string left.

There are several different ways to calculate the GCF of two numbers. Here is one way that reinforces the term Greatest Common Factor.

EXAMPLE 1

Find the GCF of 15 and 25 using a T-chart.

SOLUTION

Find the GCF of two numbers by first listing all the factors of each of the numbers. Find all the common factors of the two numbers, then choose the greatest.

15	
1	15
3	5

25	
1	25
5	5

The common factors of 15 and 25 are 1 and 5, and the GCF of 15 and 25 is 5 because it is the larger of the two common factors.

EXAMPLE 2

List the common factors of 27 and 32, then find the GCF.

Have students talk about the distinction between prime numbers and relatively prime numbers. Relatively prime numbers need not be prime numbers but it is a relation between 2 or more numbers.

The GCF of two prime numbers is 1.

PROBLEM 1

a. common factors: 1; GCF = 1; 15 and 22 are relatively prime.
b. common factors: 1, 13; GCF = 13; 39 and 65 are not relatively prime.
c. common factors: 1; GCF = 1; 7 and 11 are relatively prime.

SOLUTION

Use the same method you used in the previous example. Using a T-chart, first list the factors of each number.

27	
1	27
3	9

32	
1	32
2	16
4	8

27: 1, 3, 9, 27
32: 1, 2, 4, 8, 16, 32

In this case, there is only one common factor: 1. Therefore, the GCF of 27 and 32, written also as GCF(27, 32), is 1. There is a term to describe a relationship between numbers whose GCF is 1.

DEFINITION 3.5: RELATIVELY PRIME
Two integers *m* and *n* are **relatively prime** if the GCF of *m* and *n* is 1.

From the definition above, the numbers 27 and 32 are relatively prime. Notice that neither 27 nor 32 are prime numbers. If we consider two prime numbers like 3 and 7, what is their GCF? Check a few more examples. Make a generalization about the GCF of any two prime numbers. **See TE.**

PROBLEM 1

Find the common factors and GCF of the following pairs of numbers. State whether the numbers are relatively prime or not.

a. 15 and 22 b. 39 and 65 c. 7 and 11

All the factors of 108: (1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108)

All the factors of 168: (1, 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, 84, 168)

EXAMPLE 3

If the number is even, factor out 2 and continue to look for even numbers. After that, look for multiples of three and so forth. Encourage students to be systematic and use number sense to factor.

Find the GCF of a pair of larger numbers like 108 and 168 using the process of first finding factors, then the common factors, and eventually the greatest common factor.

First, list all the factors of 108. Then, list all the factors of 168. Make sure you have 12 factors for 108 and 16 factors for 168. There are many factors to find. If you do not have them all listed, go back and find them.

Determine all the factors the two numbers have in common. You should find that the common factors are 1, 2, 3, 4, 6, 12. From this list, you can see that the greatest common factor of both 108 and 168 is 12. As you discovered, this method for finding the GCF works well. However, the more factors the numbers have, the more time it takes to make the list of factors for each number. Fortunately, prime factorization makes finding the GCF of two numbers easier.

Here is an example of the efficiency of prime factorization.

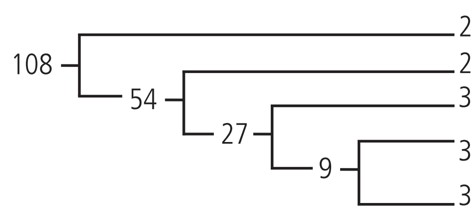
EXAMPLE 3

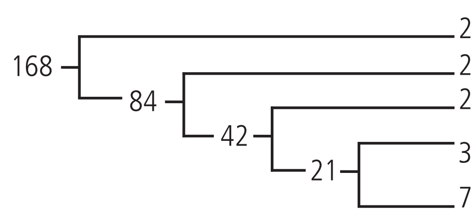
Find the GCF of 108 and 168 using prime factorization.

SOLUTION

Start by finding the prime factors of 108 and 168.

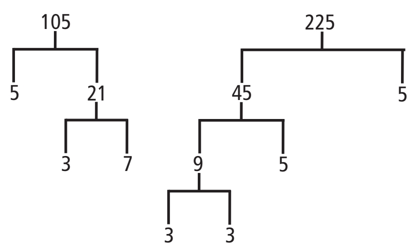
Use factor tree diagrams to do this:





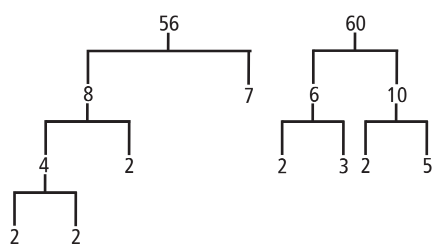
Problem 2

a.



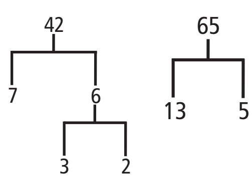
GCF (105, 225) = 3 x 5 = 15

b.



GCF (56, 60) = 2 x 2 = 4

c.



There is no GCF of 42 and 65. 42 and 65 are relatively prime.

Recall that the prime numbers are the building blocks of the integers. If you want to find the GCF of two integers, find the building blocks, or prime factors, the two numbers have in common. Remember, when doing this, it is helpful to write out the prime factors of each number in an organized way using exponents.

$$108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^3$$
$$168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 2^3 \cdot 3 \cdot 7$$

You have written the prime factors of 108 and 168 with and without exponents, lining up the equal prime factors. It is clear which factors the two numbers have in common:

$$\begin{array}{rcccl} 108 = & \boxed{2 \cdot 2} \cdot & \cdot \boxed{3} & \cdot 3 \cdot 3 & = 2^2 \cdot 3^3 \\ 168 = & \boxed{2 \cdot 2} \cdot 2 \cdot & \boxed{3} & \cdot 7 & = 2^3 \cdot 3 \cdot 7 \end{array}$$

The prime factors have two 2's and one 3 in common. So $2 \cdot 2 \cdot 3 = 12$ is the greatest common factor of 108 and 168. It is also possible to check the common factors using long division. If there were a larger common factor, the quotients would have even more common prime factors, or higher powers of common prime factors, or both. However, the quotient after dividing 108 by 12 is 9, and the quotient after dividing 168 by 12 is 14. It is easy to check that 9 and 14 have no common factor other than 1, and so 9 and 14 are relatively prime.

EXAMPLE 4

Use the prime factorization method to find the GCF of 18 and 35.

SOLUTION:

The prime factorizations are $18 = 2 \cdot 3^2$; $35 = 5 \cdot 7$. Notice that there are no common prime factors. However, we should remember that a factor of any number is 1. The GCF of two numbers with no common prime factors will have 1 as its GCF.

PROBLEM 2

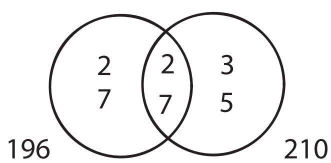
Use the prime factorization method to find the GCF of each pair of numbers:

a. 105 and 225 b. 56 and 60 c. 42 and 65

PROBLEM 3

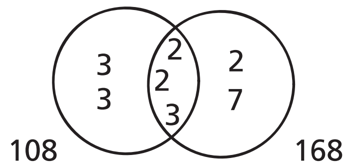
$196 = 2 \cdot 2 \quad \cdot 7 \cdot 7 = 2^2 \cdot 7^2$

$210 = \quad 2 \cdot 3 \cdot 5 \cdot 7 \quad = 2 \cdot 3 \cdot 5 \cdot 7$



A visual representation called a **Venn diagram** might help you to see how the GCF of numbers like 108 and 168 is constructed from the prime factorization. To make the Venn diagram, draw a circle for each number you are considering. Inside each circle write all its prime factors. If there are common prime factors, then the circles will intersect and the common primes will be in the area common to both, or in the intersection of the circles. If the circles have no common prime factors, then you conclude that the only common factor is 1. Remember, 1 is always a common factor for any pair of integers, and it is the greatest common factor if there are no other common factors.

The Venn diagram will look like this, and the factors for the GCF can be found in the circles' overlap. GCF (108, 168) = 2 · 2 · 3 = 12.

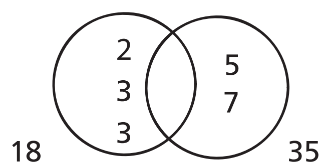


PROBLEM 3

Compute the GCF of 196 and 210. Use a Venn diagram and the method from Example 3, and decide which you prefer.

EXAMPLE 5

Let us revisit the problem of finding the GCF of 18 and 35 again. The Venn Diagram approach will show that there are no common prime factors in the two circles.



In such cases the GCF will be 1.

EXERCISES

1. a. 1,5; 5
   b. 1,2,3,4,6,12; 12
   c. 1,3; 3
   d. 1;1
   e. 1,5; 5
   f. 1,3,9; 9
   g. 1,2,4,8,16; 16
   h. 1,2,3,4,6,8,12,24;24

2. The method we want the students to discover for powers of prime factors and the GCF has two parts:
   Step 1: Include every different prime factor in all of the numbers involved.
   Step 2: Use the smallest power found in any one of the numbers involved for each factor from Step 1.
   (Teachers, always encourage your students to use what they have already done rather than recreating the wheel, which in mathematical computation might be particularly boring.)
   a. 8; 12, 4
   b. 18; 30, 6
   c. 108; 63, 9
   d. 42; 50, 2
   e. 42; 55, 1
   f. 72; 132. 12
   g. 525; 42, 21
   h. $a^3 \cdot b^2$; $a^2 \cdot b \cdot c$, a^2b

3. Check that students show prime factorization.
   a. 8
   b. 1
   c. 12
   d. 3
   e. 25
   f. 24

4. 1 because the only factors of any prime numbers are 1 and the number itself.

EXERCISES

1. For each pair of integers below, find all of the common factors of the two integers. Then find the GCF of the two integers.

    a. 15 and 10 **1, 2; 2**
    b. 12 and 36 **1, 3; 3**
    c. 21 and 24 **1, 5; 5**
    d. 30 and 41 **1, 2, 3, 6; 6**
    e. 55 and 70 **1, 2, 4; 4**
    f. 45 and 72 **1, 3, 9; 9**
    g. 64 and 80 **1; 1**
    h. 120 and 144

2. In each part of this exercise, the prime factorizations of two numbers is given. First, use the prime factorizations to find the GCF of the two numbers. Then, compute (find the value of) the two numbers from their prime factors.

    a. $2 \cdot 2 \cdot 2$ **12**
       $2 \cdot 2 \cdot 3$ **18; 6**
    b. $2 \cdot 3 \cdot 3$ **30**
       $2 \cdot 3 \cdot 5$ **8; 2**
    c. $2 \cdot 2 \cdot 3^3$ **150**
       $3 \cdot 3 \cdot 7$ **231; 3**
    d. $2 \cdot 3 \cdot 7$
       $2 \cdot 5 \cdot 5$ **1000; 100**
    e. $2 \cdot 3 \cdot 7$ **66**
       $5 \cdot 11$ **245; 1**
    f. $2^3 \cdot 3^2$ **56**
       $2^2 \cdot 3 \cdot 11$ **196; 28**
    g. $3 \cdot 5^2 \cdot 7$ **900**
       $2 \cdot 3 \cdot 7$
    h. $a^3 \cdot b^2$
       $a^2 \cdot b \cdot c$

3. For each pair of integers below, find the GCF of the two integers using prime factorization. **See TE.**

    a. 16 and 40 **2**
    b. 7 and 17 **1**
    c. 60 and 72 **1**
    d. 63 and 75 **2**
    e. 125 and 200 **3**
    f. 144 and 168 **5**
    g. State which of the above pairs of numbers is relatively prime. Explain your reasoning.

4. What is the GCF of two prime numbers *p* and *q*? Explain your reasoning.

5. Josh has 12 basketball cards, 20 football cards, and 24 baseball cards. He wants to separate them into mini albums that contain an equal number of each type of sports card.

5. a. The greatest number of mini albums Josh can make is 4.
   b. Each album will contain 3 basketball cards, 5 football cards, and 6 basebal cards.

6. 6 necklaces will each have 5 blue beads, 8 red beads, and 6 yellow beads.

7. 8 arrangements where each arrangement has 4 yellow balloons, 3 orange balloons and 2 red balloons.

8. Hiroko can divide her jelly beans among 14 friends where each friend receives 3 green jelly beans, 2 blue jelly beans and 1 orange jelly bean.

9. 7 people total, Tina herself and 6 guests.

10. a. \$0.10, \$0.20, \$0.25, \$0.30, \$0.35, \$0.40, \$0.45, \$0.50, \$0.55, \$0.60, \$0.65, \$0.70, \$0.75, \$0.80, \$0.85, \$0.90, \$0.95
    b. A quarter is worth 25 cents and a dime is worth 10 cents. The GCF of these two numbers is 5.
    c. All of the answers in part (a) are multiples of 5, which happens to be the GCF of 10 and 25 and the answer to part (b).

12. Spiral Review 7.2(G)
    Answer: B

a. What is the greatest number of mini albums he will be able to make if all cards are to be used?

b. How many of each type of sports card will he put into each mini album?

6. Julianne is making necklaces to sell at the fair. She wants to use all the beads she has left from another project. This includes 30 blue beads, 48 red beads, and 36 yellow beads. How many of each color of bead will each necklace contain if she uses all the beads and makes each necklace identical?

7. Flora's flower shop makes balloon arrangements. She has 32 yellow balloons, 24 orange balloons, and 16 red balloons. Each arrangement must have the same number of each color. What is the greatest number of arrangements that Flora can make if every balloon is used?

8. Hiroko has 42 green jelly beans, 28 blue jelly beans, and 14 orange jelly beans. She wants to share her jelly beans evenly among her friends. What is the greatest number of friends Hiroko can divide the jelly beans among?

9. Tina hosted a party for a certain number of guests. She prepared a vegetable tray with 14 celery sticks and 21 carrot sticks. If all the vegetables were eaten, and each guest had the same number of celery sticks and the same number of carrot sticks as Tina had, how many guests were at the party? **6 guests**

10. a. Make a list of all the amounts of money less than a dollar that you can make with an endless supply of quarters and dimes. **See TE.**

    b. How many cents is a quarter worth? How many cents is a dime worth? What is the GCF of these two numbers? **See TE.**

    c. What is the relationship between your answers in parts **a** and **b**?

11. Find the GCF of 10^6 and 6^{10}. $\mathbf{2^6}$

Spiral Review:

12. Holly bought breakfast for herself and 2 friends. The cost of each breakfast was between \$6 and \$7, including tax. Which of the following could not be a total cost for the breakfasts that Holly bought?

    a. \$19.20 b. \$17.80 c. \$20.50 d. \$18.25

13. Spiral Review 6.2(C)
    Answer: C

Investigation

15. This problem foreshadows the Least Common Multiple of a set of numbers. Do not give this number a name just yet as the concept is fully introduced in Section 8.5.

    There are many conjectures students could make about this problem. They could notice that whenever the GCF is equal to 1 then $p = m \times n$. This is true; however, you might challenge your brighter students to find the pattern that $(m \times n)/\text{GCF} = p$.
    Encourage students to work together and use more than one operation.

13. Bryan works at the grocery store and earns $11 per hour. If Bryan works 38 hours each week, which expression could be used to determine his total earnings for 1 year?

    a. 11(38)

    b. 11(52)

    c. 11(38)(52)

    d. 11(38)(12)

14. **Ingenuity:**

    Suppose that the product of two integers is 63,000. What is the greatest possible value of their GCF?? **14**

15. **Investigation:**

    Suppose we want to find the GCF of the numbers 8407 and 8457. Remember that in the frog-jumping model of the GCF, the GCF of 8407 and 8457 is the number of the highest-numbered frog that will land on both 8407 and 8457 on the number line, assuming that all frogs start at the number 0.

    a. Explain why no frog numbered higher than 50 can land on both 8407 and 8457.

    b. Can the 33-frog land on both 8407 and 8457? How about the 29-frog? The 25-frog? The 17-frog? The 10-frog? What do you notice?

    c. Make a list of the frogs with numbers from 1 to 50 that could land on both 8407 and 8457.

    d. Assuming that all frogs start at 0, which of the frogs you listed in (c) actually do land on both 8407 and 8457.

    e. What is the GCF of 8407 and 8457?

    f. What is the GCF of 54218 and 54418?

16. **Challenge:**

    Find the GCF of 123,456,789,011 and 314,159. **1**

Section 3.5 - Common Multiples and the LCM

Big Idea:

Finding common multiples and least common multiple (LCM)

Key Objectives:

- Formalize that skip counting produces multiples, and that multiple lists are infinitely long.
- Use listing method to find LCM.
- Use prime factorization to find LCM.
- Use Venn diagrams to find LCM.

Materials:

No extra materials needed.

Pedagogical/Orchestration:

- Use word problems such as the hot dog problem as application of LCM in real life situations.

Activity:

"Factoring with Venn Diagrams" (this can be found at the beginning of Section 3.4 in the Teacher's Edition)

Exercises:

The word problems may require guidance from the teacher for set up.

Vocabulary:

multiple, common multiple, least common multiple (LCM)

TEKS:

6.1(F); 6.12(A); 6:13(B)

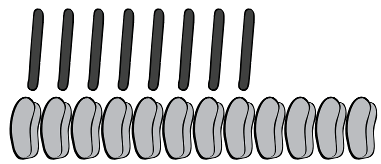
Launch for Section 3.5:

Ask students if any of them have ever volunteered at an animal shelter. If none have, ask if any would like to volunteer. Select two students as your volunteers and have them sit in chairs in front of the room facing the class. Use the students' own names: the names Jose and Kerry will be used for this example. During the summer, Jose goes to the animal shelter every three days and Kerry goes every five days. If they just saw each other today, when is the next time the two volunteers will meet at the shelter? Tell the volunteers that you will be counting days and they will need to stand every time it is his or her turn to go to the shelter. Tell the class to keep track of the days each volunteer goes to the shelter by making a list. Start counting slowly, and at 3, Jose should stand, and then sit down. At 5, Kerry should stand and then sit down. This should continue until at 15, for the first time Jose and Kerry stand together. Make a note of this to the class and continue counting at least until 30. Ask a student from the class to write on the board, the days that Jose visited the shelter, and another student to write the days that Kerry visited the shelter. Ask the class if they know a name for all the numbers in Jose's list. If they do not know tell them the numbers in Jose's list are the multiples of 3. Then ask about the numbers in Kerry's list and they should be able to tell you those are multiples of 5. Circle the 15 that is in both lists and ask what the number 15 would represent. Students may remember that this is called a common multiple. Ask students what other multiples the two lists would have in common, and if they can see a pattern. Students may notice that these are the multiples of 15. Remind students that the numbers 3 and 5 are called factors of 15. Tell students that 15, which is the smallest of the common multiples, is a special number we call the Least Common Multiple, and that it is used in working with fractions. Tell students, "Today we will be building on what we know about multiples."

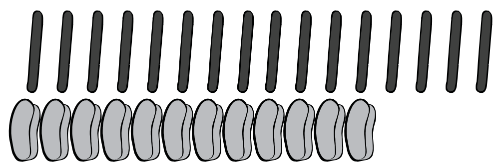
As a preliminary to finding the least common multiple, have your students rehearse the rule for finding the GCF from Section 3.4. Namely, for the GCF: the product of all the common primes raised to the lowest exponent that appears in either factorization. We will see that to find the LCM we can find the product of all the primes that occur, raised to the highest exponent that appears in either factorization. The Investigation in Exercise 12 will show the very nice relationship that the product $m \cdot n = \text{LCM}(m,n) \cdot \text{GCF}(m,n)$. Don't try to get them to see this yet, but be prepared if a student notices this pattern.

SECTION 3.5 COMMON MULTIPLES AND THE LEAST COMMON MULTIPLE

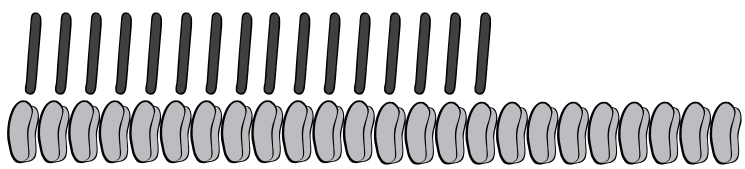
Have you noticed that hot dogs often come in packages of eight, and hot dog buns come in packages of twelve? When people plan to cook hot dogs, they tend to buy one package of hot dogs and one package of buns. But if they do this, they are left with four extra buns.



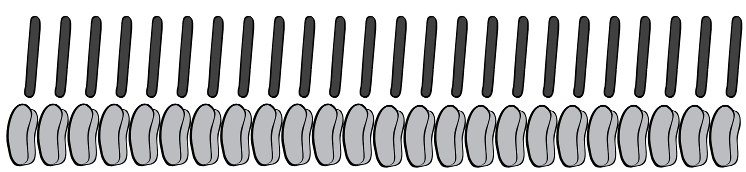
Some people who pay for the extra hot dog buns don't want to waste them. What can they do? They could buy another package of eight hot dogs:



But now there are four extra hot dogs without buns. If they buy more buns:



There are eight buns without hot dogs, even more extra buns than the first time. Will this process ever end? Try buying one more package of hot dogs:



Notice that we could begin with zero in our listing which would technically be the least number, but for our definition, the least common multiple is a positive integer.
We therefore begin with having 1 pack of each, then 2 packs of each and so on. We should note that 0 is a multiple of any number, n, because 0 = 0(n) for any integer n.

Aha! We have finally reached a point where we have exactly the same number of hot dogs and buns. Of course, in order to get there, the customers had to buy two packages of buns and three packages of hot dogs. Maybe they can freeze the rest.

What happened mathematically with the hot dogs and buns? One way to organize the number of hot dogs and the number of buns is:

Hot dogs:	8	16	**24**	32	40	48	56	64	72…
Buns:		12	**24**	36		48	60		72…

Look for some positive integer *N* where anyone could buy exactly *N* hot dogs and *N* buns. Finding the number gives the consumer the number of hot dogs and the number of buns to buy so that they come out evenly.

It is easy to see that 24 is the smallest positive integer that is in both lists. So, the smallest value of *N* for both items is 24. Notice that the numbers in the first list are the multiples of 8. This makes sense because it is only possible to buy 8 hot dogs at a time. The numbers in the second list are the multiples of 12 because buns come only in packages of 12. So, 24 is the smallest positive integer that is a multiple of both 8 and 12. Mathematicians have a term for this:

DEFINITION 3.6: COMMON MULTIPLE AND LCM

Integers *a* and *b* are positive. An integer *m* is a **common multiple** of *a* and *b* if *m* is a multiple of both *a* and *b*. The **least common multiple**, or **LCM**, of *a* and *b* is the smallest positive integer that is a common multiple of *a* and *b*. We write the LCM of *a* and *b* as LCM(*a, b*).

Notice that 48 and 72 are also common multiples of 8 and 12, but not the least.

As we found with the GCF, there are several ways to find the LCM of two numbers. Try some examples:

EXAMPLE 1

Find the LCM of 5 and 7.

EXAMPLE 1

Remind students that multiples are infinitely long so if a common multiple does not appear with the first 10 multiples, the lists should be expanded.

SOLUTION

One way to find the least common multiple of two numbers is to list the positive multiples of each number in increasing order until you find an integer that is in both lists. For example, start by writing the first fourteen positive multiples of 5 and the first 10 positive multiples of 7:

Multiples of 5:

5, 10, 15, 20, 25, 30, **35**, 40, 45, 50, 55, 60, 65, 70, …

Multiples of 7:

7, 14, 21, 28, **35**, 42, 49, 56, 63, 70, …

Notice that 35 is in both lists. 70 is also on both lists. In fact, if we continued listing the multiples, we would find other common multiples of 5 and 7. However, the smallest positive integer that is a multiple of both 5 and 7 is 35. That means 35 is the LCM of 5 and 7.

EXAMPLE 2

Radio station KISS broadcasts a commercial every 22 minutes. WILD broadcasts a commercial every 12 minutes. If the two stations broadcast their commercials at 3:20 , when is the next time their commercials will air at the same time?

SOLUTION

Try the same method on 22 and 12 as you did in Example 1. Start by writing the first ten multiples of each number:

Multiples of 22:

22, 44, 66, 88, 110, 132, 154, 176, 198, 220, …

Multiples of 12:

12, 24, 36, 48, 60, 72, 84, 96, 108, 120, …

There appear to be no common multiples from the lists. However, the two lists do not stop after ten entries; they go on forever. So, to find the common multiple of 22 and 12, try extending the two lists:

Multiples of 22:

22, 44, 66, 88, 110, **132**, 154, 176, 198, 220, 242, 264, 286, 308, …

EXAMPLE 3

The prime factorization method for finding the LCM can be introduced to 6th graders, but most students are more comfortable using the listing method.

Multiples of 12:

12, 24, 36, 48, 60, 72, 84, 96, 108, 120, **132**, 144, 156, 168, …

The number 132 is in both lists. Therefore, 132 is the LCM of 22 and 12. The LCM was in the original list of multiples of 22, but it was necessary to extend the list of multiples of 12 to reach 132.

Remember that there are 60 minutes in 1 hour. 132 minutes is equal to 120 minutes + 12 minutes or 2 hours and 12 minutes. We add this time to 3:20 and get 5:32 as the next time their commercial will air at the same time.

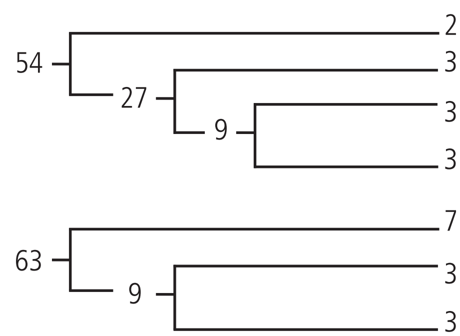
As you have seen, with very large numbers, it might be necessary to write many multiples to find their LCM. This can be very time-consuming, so it would be better to have a more efficient method for computing the LCM of two numbers, just as you did when looking for the GCF. Fortunately, prime factorization comes to the rescue again.

EXAMPLE 3

Find the LCM of 54 and 63.

SOLUTION

First factor 54 and 63, using prime factorization:



The prime factorizations are $54 = 2 \cdot 3^3$, and $63 = 3^2 \cdot 7$. Again, it is useful to express the prime factors with the exponents, aligning like factors:

$$\begin{aligned} 54 &= 2 \cdot 3^3 \\ 63 &= \phantom{2 \cdot{}} 3^2 \cdot 7 \end{aligned}$$

To find a number that is a multiple of both 54 and 63, the multiple must have the prime building blocks that both 54 and 63 have. Before continuing, consider how you might be able to construct a multiple of 54 and 63 using the prime factors for each of the numbers. How can you construct the smallest such multiple?

Now look at the method for finding the LCM of numbers using their prime factorization. The factors of each are as follows: 54: $3 \cdot 3 \cdot 3 \cdot 2$ and 63: $3 \cdot 3 \cdot 7$. By examining the two sets of prime factors, you can see that a common multiple must include 2, 3, and 7. However, $2 \cdot 3 \cdot 7 = 42$ is not a multiple of 54 or 63. Because 54 has three factors of 3 and 63 has two factors of 3, to include both numbers, use three factors of 3. Why won't two factors of 3 be enough? Now multiply the factors 2, 3^3, and 7.

$$2 \cdot 3^3 \cdot 7 = 378$$

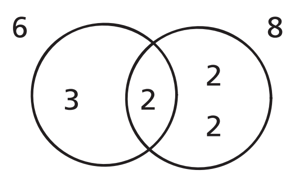
This is the smallest integer that contains all of the building blocks, or prime factors, in both sets of prime factors. Therefore, 378 is the LCM of 54 and 63.

PROBLEM

Find the LCM of each pair of numbers below using the prime factorization method:

a. 36 and 42 **252** b. 17 and 18 **301** c. 13 and 52 **52**

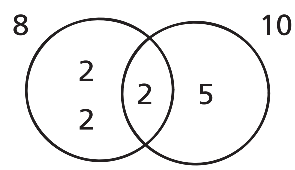
Another approach to solving for the LCM is to look at the Venn diagram, as you did in Section 3.4, when you worked with GCFs. Examine the prime factors of 6 and 8. The Venn diagram includes the prime factors for each number in the respective circles. Note the common factors in the overlapping part of the circles.



$\text{LCM}(8, 10) = 2^3 \cdot 5 = 40$

A multiple of 6 requires a factor of 2 and 3; a multiple of 8 requires 2, 2, and 2. So, a multiple of 6 and 8 requires one 3 and three 2s as factors. The shortest list of factors that will produce a multiple of both is $2 \cdot 2 \cdot 2 \cdot 3$. The greatest common factor is 2 and we avoid double counting it in finding the LCM. Thus, to get the LCM of 6 and 8, take the product of the highest power of all the factors that occur in either number, that is $2^3 \cdot 3$, to get the LCM of 24. *Remember to use the highest power of any prime for the LCM just as you used the lowest power of any common prime for the GCF.*

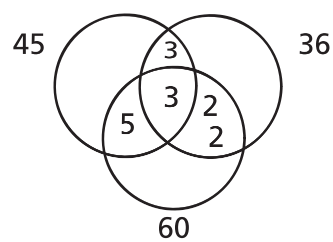
Try the Venn diagram approach to find the LCM of 8 and 10.



You will find that the Venn diagram is more practical when the numbers are larger and there are three numbers or more. For example, to find the least common multiple of 45, 36, and 60, write all the prime factors and notice which factors are common.

$$45 = 3^2 \cdot 5$$
$$36 = 2^2 \cdot 3^2$$
$$60 = 2^2 \cdot 3 \cdot 5$$

We can represent this information as a Venn diagram:



EXERCISES

1. a. 28
   b. 15
   c. 24
   d. 99
   e. 60
   f. 91

2. Make sure your students show enough work.
   a. 60
   b. 72
   c. 80
   d. 168

3. Yes, the pairs of integers in the previous exercises that are relatively prime: 9 & 11 and 7 & 13. Answers will vary as to how far you go using this method to find whether or not two integers are relatively prime. But, the idea is using prime factorization is more efficient. If p and q are relatively prime then their LCM is their product, pq.

Both 45 and 60 have a common factor of 5, so 5 is the intersection of the 45 circle and the 60 circle. The factor 2^2 is in both 60 and 36, so 2^2 is the intersection of the 60 circle and the 36 circle. There is a factor of 3 common to 45 and 36, so their intersection contains a 3, and the intersection of all three circles contains a 3 because it is a common factor of all three numbers.

After you have separated the factors into the different regions in the Venn diagram, by multiplying all the numbers in the circles and their intersections you will have the LCM of 45, 36, and 60. Did you get 180? If so, you are correct.

To summarize these investigations, the LCM of two integers a and b is equal to the product of all the primes that occur in the prime factorizations, raised to the highest exponent that appears in either factorization.

EXERCISES

1. Find the LCM of the following pairs of numbers by listing multiples of the two numbers until you find the first multiple common to both lists.

   a. 4 and 7
   b. 5 and 15
   c. 6 and 8
   d. 9 and 11
   e. 12 and 60
   f. 7 and 13

2. Find the LCM of the given numbers either by listing multiples of the two numbers until you find the first multiple common to both lists or by using the prime factorization of each number.

   a. 12 and 20
   b. 8 and 9
   c. 16 and 40
   d. 24 and 56

3. Were there any pairs of integers in the previous exercises where the integers were relatively prime? Explain whether you have to check every prime factor of both integers to find that the two integers are relatively prime.

   Based on the evidence in the previous exercises, if p and q are different prime numbers, what is the LCM of p and q? Use what you know about the common factor of two relatively prime numbers to explain why your previous answer makes sense.

If you have not discussed the word conjecture in class yet, this would be a perfect opportunity to do so.

4. a. 88
   b. 84
   c. 39
   d. When one number is a multiple of the other, the larger is the LCM.

5. a. LCM = 24
   b. LCM = 210
   c. LCM = 60
   d. LCM = 432

6. a. GCF = 6 and LCM = 36
   =12; =18
   b. GCF = 45 and LCM = 675
   =45; =675
   c. GCF = 2 and LCM = 1980
   =44; =90
   d. GCF = 90 and LCM = 9000
   =360; =2250

7. a. GCF = 24 and LCM = 144
   b. GCF = 30 and LCM = 180
   c. GCF = 24 and LCM = 720

Check student work for prime factorizations.

8. 21 days

9. Days 30, 60, and 90

4. Find the LCM for each of the following pairs of numbers:

   a. 11 and 88 **60** b. 7 and 84 **72** c. 39 and 13 **45**

   d. Look for a pattern in computing the LCM for **a-c**. Make a conjecture that explains this pattern. **See TE.**

5. Prime factors of two integers appear in each part below. Compute the two integers from their prime factors. Use prime factorization method from the Exploration to find the LCM of the two numbers. **See TE.**

   a. $2 \cdot 2 \cdot 2$
      $2 \cdot 3$

   b. $2 \cdot 5$
      $3 \cdot 7$

   c. $2 \cdot 2 \cdot 5$
      $2 \cdot 2 \cdot 3$

   d. $2 \cdot 2 \cdot 2 \cdot 2$
      $3 \cdot 3 \cdot 3$

6. The prime factors of two integers appear below. Compute the two integers from their prime factors. Find the GCF and LCM for each pair of numbers.

   a. $2^2 \cdot 3$
      $2 \cdot 3^2$

   b. $3^2 \cdot 5$
      $3^3 \cdot 5^2$

   c. $2^2 \cdot 11$
      $2 \cdot 3^2 \cdot 5$

   d. $2^3 \cdot 3^2 \cdot 5$
      $2 \cdot 3^2 \cdot 5^3$

7. For each pair of integers below, use prime factorization to find the GCF and LCM. **Check student work for prime factorizations.**

   a. 48 and 72 **GCF = 36 and LCM = 540**

   b. 90 and 60 **GCF = 1 and LCM = 5544**

   c. 120 and 144 **GCF = 10 and LCM = 880**

8. Jackie attends dance class every 7 days, and Lorianne attends every 3 days. If they both attended class today, in how many days will they both attend class together again?

9. Kristi, Nama, and Karen like to go work out at the recreation center. Kristi goes every other day, Nama goes every third day, and Karen goes every fifth day. They made an arrangement to carpool on the days they all go on the same day. If they all start today, find the next 3 times they will go together. Make a table to help you find the answers. **ee TE.**

10. 3 packages of invitations and 5 packages of party favors.

11. a. LCM = 180 items; 4 containers of erasers and 3 containers of pencils
    b. 180 students.

12. Greatest Common Factor. There will be either 5 or 6 people at the party. List multiples of 5 and multiples of 6. The first common multiple will be the first number of cookies that will serve both 5 or 6 people evenly, with no cookies left over.
    30 Cookies

13. Encourage your students to be careful when setting up this problem. They need to read carefully and notice that Randy counts as one of the "eaters" in this problem.
    24 Cookies

14. GCF(15, 36, 48) = 3
    LCM(15, 36, 48) = $2^4 \cdot 3^2 \cdot 5$

15. a. LCM = 24 items; 2 boxes and 3 cartons
    b. 24 students
    c. 10 boxes and 15 cartons.

10. For your birthday party, you are planning to invite 45 friends. Invitations come in packages of 15, and party favors come in groups of 9. What is the least number of packages of invitations and party favors you need to buy in order to have enough for each friend and nothing left over?**e**

11. Mr. Sweeney wants to reward all the students at his school for their behavior by giving them each a fancy pencil and an eraser. Erasers come in containers of 45, while pencils come in containers of 60.

    a. What is the least number of containers of each he needs to purchase in order to have enough to give each student a pencil and eraser?

    b. How many students are at Mr. Sweeny's school? **See TE.**

12. Thomas is baking cookies for a party. Alice, Brad, Carlos, and Diana told Thomas that they definitely plan to attend. Eric told Thomas that he would like to attend, but he is not sure he can make it. Thomas wants to bake enough cookies so that he and all his guests can have the same number of cookies, whether or not Eric shows up. What is the smallest number of cookies that Thomas can bake for his party? **30**

13. Randy is baking cookies for a family get-together at his house. He wants to bake enough cookies so that each person at the party, including himself, gets the same number of cookies. Randy knows that there are seven family members who will definitely come to the party. In addition, Randy has an aunt, an uncle, and two cousins who might or might not come. Because they are in the same family, either all four of these people will show up, or none of them will. What is the smallest number of cookies that Randy can cook if he wants to guarantee that everyone gets the same number of cookies, with no leftovers? **24 cookies**

14. Use a Venn diagram to find the GCF and LCM for 15, 36, and 48. **S**

15. For Junior Math Camp, the director had to purchase snacks for 100 students. She plans to buy granola bars and juice bottles. The granola bars come in boxes of 12, and juice bottles come in cartons of 8. She needs to purchase the smallest number of boxes of granola and cartons of juice bottles so that each student gets exactly one granola bar and one juice box with the same number of granola bars and juice boxes left over.

16. Spiral Review (6.1 E)
    2 groups with 14 desks; 4 groups with 7 desks; 7 groups with 4 desks; 14 groups with 2 desks.

17. Spiral Review (6.1 E)
    a. 8
    b. 167

Ingenuity

18. a. Answers will vary among any of the following pairs of numbers: (1,4), (2,4), (4,4). There are 3 such ordered pairs.
    b. Answers will vary among any of the following pairs of numbers: (1,12), (2,12), (3,12), (4,12), (6,12), (12,12), (3,4), (4,6). There are 8 such ordered pairs.
    c. Answers will vary among any of the following pairs of numbers: (1,120), (2,120), (3,120), (4,120), (5,120), (6,120), (8,120), (10,120), (12,120), (15,120), (20, 120), (24,120), (30,120), (40,120), (60, 120), (120, 120), (3, 40), (6,40), (10, 24), (12, 40), (8,15), (8, 30), (8,60), (5,24), (40,60), (30,40), (15,40), (15,24), (20,24), (24,30), (24,40), (24,60). There are 32 such ordered pairs.

An example of how to approach part (b):

Students may approach this problem by trying to make an exhaustive list, but this can often lead to missing answers. Encourage students to use prime factorization and to choose which factors to include in the pair of numbers to make a LCM of 4, 12 or 120. Let us look more closely at part (b). We know that the prime factorization for 12 is $2^2 3$ so one could think about factors of 12 in the following way.

Each factor can be represented as such: __ · __ where the first slot may or may not contain one or two 2s and the third slot may or may not contain a 3 (since we are looking for prime factors of 12 we break down into 2s and 3s, but if we are looking for another number we would have other slots, like 15 is composed of 3 and 5 as factors so our slots would have 3s and 5s in them). All of the (n, 12) pairs can be generated in this way, where n = __ · __.

Since each of the slots can have a particular factor or it cannot have a particular factor, you can think about the slots as lights that turn off and on, where the first slot has 3 different settings: high, low, and off. What are the different combinations of ways for the lights to be on? You can have both lights off (this represents $n = 1$), the first light can be on low and the second off ($n = 2$), or the first light can be on high and the second off ($n = 4$). You can see the results in the following table:

first light	second light	# of 2s	# of 3s	n
off	off	0	0	1
low	off	1	0	2
high	off	2	0	4
off	on	0	1	3
low	on	1	1	6
high	on	2	1	12

After students have discovered all of the pairs of the form (n, 12), they can use a similar method with what they know about LCMs to find all of the other pairs.

a. What is the fewest number of boxes of granola bars and cartons of juice bottles she needs to buy to have an equal amount of each?

b. How many students will this feed? Is it enough?

c. How many total boxes and cartons need to be purchased in order to feed all students and have the same number of granola bars and juice boxes left over?

Spiral Review:

16. Mrs. Waters has 28 desks in her classroom. She wants to arrange the desks into groups so that each group has the same number of desks. List all the possible number of desks that could be in each group.

17. What is the value of each expression below?

a. $48 - 5 \cdot (3+5)$ b. $3(9 - 2)^2 + 4(5)$

18. **Ingenuity:**

Suppose that m and n are positive integers such that the GCF of m and n is20, and the LCM of m and n is 1800. What are the possible values of m and n? Find as many possible values as you can.

19. **Investigation:**

The table below lists two variables m and n. Copy and fill out this table. In the column labeled GCF(m, n), write the GCF of m and n. In the column labeled LCM(m, n), write the LCM of m and n. As you fill out the table, what do you notice?

Investigation - SEE TABLE

m	n	GCF (m, n)	LCM (m, n)
3	7	**1**	**21**
4	6	**2**	**12**
5	15	**5**	**15**
8	15	**1**	**120**
10	36	**2**	**180**
20	48	**4**	**240**
30	50	**10**	**150**
30	65	**5**	**390**
77	81	**1**	**6237**
96	100	**4**	**2400**

20. **Challenges:**
    Ans: 2

m	n	GCF(m, n)	LCM(m, n)
3	7		
4	6		
5	15		
8	15		
10	36		
20	48		
30	50		
30	65		
77	81		
96	100		

20. **Challenge:**

The LCM of 1, 2, 3, ..., 98, 99 ends in how many zeros?

1. a.

36	
1	36
2	18
3	12
4	9
6	6

b.

64	
1	64
2	32
3	24
4	16
8	8

c.

57	
1	57
3	19

d.

29	
1	29

2. Answer will vary. They may include 56, 64, 72, 80, 88, etc.

3. a. Twenty is divisible by 4 (or 5). b. Four (or 5) is a factor of 20. c. Twenty is a multiple of 4 (or 5).

4.

	2	3	5	6	9	10
873	**No**	**Yes**	**No**	**No**	**Yes**	**No**
78	**Yes**	**Yes**	**No**	**Yes**	**No**	**No**
280	**Yes**	**No**	**Yes**	**No**	**No**	**Yes**

5. a. Composite, 3 x 29 b. Composite, 2 x 62 c. Prime d. Composite 11 x 11
e. Prime

6. a. $2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$ b. $2 \times 5 \times 5 \times 7 = 2 \times 5^2 \times 7$
c. $13 \times 13 = 13^2$ d. 67 Already prime

7. a. 5
b. 15
c. 12
d. 1

REVIEW PROBLEMS

1. List the factors of each of the following numbers using T-charts:

   a. 36 b. 64 c. 57 d. 29

2. List five multiples of 8 that are greater than 50.

3. Write a true statement for each of the following words (multiple, factor, and divisible) using the numbers 4, 5, and 20.

   a. ____________________

   b. ____________________

   c. ____________________

4. Using the table below, check to see if each of the numbers in the left column is divisible by 2, 3, 5, 6, 9, and 10.

	2	3	5	6	9	10
873						
78						
280						

5. Determine whether each number is prime or composite. If composite, write a factor pair that proves your answer.

   a. 87 b. 124 c. 91

   d. 121 e. 79

6. Find the prime factorization of each using a prime factor tree. Write your answer using exponents if possible. If not possible, explain why.

   a. 72 b. 350 c. 169 d. 67

7. Find the greatest common factor of each pair of integers using one of the methods studied in Chapter 3.

   a. 20 and 25 b. 45 and 60

   c. 12 and 36 d. 16 and 17

8. a. 24
   b. 90
   c. 42
   d. 60

9. a. GCF: 16 and LCM: 32
   b. GCF: 6 and LCM: 120
   c. GCF: 12 and LCM: 60

10. She can invite 23 people and with herself that would be 24 people.

11. They will clean the yard together on the 28th day.

12. There will be 6 people playing. Each will have 2 water guns and 3 water balloons.

13. She can make 12 bouquets. Each will contain 2 yellow, 3 red, and 5 blue balloons.

8. Find the least common multiple of each pair of integers using one of the methods studied in Chapter 3.

   a. 8 and 12 b. 10 and 18

   c. 14 and 21 d. 15 and 20

9. Find the greatest common factor and the least common multiple for each of the following factorizations.

   a. $2 \cdot 2 \cdot 2 \cdot 2$ and $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ GCF: LCM:

   b. $2 \cdot 2 \cdot 2 \cdot 3$ and $2 \cdot 3 \cdot 5$ GCF: LCM:

   c. $2 \cdot 2 \cdot 3 \cdot 5$ and $2 \cdot 2 \cdot 3$ GCF: LCM:

10. Chandra is planning to have a party. Cookies are sold in packages of six. Juice drinks are sold in packages of eight. She is trying to figure out how many friends she is able to invite so that each friend gets exactly one cookie and one drink with nothing left over. How many friends can Chandra invite to her party?

11. One day, Nathan and Jeremy decide to help their neighbor Carlos keep his yard clean over the summer. Since Jeremy has more free time, he is planning to continue cleaning the yard every 4 days, while Nathan is planning to clean the yard every 7 days. When will be the second time both Jeremy and Nathan clean Carlos' yard on the same day?

12. Christopher is having friends over for a fun day of water games. He has 12 water guns and 18 water balloons. If all the water guns and water balloons are handed out and each person has the same number of water guns and water balloons, how many people were playing? How many water guns and water balloons did each person have?

13. Veronica is making balloon bouquets for a party. She has a package of 24 yellow balloons, 36 red balloons, and 60 blue balloons. If she makes identical bouquets using all the balloons, what is the largest number of balloon bouquets she can make? How many balloons of each color will each bouquet have?

MATH EXPLORATIONS

Fractions

Note: the page numbers above correspond with the Student Edition

CHAPTER PREVIEW

Section 4.1 addresses several key aspects of fractions. The visual models that are linear, area, and discrete are presented through number line, paper folding, and sets. Equivalence of fractions are then reinforced visually and informally through the paper folding and then formalized as the Equivalent Fraction Property. Students simplify fractions and also create numerous equivalent fractions using the property. Many activities (e.g. Fraction Friend) are included in the Teacher's Edition for classroom use. Section 4.2 makes use of the linear and area models to compare and order fractions. Common denominators are also used to make comparisons but we encourage use of benchmarks of 0, ½ and 1 to get a sense of the magnitude of the fraction. Section 4.3 extends from proper fractions to mixed fractions and write these as improper fractions as well. The fractional notation of numerator and denominator can also be thought of as composed of a certain number of unit fractions. Addition and subtraction of fractions with common and uncommon denominators are presented in section 4.4. The area and linear models are again used to visualize the operations. Section 4.5 works with mixed fractions and addresses the regrouping that may be needed to add and subtract. A common denominator method as well as the least common denominator method are presented in this section.

Section 4.1 - Models for Fractions

Big Idea:

Recognizing and using fractional models to represent part of a group and part of a whole

Key Objectives:

- Use linear models, area models, and discrete models to represent fractions.
- Understand how to find equivalent fractions.
- Understand the process of simplifying a fraction and be able to write a fraction in simplest form.

Materials:

Paper for folding (each sheet should be the same size to represents one whole), grid paper, inch-fractions strips (pre-cut, halves, thirds, fourths, fifths, sixths, eighths) 9 per student

Pedagogical/Orchestration:

- This section presents fractions as parts of groups and wholes, with basic vocabulary.
- The Pizza Bread Stick Activity in the Launch, and the Folding Paper Activity in the lesson are excellent ways for students to see a visual representation of equivalent fractions.

Activity:

"Connect Three", "Fraction Friend", "Friendly Fractions: Learning with Licorice", "Hot Pizza" and "Tick-Tock Fractions"

Exercises:

Exercise 13 prerequisite of being able to tell time.

Exercises 17-19 foreshadows comparing and ordering fractions.

Vocabulary:

area model or brownie model, numerator, denominator, discrete model, equivalent, equivalent fractions, simplifying, simplest form

TEKS:

6.1(A,B); 6.12(B); 6.13(A,B) New: 6.2(E,F); 6.4(G)

Launch for Section 4.1:

Start the class by saying, "Today we will be making some visual fractions."

1. Make a line of students (12 is best). Tell them they are a pizza bread stick that needs to be divided into pieces.
2. Suppose there are_______people who want to split the bread stick. Have someone not in line divide the giant bread stick into ______ equal pieces. Ask the class what fraction of the whole bread stick each piece represents. It is a good foreshadowing of equivalent fractions that some students may say $\frac{1}{2}$ and some may say $\frac{6}{12}$.
3. Suppose ______ people want to divide the bread stick equally. Have someone do this. Again, ask the class what fraction of the whole bread stick each piece represents. Students will respond with $\frac{4}{12}$ or $\frac{1}{3}$.
4. Do the same process as above by dividing the bread stick for _____ people and then _____ people. Have different students do the dividing to get as many students involved as possible.
5. Ask for student volunteers to come to the board and write any fractions they saw that seemed to represent the same portion of the bread stick. For instance a student may go to the board and write $\frac{2}{12}$ $\frac{1}{6}$ and others. Tell students, "These types of fractions that are different forms of the same fraction are called 'equivalent fractions' because they are equal to each other. Today we will be focusing on writing equivalent fractions, so look for patterns that will make the job easier."

Connect Three

Materials:

Connect Three Board Game (from the Math Explorations 6th Grade CD)
Red/Yellow Chips (16)
2 Dice

Instructions:

1. Divide the 16 chips so that one player will have 8 red chips and the other will have 8 yellow chips.
2. Each player should roll 1 die; the player with the highest roll goes first.
3. Player 1 rolls the dice and makes a proper fraction using the numbers rolled.
4. Player 1 then finds its equivalent form on the board game and covers it with his/her chip.
5. Player 2 now repeats steps 3 and 4. *If a space is covered the player may remove the chip and replace it with their own.
6. If a player rolls double, he/she loses a turn.
7. A player wins the game by connecting three in a row vertically, horizontally or diagonally.

Connect Three Board

$\frac{8}{20}$	$\frac{7}{14}$	$\frac{12}{20}$	$\frac{4}{12}$
$\frac{4}{20}$	$\frac{9}{12}$	$\frac{8}{12}$	$\frac{12}{15}$
$\frac{5}{10}$	$\frac{9}{15}$	$\frac{6}{12}$	$\frac{3}{18}$
$\frac{20}{24}$	$\frac{5}{15}$	$\frac{6}{9}$	$\frac{3}{12}$

Fraction Friend Activity

Materials:

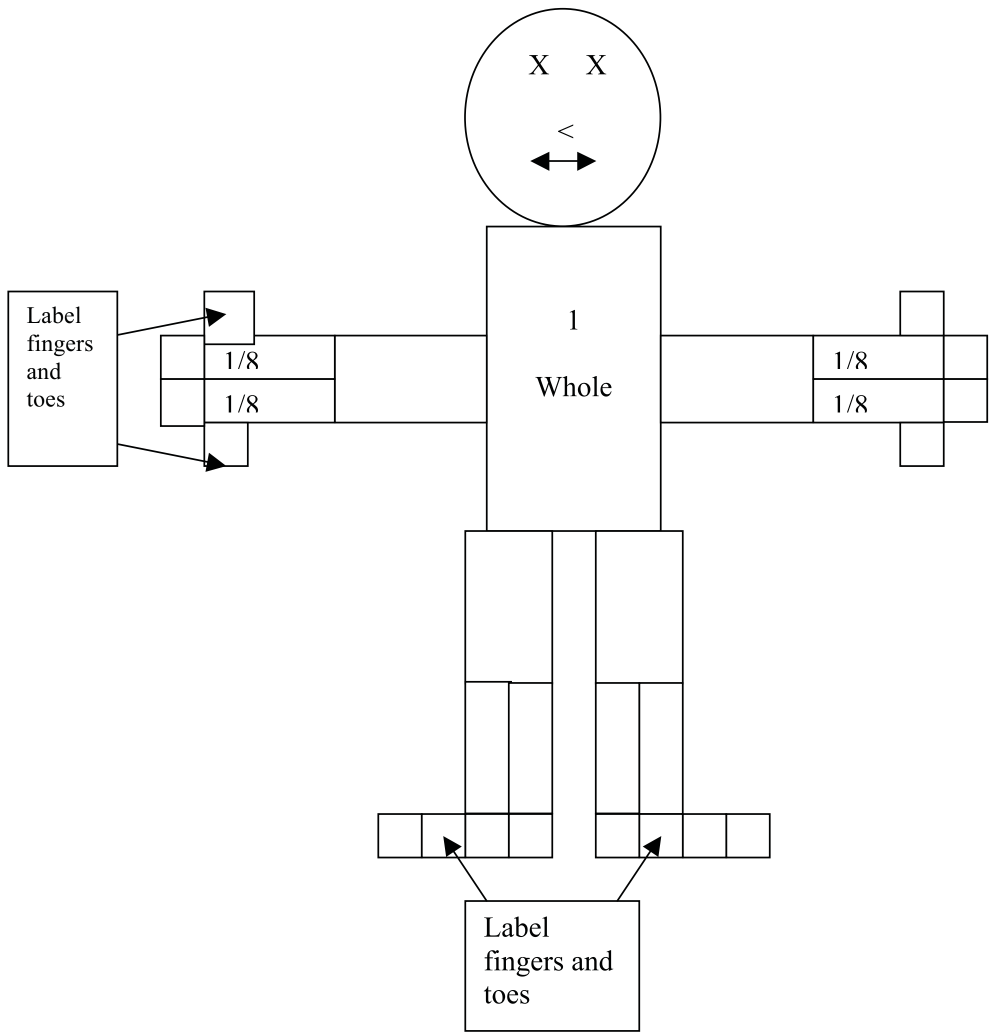
Construction paper (5 sheets of different color per student)
Scissors
Glue sticks
Markers or crayons
Yarn (optional)

Instructions:

1. Choose a color and discuss that it will represent the body or one whole. Label it as "1 whole" in big print.
2. Choose a second color. Fold it into fourths and cut them out. These parts represent the arms and legs. Label each part "$\frac{1}{4}$" in large print, then cut and glue them to the whole (body).
3. Choose a third color. Fold it into eighths by starting with $\frac{1}{2}$, then $\frac{1}{4}$, then $\frac{1}{8}$. Label each part "$\frac{1}{8}$" in large print. These will be the calves and forearms. Cut and glue 2- eighths to each fourth (arms and legs).
4. Choose a fourth color. Fold it into sixteenths. Start with folding the paper in half and repeat this four times to make sixteens ($\frac{1}{2}$, $\frac{1}{4}$. $\frac{1}{8}$, $\frac{1}{16}$). Label each part"$\frac{1}{16}$" then cut and glue to eighths (fingers and toes).
5. Using a gallon ice cream lid have students trace the circle and cut it out and use it to decorate a face for their fractions friend using only math symbols ^+ – # ' = # %....h. They can use yarn to add hair, mustache or eyelashes. Some students have used scraps to make boxes, ties, hats, etc. (using the math symbol criteria).

Extension: students can write the decimal and percent equivalent form for each part.

Question: How many "wholes" were used to create "Fraction Friend" ?


X X
<
1
Whole
Label
fingers
and
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1/8
1/8
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1/8
Label
fingers and
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Friendly Fractions: Learning with Licorice

Materials:

Rainbow colored licorice (3 different colors for each group of fractions)
Learning with Licorice worksheet
Plastic knives or scissors
Pencils
Map Pencils or crayons

Activity Instructions:

1. Teacher explains that "friendly fractions" are commonly used fractions (halves, thirds, fifths). We call the "friendly fractions" because other fractions can easily be made from them.
2. Have students take a licorice and cut it to the size of a row in the box. Now cut this in half. Place pieces in the second licorice box and sketch what you see on your worksheet. Shade in $\frac{1}{2}$. How many halves equal 1 whole?
3. Next, cut each half in half. How many pieces do you now have? So we see that $\frac{1}{2}$ of $\frac{1}{2}$ is equal to $\frac{1}{4}$. How many fourths will equal one whole? Place pieces in the third licorice box and sketch what you see on your worksheet. Shade in $\frac{1}{4}$. How many fourths equal one whole?
4. Divide your fourths in half. How many pieces do you have? We can make eighths out of fourths by dividing them in half. Place pieces in the licorice box and sketch what you see on your worksheet. Shade in $\frac{1}{8}$. How many eighths equal one whole?
5. How many pieces do you think we will have if we cut the eighths in half? Prove it! Place pieces in the licorice box and sketch what you see on your worksheet. Shade in $\frac{1}{16}$. How many sixteenths equal one whole?
6. What pattern do you see in your fractions? What can you conclude about the denominator as the fraction is being divided in half? We say that these fractions are "friendly" because:

   $\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$ $\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$ $\frac{1}{2}$ of $\frac{1}{8} = \frac{1}{16}$
7. Without using your model do know what is $\frac{1}{2}$ of $\frac{1}{16}$?
8. Divide the pieces amount team members and eat your licorice.

Licorice Box
1 Whole

Halves

Ex. $= \frac{1}{2}$ $\frac{2}{2} =$ 1 whole

= ____ ____ = 1 whole

= ____ ____ = 1 whole

= ____ ____ = 1 whole

*Repeat steps 1-6 with **"Thirds"** and **"Fifths"**, using a different color licorice for each.
Thirds = 1/3; ½ of 1/3 = 1/6, ½ of 1/6 = 1/12 **Fifths** = 1/5; ½ of 1/5 = 1/10;

Thirds

=____ ____ = 1 whole

=____ ____ = 1 whole

=____ ____ = 1 whole

Fifths:

=____ ____ = 1 whole

=____ ____ = 1 whole

Hot Pizza Activity

Materials:

Hot Pizza Worksheet
Crayons
Ruler
Paper

Activity Instructions:

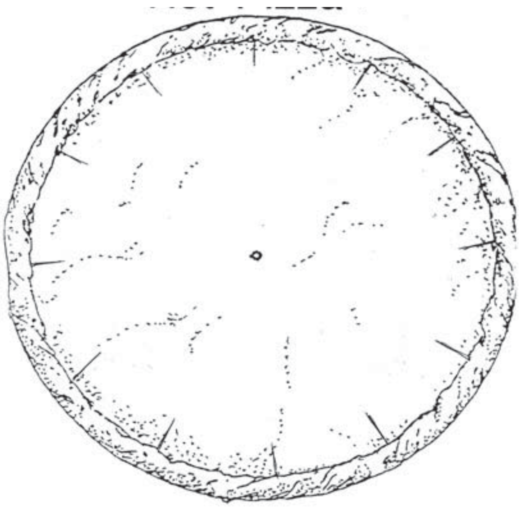
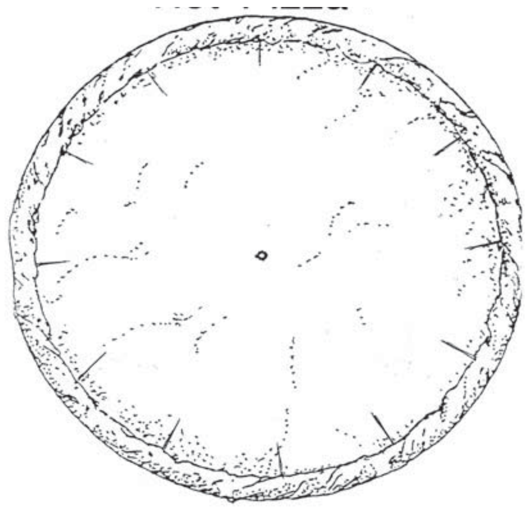
Each student should have a worksheet and students should work with a partner.

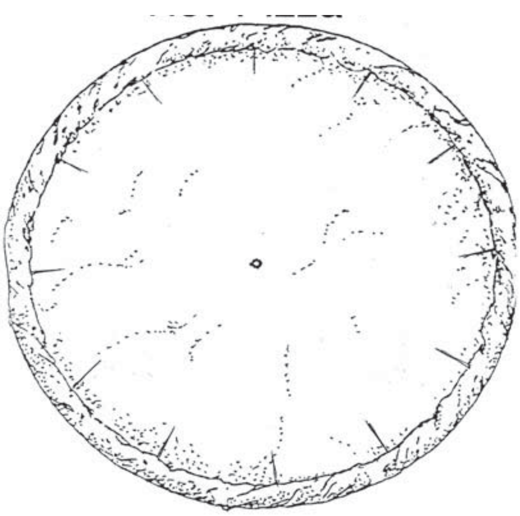
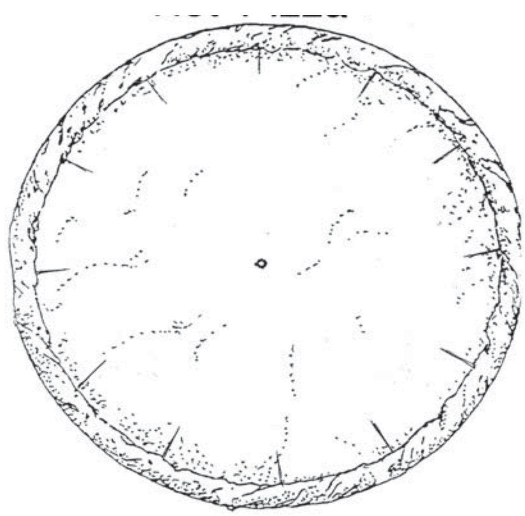
1. Instruct student A to divide the pizza into fourths and shade in $\frac{1}{4}$ of the pizza. Next have student B cut the pizza into twelfths and shade $\frac{3}{12}$ of the pizza. What does this say about $\frac{1}{4}$ and $\frac{3}{12}$?
2. Now instruct student A to divide and shade in $\frac{1}{2}$ of the next pizza. Student B is to divide and shade $\frac{3}{9}$ of the pizza. What can you conclude about $\frac{1}{2}$ and $\frac{3}{9}$?
3. Student A divides and shades in $\frac{1}{2}$ of the next pizza and student B shades $\frac{3}{6}$ of the pizza.
4. As a final step, student A shades $\frac{2}{3}$ of the final pizza and student B shades $\frac{8}{12}$ of the pizza.

Extension: Have students divide into groups of 4. Have each group draw 3 equivalent factions for a given fraction on butcher paper with crayons or markers. Every group should be able to explain their drawings. Do you notice anything about the numerator and denominator of the equivalent fractions that each group chose? Can each group find one more equivalent fraction for each of the other groups without having to draw a picture?

Name:____________________________

 Hot Pizza 

Tick-Tock Fractions Activity:

Materials:

3 crayons (different colors)
Clock Worksheet
Rulers

Activity Instructions:

Let's look at how fractions are used with the concept of time. Remember that 60 minutes is equal to one hour. Look at your clock.

1. What part of an hour is 30 minutes? Shade that part one color. What do you notice? Did you notice that 30 minutes is 30 parts out of 60 parts or 30/60 of an hour? We often say 30 minutes is half an hour. That means that 30/60 = $\frac{1}{2}$.
2. What fraction of an hour is 15 minutes? Shade that part a second color. 15/60 is correct. This fraction represents 15 minutes out of an hour or 60 minutes total. If we think of 15 minutes as a quarter of an hour, there are four quarters in one hour so $\frac{1}{4}$ is also correct. 15/60 = $\frac{1}{4}$.
3. What fraction of an hour is 20 minutes? Shade in a 20 minute representation on your clock using a third color. Can you find two ways to write the fraction?
4. Can you write 10 minutes as a fraction of an hour without drawing a clock? Are there two ways to express this fraction?

Clock Worksheet
Practice Exercises

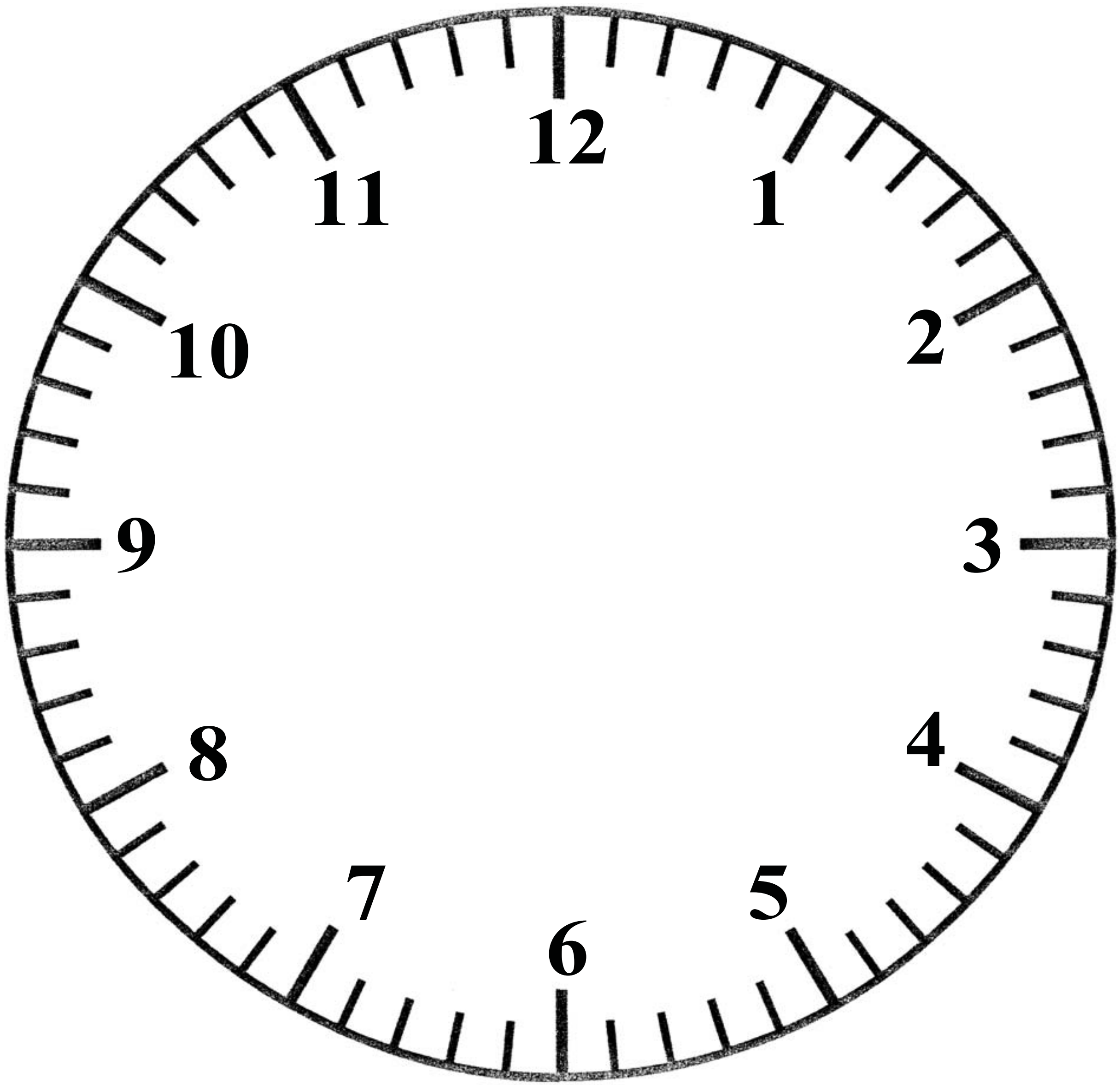
1. Jeremy practiced juggling for 40 minutes. For what fraction of an hour did Jeremy practice?

2. Daniel practiced trombone for 45 minutes. For what fraction of an hour did Daniel practice?

3. Mike played the piano for 1.5 hours. For how many minutes did Mike play?

4. Nicole spent $\frac{3}{4}$ of a of an hour making dinner. After dinner, Nathan spent $\frac{1}{3}$ of an hour cleaning up. How long did the two of them take in their work? Write your answer in minutes and as a fraction.

Clock Worksheet



Activity

Brainstorm a list of fractions on the board. Have each student write out in words an explanation of what a fraction is. (Brainstorm on the board, using words and then symbols, the fractions and a description of how they are used, i.e., one third of a cup of sugar. This is a review of 5th grade concepts.)

In discussing visual models of fractions, we are stressing how to describe and talk about them in words first. The numerical names, such as $\frac{1}{2}$ or $\frac{2}{3}$, will come on the next page.

DECIMALS, FRACTIONS AND PERCENTS

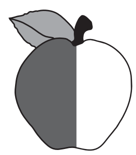
4

SECTION 4.1 MODELS FOR FRACTIONS

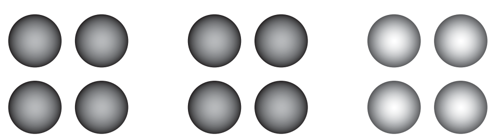
In this section you will examine some models that represent fractions.

First, think of examples of fractions that you have seen.

There are two basic ways to represent fractions: as a part of a whole or part of a group. Picture one apple divided into two equal parts. The shaded part represents half the apple.

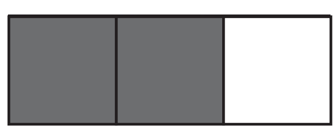


Next, divide a number of marbles into three equal groups. Two-thirds of the marbles is two of these three equal groups:



Now think of a fraction as part of a pan of brownie. This is called the **area** or **brownie model** for a fraction. Pictures representing the numbers one-half and two-thirds can look like these:

and



Of course, the pan need not always be a rectangle.

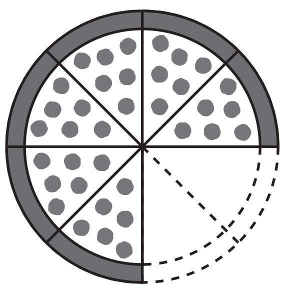
The missing part of the pizza can be described as 2 out of 8 slices of pizza or $\frac{2}{8}$. Don't be too quick to try to get them to see that it is also $\frac{1}{4}$ of the pizza.

We write the two fractions mathematically as

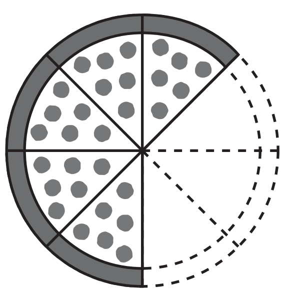
$$\text{One half} = \frac{1}{2} \qquad \text{Two thirds} = \frac{2}{3}$$

We use two numbers in writing a fraction, the **numerator** and the **denominator**. The numerator is above the denominator; it indicates how many parts are shaded. "Numerator" comes from the same root as "number." It counts the number of parts. The denominator tells how many parts the whole or group is divided into. "Denominator" comes from the same root as "name." It names the parts that the whole is divided into, like halves or thirds.

If you've ever had pizza, you probably noticed that it is already divided into several pieces that are roughly equal. Draw a circle to represent a pizza and draw lines to subdivide it into 8 equal pieces. Depending on how hungry you are, you choose to eat a certain number of pieces of pizza. You might have just two pieces, or two eighths of the pizza:



If you're a little hungrier, you might have three pieces, or three-eighths of one pizza:



Discuss with the students that the AREA MODEL is not restricted to just rectangles or brownies. For example, we can look at other geometric shapes like circles that we see in pizzas. When this book refers to the AREA MODEL, we mean looking at the area of the shape but not restricting ourselves to any particular shape. With that said, we often use the rectangle as a convenient shape.

Activity: Folding Paper

Watch to see if students use the same size sheet to represent the whole if they compare the parts or fractions of these wholes with each other. The issue is discussed on the next page in the text.

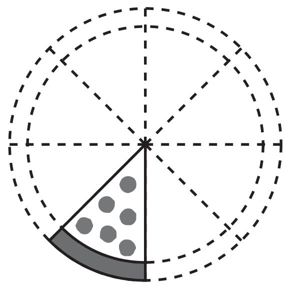
Equivalence in fractions is an essential concept in understanding how fractions work. Don't assume all your students understand equivalence until they can demonstrate it with **Folding Paper** in this section.

Step 2: We fold the paper twice to make fourths. All four parts on the sheet are equal to $\frac{1}{4}$th. The fraction that represents 3 of these parts is $\frac{3}{4}$. There are two possible fractions that represent 2 parts of the paper: $\frac{2}{4}$ or $\frac{1}{2}$

Step 3 & 4:

Let your students record how many folds it takes to half a sheet, fold it into fourths, eighths and sixteenths. Then ask them to compare the number of times they folded the sheet to the number of parts they produced with that fold. Students may notice that as you fold the paper in half each time that the number of pieces increases as a power of two: 2, 4, 8, 16, etc. Later your students will discover they were producing fractional equivalents to the negative powers of two.

And if you're extremely hungry, you might eat seven pieces, or seven-eighths of a pizza.



Use the area model to draw the fractions below. Draw the whole as a rectangle because it is easier to divide into equal pieces. For each fraction, identify the numerator and denominator. Then write the fraction mathematically.

a. Three-fourths
b. Two-fifths
c. One-fifth
d. Three-tenths

ACTIVITY: FOLDING PAPER

Materials: You will need several sheets of paper for this activity. Each sheet represents one whole.

Step 1: Fold the paper to represent the number $\frac{1}{2}$. Write $\frac{1}{2}$ on each of the two parts of the folded paper. Is there more than one way to represent $\frac{1}{2}$?

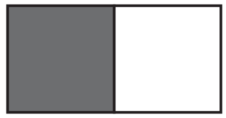
Step 2: Use a new sheet to create $\frac{1}{4}$. How many parts equal to $\frac{1}{4}$ are there in the whole sheet of paper? What fraction represents three of these parts? What represents two parts of the paper?

Step 3: Use a new sheet of paper to make a folded piece that has eight equal parts. Identify and make a list of as many fractions involving the denominator 8 as you can. Which of these fractions represent the same fractional part as the fractions in Step 1 and 2 with different denominators?

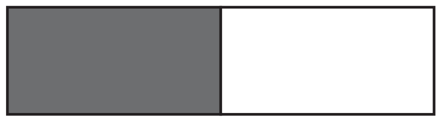
Step 4: Fold this same sheet of paper once more to make sixteenths. How many times did you fold the paper?

They are both 2 parts out of 4 parts. The difference is the size of the whole or “1”. Ask your students for another example of 2 parts out of 4 parts that looks different.

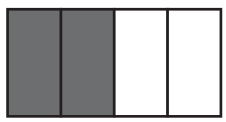
It is possible for two fractions with different numerators and denominators to represent the same amount. Consider the following example:

$\frac{1}{2} =$ 

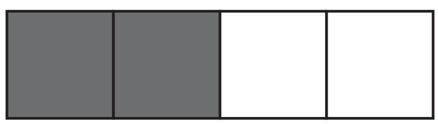
Notice that we can draw a larger picture, but it still represents the same part of the whole.

$\frac{1}{2} =$ 

If we divide a whole into 4 equal parts, 2 of the 4 parts can be written as the fraction

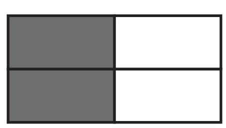
$\frac{2}{4} =$ 

or we can draw it as the larger picture

$\frac{2}{4} =$ 

Are these both a representation of $\frac{2}{4}$? They do not look the same. So what is the difference?

Shading 2 equal parts out of 4 is equivalent to shading 1 part out of 2. This means the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are **equivalent**. Another way to show that the fraction $\frac{1}{2}$ is equivalent to the fraction $\frac{2}{4}$ is to take the picture representing $\frac{1}{2}$ and draw a horizontal slice as shown below:

$\frac{2}{4} =$ 

Activity: Equivalent Fraction Chart

As the class compiles lists of equivalent fractions, record these on poster paper or on the board so that they can be kept as a reference. Take for example the chart below:

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$
$\frac{2}{4}$	$\frac{2}{8}$	$\frac{6}{8}$	$\frac{2}{16}$	$\frac{6}{16}$	$\frac{10}{16}$	$\frac{14}{16}$
$\frac{4}{8}$	$\frac{4}{16}$	$\frac{12}{16}$				
$\frac{8}{16}$						

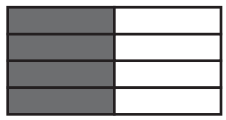
Example 1

This discussion can easily be one you lead orally.

The whole class has 12 students. The fraction of the class that is girls is $\frac{8}{12}$ and the fraction of the class that is boys is $\frac{4}{12}$. Ask if these fractions are equivalent to simpler fractions, that is, a fraction with a smaller denominator? Yes, $\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$ and $\frac{4}{12} = \frac{2}{6} = \frac{1}{3}$. Add these to the class chart.

The horizontal slice doubles the numerator and also doubles the denominator, the number of parts the whole is divided into.

Suppose we make three horizontal cuts in the original rectangular model for $\frac{1}{2}$ to form equal sized pieces. What fraction is shaded?



Like the example above, the picture represents both $\frac{1}{2}$ and $\frac{4}{8}$. These fractions are equivalent. We write $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$ because the fractions represent the same part of a whole.

Remember, the "whole" is not just a geometric shape that represents one whole, like a circle or a rectangle, subdivided into equal parts. For example, the whole might be a class that has 8 girls and 8 boys. What fraction of the class is female? Male?

ACTIVITY: EQUIVALENT FRACTION CHART

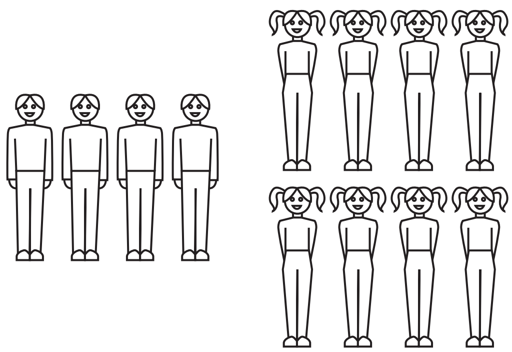
Start a chart of equivalent fractions. In each column, write fractions that are equivalent to each other. Be sure to use the fractions from the paper folding activity as well as any from now on in our discussions. You can also use a number line and label the corresponding points with the equivalent fractions. This may be a useful reference as you work with fractions in the sections to follow.

EXAMPLE 1

A class of 12 consists of 4 boys and 8 girls. We know that the fraction of the class that is male can be written $\frac{4}{12}$ and the fraction of the class that is female can be written $\frac{8}{12}$. Write another way to express the fraction of the class that is boys as well as the fraction of the class that is girls, using equivalent fractions.

SOLUTION

If you said $\frac{4}{12} = \frac{1}{3}$ as another way to express the fraction of the class that represents the boys, then you were correct. In the original fraction, the denominator 12 represented the number of people in the class and the numerator 4 represented the number of boys in the class. Another way to write the fraction of the class that is girls is $\frac{8}{12}=\frac{2}{3}$.



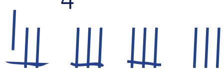
Remember, two fractions that represent the same part of a whole are called **equivalent fractions**.

EXAMPLE 2

Sandra buys a pack containing a dozen pencils. She sharpens $\frac{3}{4}$ of the pack of pencils. How many pencils did Sandra sharpen?

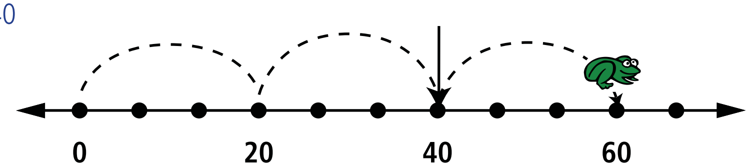
This is an example of the **discrete model** for fractions. Discrete means being made up of individual parts, just like groups of students. In this model, we assume we have a collection or group of n objects and subgroup of m objects. The fraction $\frac{m}{n}$ represents the part of the whole group that is in the subgroup. Though $\frac{4}{12} = \frac{1}{3}$ and the two fractions are equivalent, we do not want to leave the impression that there are only 3 students in the class now and that the number of boys decreased from 4 to 1.

Example 2: $\frac{3}{4}$ of 12 = 9

 = 9 pencils

Problem 1:

$\frac{2}{3}$ of 60 = 40



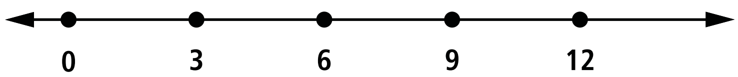
She has raised \$40. She needs to raise to \$20 more.

EXPLORATION

Students should discover or remember that multiplying the numerator and denominator by the same number forms a new equivalent fraction, i.e., $\frac{1}{2} = \frac{\hat{}1h\hat{}4h}{\hat{}2h\hat{}4h} = \frac{4}{8}$. Some will prefer the picture explanation, but the majority should be using the symbolic method. The symbolic method is useful for computation.

A number line is another way to find equivalent fractions.

You can use a number line and make 4 jumps like this:



Notice that if each jump represents a grouping of the dozen pencils into four groups, then one jump lands at 3 pencils, two jumps at 6, 3 jumps land on 9 pencils, and 4 jumps land on 12 pencils. 9 pencils is equivalent to $\frac{3}{4}$ of the dozen pencils. We can say $\frac{9}{12} = \frac{3}{4}$.

PROBLEM 1

Pat wants to buy a video game that costs \$60. So far, Pat has raised $\frac{2}{3}$ of the cost. How much money has she raised? How much more money does she need to raise?

EXPLORATION

Find three equivalent fractions for each of the fractions below. You may use paper folding or any other model to visually show your fractions are equivalent fractions.

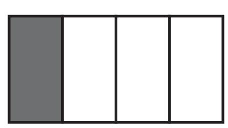
a. $\frac{1}{2}$ b. $\frac{1}{8}$ c. $\frac{1}{4}$ d. $\frac{2}{5}$ e. $\frac{2}{3}$

Do you see a pattern when two fractions are equivalent? How can you make an equivalent fraction from a given fraction without a model?

In general, we can find equivalent fractions by multiplying the numerator and denominator by the same number. For example,

$$\frac{1}{4} = \frac{(2)(1)}{(2)(4)} = \frac{(1)(2)}{(4)(2)} = \frac{2}{8}.$$

Pictorially,

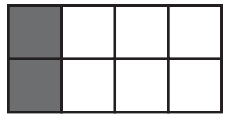
$\frac{1}{4} =$ 

Again, a basic concept in understanding fractions is the ability to simplify them so that the numerator and denominator are relatively prime. Although the concept of relatively prime numbers is not discussed until Section 8.2, these exercises should help all students either attain or reclaim the ability to recognize when a fraction is not simplified and to simplify it.

Sometimes we say that a fraction is in simplest form when you cannot simplify it any more. However, this statement is not precise. How do you know you cannot simplify it any more? When factoring the numerator and denominator into their prime factorizations, if they have no factors in common, then the fraction is in simplest form.

In simplifying a fraction, you can think of it two ways. One is to divide the numerator and denominator by the same factor, repeatedly if necessary. For example: $\frac{12}{36} = \frac{12 + 2}{36 + 2} = \frac{6}{18} = \frac{6 + 2}{18 + 2} = \frac{3}{9} = \frac{3 + 3}{9 + 3} = \frac{1}{3}$. However, a more straightforward way of using Property 4.1 is as follows: $\frac{12}{36} = \frac{62}{182} = \frac{6}{18} = \frac{32}{92} = \frac{3}{9} = \frac{13}{33} = \frac{1}{3}$

Multiplying the numerator and denominator by 2 has the effect of doubling the number of slices:

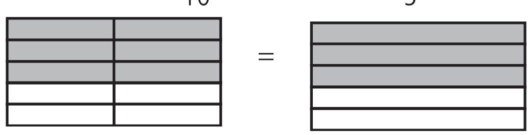
$\frac{1}{4} = \frac{2}{8} =$ 

Multiplying the numerator and denominator by the same number changes the number of shaded parts and the total number of parts by the same factor, yielding an equivalent fraction.

PROPERTY 4.1: EQUIVALENT FRACTION PROPERTY
For any number a and nonzero numbers k and b $$\frac{a}{b} = \frac{k \cdot a}{k \cdot b} = \frac{a \cdot k}{b \cdot k} = \frac{ak}{bk}$$

We have generated equivalent fractions using the area model by dividing a given representation into smaller equal pieces, and converting 1 part out of 4 parts into 2 parts out of 8 parts. $\frac{1}{4} = \frac{2}{8} = \frac{3}{12} = \frac{4}{16}$ and so on. Notice that in this case, the new denominator is a multiple of the original denominator. Must this always be true?

Many times we will want to find an equivalent fraction with a smaller denominator, if possible. We call this process **simplifying** a fraction. We will do this by using Property 4.1 in reverse. For example, to simplify the fraction $\frac{6}{10}$, we first recognize that $\frac{6}{10} = \frac{2 \cdot 3}{2 \cdot 5}$. The numerator 6 and the denominator 10 have a *common factor*, 2. Dividing both the numerator and the denominator by the common factor of 2 produces an equivalent fraction. Using the Equivalent Fractions Property, we see that $\frac{6}{10}$ is equivalent to $\frac{3}{5}$.



$$\frac{6}{10} = \frac{3 \cdot 2}{5 \cdot 2}$$

$$\frac{6}{10} = \frac{3}{5}$$

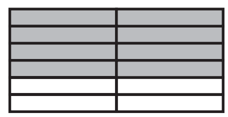
So, we have simplified $\frac{6}{10}$ to the form $\frac{3}{5}$. Notice that we are using the Equivalent Fraction Property, $\frac{a \div k}{b \div k} = \frac{a}{b}$. A fraction is said to be in **simplest form** if the numerator and denominator have no common factors except 1, in other words, a and b are relatively prime.

EXAMPLE 3

Write $\frac{8}{12}$ in simplest form.

SOLUTION

Look at a rectangular model with $\frac{8}{12}$ shaded in.



Notice that we can also view this fraction as $\frac{4}{6}$.

$$\frac{8}{12} = \frac{4 \div 2}{6 \div 2} = \frac{4}{6}$$

However, notice that this fraction can also be written equivalently as $\frac{2}{3}$.

$$\frac{8}{12} = \frac{4}{6} = \frac{2 \div 2}{3 \div 2} = \frac{2}{3}$$

Once you find the factors of each numerator and denominator as shown below,

8	
1	8
2	4

12	
1	12
2	6
3	4

Problem 2

1. $\frac{2}{5}$ 2. $\frac{1}{3}$ 3. $\frac{3}{7}$ 4. $\frac{9}{10}$

EXERCISES

1. Have students share different ways they find equivalent fractions. Answers may vary. Some answers may include:
   a. $\frac{3}{5}, \frac{12}{20}, \frac{18}{30}, \frac{24}{40}$ b. $\frac{2}{18}, \frac{3}{27}, \frac{4}{36}, \frac{5}{45}$

   c. $\frac{8}{16}, \frac{12}{24}, \frac{16}{32}, \frac{2}{4}$ d. $\frac{4}{5}, \frac{16}{20}, \frac{24}{30}, \frac{32}{40}$

   e. $\frac{5}{6}, \frac{10}{12}, \frac{30}{36}, \frac{45}{54}$ f. $\frac{1}{2}, \frac{2}{4}, \frac{12}{24}, \frac{18}{36}$

2. Answers may vary. Some answers may include: $\frac{2}{20}, \frac{3}{30}, \frac{4}{40}$, etc.
4. Some of these fractions have several simpler equivalent fractions. Watch to see if your students understand when a fraction is in its simplest form. If they don't, discuss what criteria they might use to determine whether a fraction is in its simplest form. (cf = common factor; gcf = greatest common factor)
   a. $\frac{3}{4}$; cf 2, 3, 6 and gcf 6   c. $\frac{4}{5}$; cf 2, 3, 6 and gcf 6

   b. $\frac{3}{4}$; cf 7 and gcf 7   (Rest of the parts can be solved in a similar way.)

You can identify the greatest common factor, GCF, of the numerator and the denominator and use it to simplify the given fraction by applying the equivalent fraction property so that $\frac{8}{12} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{2}{3}$.

PROBLEM 2

Write the following problems in simplest form.

1. $\frac{24}{60}$ 2. $\frac{14}{45}$ 3. $\frac{27}{63}$ 4. $\frac{9}{10}$

The GCF is useful in finding the simplest equivalent fraction.

We will illustrate the usefulness of the prime factorization in computing the simplest form for a fraction. Examine the fraction $\frac{108}{168}$. Knowing the GCF of 108 and 168, we can write the fraction using the equivalent fraction property as

$$\frac{108}{168} = \frac{2 \cdot 2 \cdot 3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3 \cdot 7} = \frac{(12)9}{(12)14} = \frac{9}{14}$$

Simplification of fractions is only one useful application of prime factorization.

EXERCISES

1. Find three equivalent fractions for each of the fractions below. You may use paper folding or any other model to determine the equivalent fractions.

   a. $\frac{6}{10}$ b. $\frac{1}{9}$ c. $\frac{4}{8}$ d. $\frac{8}{10}$ e. $\frac{15}{18}$ f. $\frac{6}{12}$

2. Find four fractions that are equivalent to $\frac{1}{10}$.

3. Find three equivalent fractions for $\frac{14}{24}$.

5. a. $\frac{1}{2}$ c. $\frac{5}{8}$ e. $\frac{6}{18}$ or $\frac{1}{3}$

b. $\frac{1}{4}$ d. $\frac{3}{8}$ f. $\frac{4}{18}$ or $\frac{2}{9}$

6. A represents $\frac{1}{4}$

B represents $\frac{1}{16}$

C represents $\frac{1}{32}$

D represents $\frac{1}{8}$

E represents $\frac{1}{16}$

F represents $\frac{1}{8}$

G represents $\frac{1}{16}$

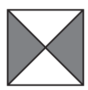
8. Juan practiced trombone for $\frac{45}{60}$ of an hour. This fraction is equivalent to $\frac{3}{4}$. Answers may vary for explanations.

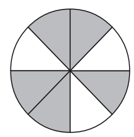
4. For each of the following fractions, find a common factor in the numerator and denominator. Then, use the Equivalent Fraction Property to simplify the fraction.

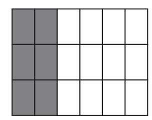
a. $\frac{18}{24}$ c. $\frac{24}{30}$ e. $\frac{35}{45}$

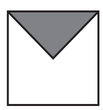
b. $\frac{21}{28}$ d. $\frac{28}{42}$ f. $\frac{51}{72}$

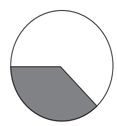
5. What fraction is represented by the shaded portion of each figure below?

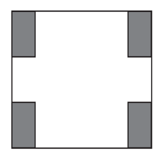
a. 

c. 

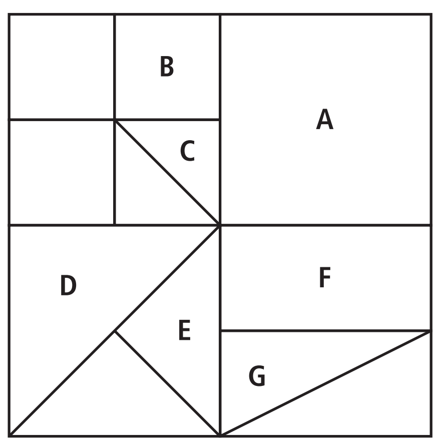
e. 

b. 

d. 

f. 

6. Determine the fraction that represents each of the labeled regions assuming the large square represents the whole or 1.



For the following exercises, when applicable, use an area or linear model to solve.

7. Ans: $\frac{1}{3}$

9. $\frac{20}{32}$ of the class is female, $\frac{5}{8}$ is an equivalent fraction. $\frac{12}{32}$ of the class is male, $\frac{3}{8}$ is an equivalent fraction. Answers may vary for justifications.

10. 2 kittens. Have the students use the fraction bar model.

For Exercises 11 and 12 have students share approaches that are different and reflect on how they get their answers.

11. 18
12. 240

13. a. 2:10 p.m. b. 1:55 p.m. c. 2:00 p.m. have students use an analog clock to explain their answer.

14. Answers may vary for equivalent fractions.

15. $\frac{4}{24}$ students do not like peanut butter. Answers may vary, but some answers may include: $\frac{16}{48}, \frac{1}{6}, \frac{3}{18}, \frac{2}{12}$. At this stage, some students might recognize or remember that equivalent fractions with smaller denominators are preferable to equivalent fractions with larger denominators

7. Jonathan practiced playing the trumpet for 20 minutes. For what fraction of an hour did he practice? Write this fraction in simplest form.

8. Juan practiced trombone for 45 minutes. For what fraction of an hour did Juan practice? What is the simplest equivalent fraction to your answer? Explain.

9. In Mrs. Garcia's class there are 20 girls and 12 boys.

   a. What fraction of the class are girls?

   b. What fraction of the class are boys?

   c. Write both answers as equivalent fractions by simplifying your answers.

10. Mindy, Andrew's cat, had 8 kittens. He says that $\frac{1}{4}$ of them are white. Use the fraction bar model to show how many kittens are white.

11. Sandra has a box of M & M candies. She counted 6 brown M & Ms and says that is $\frac{1}{3}$ of all the M & Ms she has left. How many M & Ms does Sandra have?

12. A new spirit shirt is being sold at Brilliant Middle School. Of the 360 sixth grade students, $\frac{2}{3}$ of them have purchased the new shirts. How many sixth graders have purchased the new shirts? Use a fraction bar model to help you find the answer.

13. If the time is 1:40 p.m. right now, what time will it be in the following situations: (Hint: You might want to use the circular model to help you.)

    a. half an hour later?

    b. $\frac{1}{4}$ of an hour later?

    c. $\frac{1}{3}$ of an hour later?

14. Find an equivalent fraction for $\frac{4}{12}$ that has a larger denominator. Then find 2 equivalent fractions for $\frac{4}{12}$ that have smaller denominators.

15. A class with 24 students has 4 students who do not like peanut butter. What fraction of the class does not like peanut butter? Give at least 3 equivalent fractions for this answer.

18. Answers may vary, some answers may include: $\frac{5}{10}$ which is equivalent to $\frac{1}{2}$, $\frac{7}{15}$, $\frac{8}{15}$, $\frac{9}{20}$, $\frac{11}{20}$.
19. Answers may vary, some fractions in between $\frac{1}{2}$ and $\frac{2}{3}$ are $\frac{3}{5}$, $\frac{4}{7}$ and $\frac{5}{9}$. Some students will convert $\frac{1}{2}$ and $\frac{2}{3}$ to equivalent fractions and use those to compute fractions that are in between $\frac{1}{2}$ and $\frac{2}{3}$. For example, a student might realize that $\frac{1}{2}$ and $\frac{2}{3}$ are equivalent to $\frac{6}{12}$ and $\frac{8}{12}$, which would lead them to discover that $\frac{8}{12}$ is in between those two values.

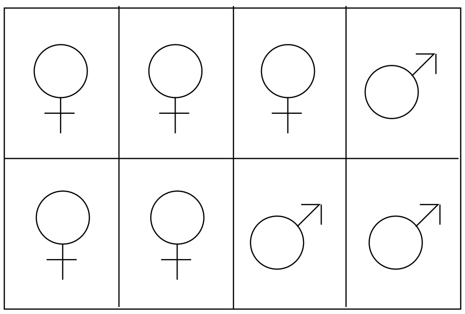
20. Spiral Review (6.5A) Answer: a.

21. Spiral Review (7.1A)

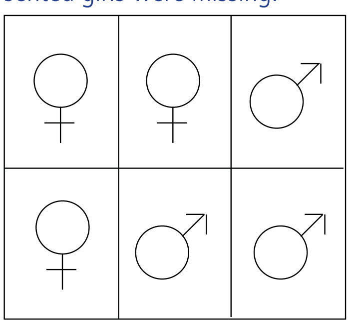
    -35, -27, -6, 50, 52

22. **Ingenuity:**

Suppose we divide a rectangle into eighths and use this picture to represent Ms. Garrett's class:



On Friday, there were exactly as many girls as there were boys in class; therefore, two of the eighths that represented girls were missing:



But we know that there were six girls missing; therefore, each piece of the rectangle must represent three students. Since there were six "pieces" or six eighths in class on Friday, there were 6 x 3 = 18 students in class that day. Note that this picture enables us to visualize the problem and solve it efficiently without setting up an equation and solving. The area model or "fudge model" for fractions is a very potent tool that can allow us to solve problems that might otherwise require us to write and solve equations.

16. Write the following fractions in simplest form, if possible.

a. $\frac{42}{72}$ b. $\frac{125}{1000}$ c. $\frac{25}{100}$ d. $\frac{7}{5}$ e. $\frac{17}{51}$

f. $\frac{12}{13}$ g. $\frac{9}{16}$ h. $\frac{12}{25}$ i. $\frac{15}{90}$ j. $\frac{8}{52}$

17. Use paper folding to discover a fraction that represents a part that is larger than $\frac{1}{4}$ and less than $\frac{1}{2}$. Find 2 more fractions between $\frac{1}{4}$ and $\frac{1}{2}$.

18. Find and write a fraction that represents more than $\frac{2}{5}$ and less than $\frac{3}{5}$ of a whole. Explain how you found your answer and why your fraction is correct. Find two more fractions between $\frac{2}{5}$ and $\frac{3}{5}$.

19. Find and write a fraction that represents more than $\frac{1}{2}$ and less than $\frac{2}{3}$. Show that your answer is right.

Spiral Review:

20. Mrs. Murphy handed out 28 test booklets equally among 7 groups of students. Which equation can be used to find t, the number of test booklets each group received?

a. $t = 28 \div 7$ c. $t = 28(7)$

b. $t = 28 - 7$ d. $t = 28 + 7$

21. List the following integers in order from least to greatest.

52 50 -6 -27 -35

22. **Ingenuity:**

Of the students in Ms. Garrett's algebra class, exactly $\frac{3}{8}$ are boys and $\frac{5}{8}$ are girls. On Friday, six girls were absent due to a basketball tournament. None of the boys were absent. That day, there were exactly as many girls as there were boys in class. How many students were in Ms. Garrett's algebra class on Friday?

23. **Investigation:**
The purpose of this Investigation is to foreshadow decimals and percents, which we will discuss in Chapter 5.

(a) 50

(b) 25

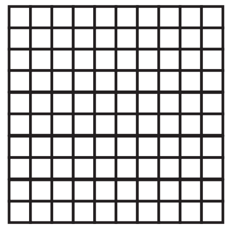
(c) 75

(d) 20

(e) 30

23. **Investigation:**

Willy makes an enormous chocolate bar in the shape of a square composed of 100 bite-size pieces.



He then eats a certain fraction of the bar. For each of the following fractions, determine how many of the 100 bite-size pieces Willy would eat if he ate the given fraction of the bar.

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{3}{4}$

d. $\frac{1}{5}$

e. $\frac{3}{10}$

Section 4.2 - Comparing and Ordering Fractions

Big Idea:

Identify benchmark fractions. Compare, and order fractions.

Key Objectives:

- Determine if a fraction is closest to 0, ½ or 1.
- Compare two fractions to see if they are equal or if one is larger than the other.
- Order fractions from largest to smallest or visa versa.

Materials:

Paper for folding (sentence strips or adding machine tape) Grid paper, Fraction Chart from the CD

Pedagogical/Orchestration:

- Students will practice using the linear model for fractions through paper folding, and using the area model for fractions by shading regions on a grid. Students can be shown the Fraction Chart from the CD as it is an extension of the linear model that makes ordering common proper fractions visually very simple.

Activities:

Fraction Chart, "Tic-Tac-Toe"

Vocabulary:

fraction, equivalent, rounding, benchmark fraction

TEKS:

6.1(A,B); 6.3(A); 6.2(D) New 6.2(D, E)

Tic-Tac-Toe

Objective: Students will review skills involving: LCM, GCF, prime, composite, even, odd, simplifying fractions, add / subtract fractions, convert improper fractions to proper fractions, equivalent fractions, factors, multiples, etc.

Materials:

2 -Tic-tac toe board games made out of tape on the floor
Colored chips or small bean bags for tossing
Dry erase boards
Dry erase markers

Activity Instructions:

1. Present a problem over covered concepts to first two players. (you may use the problems included on the CD or your own)
2. Player that gets it right gets to toss the chip on the board game. If the player misses the shot, then the second player may toss if they also answered the question correctly.
3. Continue playing until a team gets a tic-tac-toe on their board.

*Note that this activity may be used to review other topics of the teacher's choice as well

Our numeration system is often called the base-ten system or the Hindu-Arabic system. The base-ten refers to the place-values being powers of ten. The numerals evolved from the Asian and Arabic cultures.

You can simultaneously make a large number line model on the board or on the floor using string that does not stretch and masking tape. But this is no substitute for a students' own number line model. Place the fractions above the line and decimals below the line. As you build the model, we want students to use their knowledge about converting fractions to decimals and decimals to fractions to build the number line.

1. Expect responses like half a dollar, 50 cents or \$.50. Notice that some points will have multiple labels, like $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$. Remind your students that they learned that $\frac{1}{2} = 0.5$ in Section 8.1. Students may also use a piece of string because it is easier to fold into thirds and fifths. Have students make their number line with care because this number line is the beginning of a master number line that they will amend over the next couple of activities.

2. Teachers, make connections between decimals and their monetary equivalents. For example, 0.25 is equal to \$0.25.

3. What is the decimal equivalent of $\frac{1}{8}$? ($\frac{1}{8} = \frac{1}{2}$ of 0.25 or 0.125) Use this method to locate, mark and label all the eighths on the number line. TE: The number line should now have the following points labeled: 0, 1, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, $\frac{7}{8}$, $\frac{8}{8}$ above the number line.

4. Some possible comparisons the students might notice include:
   - The ruler and yard stick are divided into fractional amounts but their number line has labels for each fractional amount.
   - The yardstick, ruler and their own number line are using different scales, i.e., half of a yard is different than half of a foot or half of their own unit.

SECTION 4.2 COMPARING AND ORDERING FRACTIONS

You often see fractions as being parts of a whole. Can you think of any fractions that represent a very small part of a whole? At other times fractions are almost as big as the whole or larger. What are some examples of such fractions? You may find it useful to be able to estimate whether a fraction is very small, in which case we say that the fraction is close to 0. Or a fraction may be nearly a whole, in which case we say the fraction is close to 1. In another case, a fraction may be very close to $\frac{1}{2}$. Being able to use these estimations will be helpful in determining reasonableness of answers.

We begin with an activity that will help you to compare fractions.

LINEAR MODEL FOR FRACTIONS ACTIVITY

On a number line, each integer corresponds to a point. Recall that there are many other points between each pair of integers on the number line, and each of these points also corresponds to a number. We will now construct a number line from 0 to 1.

Materials: You will need a long strip of paper like a sentence strip or an 18-inch piece of adding machine paper.

1. Mark the left end point of the strip as 0 and the right end point as 1. You may fold this strip or use another strip to fold and transfer points to this master number line. Fold the strip end to end into two equal parts and mark the crease as the midpoint between 0 and 1. What fraction is this midpoint equivalent to? Label the points on the number line as fractions above the line.
2. Fold the strip again and use the creases to mark and label points on the master number line. Because the strip is now folded into 4 equal parts, we label the first point as $\frac{1}{4}$. Label the other points as $\frac{2}{4}$ and $\frac{3}{4}$. Add the equivalent fraction $\frac{2}{4}$ under $\frac{1}{2}$.
3. Repeat this method by folding the strip again into 8 equal parts, transferring the locations to the master number line and labeling the points with fractions. Use this method to locate, mark and label all the eighths on the number line.
4. Compare the number line with a typical foot ruler or yardstick.

5. Talk to your students about how they can fold an unmeasured strip into fifths. Solicit ideas and lead them, if necessary, to realize that they will make four equally-spaced creases in their number line to do that. The next hard part is thirds, but having made fifths and applying the lessons there, this should be easier. Sixths and ninths will follow.

   Teachers, switch to the number line from the packet in the teacher's folder (or one that you printed from the CD) that has hundredths marked but not labeled. Transfer the fractions (from halves to sixteenths) from the folded number line to this number line. Continue locating other fractions below using this new number line.

6. Talk about how to estimate the decimal form from what you know. If it is valuable, have some student with a calculator check the class' guesses. If necessary, remind students of the fraction-dollar relationship.

a.	b.	c.	d.
0.05	≈ 0.08	≈ 0.06	0.04
0.35	≈ 0.4	≈ 0.3	0.08
0.025	≈ 0.04	≈ 0.03	0.02
0.075	0.8	≈ 0.15	0.1

5. Label these fractions on the number line.
   a. $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$

   b. $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, \frac{10}{10}$

   c. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$

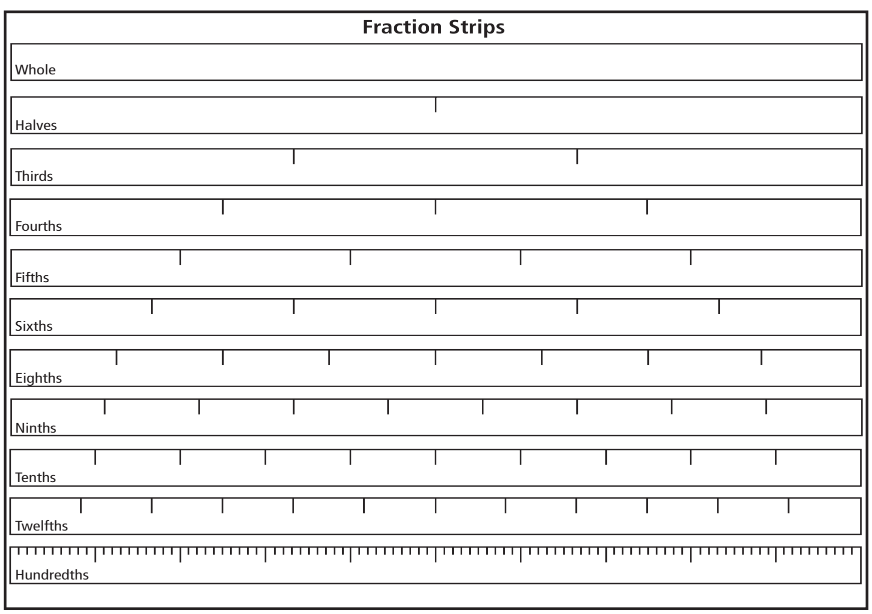
   d. $\frac{1}{6}, \ldots, \frac{5}{6}, \frac{6}{6}$

   e. $\frac{1}{9}, \ldots, \frac{8}{9}, \frac{9}{9}$

6. Use your new number line to determine which benchmark (0, $\frac{1}{2}$, or 1) each fraction is closest to.
   a. $\frac{4}{12}$ c. $\frac{5}{16}$ e. $\frac{4}{5}$

   b. $\frac{2}{3}$ d. $\frac{8}{12}$ f. $\frac{5}{50}$

The rows of the Fraction Chart below represent the result of folding the whole into equal parts to represent fractions as we did in the Linear Model Activity. The computer that created it allows the folds to be very accurate. For example, using the Fraction Chart, determine which fraction is greater: $\frac{2}{5}$ or $\frac{3}{8}$.



Students can compare the numerators of equivalent fractions with the same denominators or decimal representations to decide which fraction is greater. We will focus on using two visual models first, then discuss the advantage of finding equivalent fractions with the same denominator. We also verify order of fractions with decimals.

COMPARING AND ORDERING FRACTIONS: TWO METHODS

I. Linear Model

We want students to locate these fractions on the master number line or the Fraction Chart.

A. $\frac{3}{5}$ is greater.
This is not so easy to do with a number line and is a weakness of the linear model. If they recognize that each has an equivalent fraction in fourty-fifths, they can find the difference. Don't suggest subtraction yet. The area model will enable students to answer this question.

B. $\frac{4}{5}$ is greater.
Don't get bogged down by the second question. It is good for them to think about distance without resolving the issue yet.

C. From smallest to largest: $\frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}$. Looking at the Fraction Chart, we see what fraction is farthest left (least) then build from there rightward.

D. From largest to smallest: $\frac{7}{10}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}, \frac{5}{9}$. Looking at the Fraction Chart, we see what fraction is farthest right (greatest) then build from there leftward.

II. Area Model

A. Teachers, you might want to review the Area Model and Fudge Model to help your students remember the concept. If they draw the two fractions separately, they must overlay one on the other to compare them. This is foreshadowing common denominator. If your students don't use the term or concept, don't mention it yet.

A. With vertical folds for fourths and horizontal cuts for thirds, the area model has produced twelfths. We see that $\frac{1}{4} = \frac{3}{12}$ and $\frac{1}{3} = \frac{4}{12}$ so that $\frac{1}{3}$ is greater by $\frac{1}{12}$. Note that students may say $\frac{1}{3}$ is larger than $\frac{1}{4}$ by 1 more. However, the students should note that $\frac{1}{3}$ is $\frac{1}{12}$ more than $\frac{1}{4}$. In other words, the students should respond the fractional rather than whole numbers difference.

B. The area model tells us that $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$ so that $\frac{3}{4}$ is greater by $\frac{1}{12}$.

C. The area model tells us that $\frac{2}{5} = \frac{14}{35}$ and $\frac{3}{7} = \frac{15}{35}$, so $\frac{3}{7}$ is greater by $\frac{1}{35}$.

Reflect on how useful it is to rewrite fractions with a common denominator in order to compare them. Contrast this to their decimal representations. For instance: Which decimal is greater, 0.37 or 0.38, and by how much?

Getting your students who have a hard time with fractions to use and understand the area model will be a great service to them. Teacher will guide students to do the paper folding activity to demonstrate the fractional parts.

Given two fractions, how can you determine which of them is greater? We can now locate fractions on the number line. The fraction that is to the right of the other is the greater. What problems might arise from this method? For one, its accuracy depends on the quality of the comparative number lines. It becomes harder as the fractions get closer to the same value.

Use the master number line that you constructed or the Fraction Chart to decide which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$. Explain your answer. Is there another way to explain which is greater?

COMPARING AND ORDERING FRACTIONS

I. Linear Model:

Use the number line from 0 to 1 or the Fraction Chart to answer the following.

A. Which fraction is greater: $\frac{3}{5}$ or $\frac{4}{9}$? Can you tell how much greater using only the number line or the Fraction Chart?

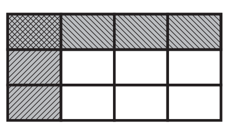
B. Which fraction is greater, $\frac{3}{4}$ or $\frac{4}{5}$?

C. Which fraction is greater, $\frac{2}{5}$ or $\frac{3}{8}$?

II. Area Model:

Use the area model to compare the following pairs of fractions. Use vertical cuts for one fraction and horizontal folds for the second fraction.

A. Which fraction is greater, $\frac{1}{4}$ or $\frac{1}{3}$? How much greater?



B. Which fraction is greater, $\frac{2}{3}$ or $\frac{3}{4}$? How much greater?

C. Which fraction is greater, $\frac{2}{5}$ or $\frac{3}{7}$? How much greater?

Have the students explain the process for the common denominator method. Make sure the students connect the common multiple and the denominator. Discuss how the least common multiple may be useful if the final answer is to be in simplest form.

PROBLEM 1

a. $\frac{3}{5}$ b. $\frac{7}{8}$ c. $\frac{5}{7}$

III. Common Denominator Method

The common denominator method compares the two fractions by rewriting the given fractions equivalently with the same denominators.

A. Which fraction is greater $\frac{2}{5}$ or $\frac{3}{8}$? How much greater?

Write equivalent fractions using a common denominator for each:

$\frac{2}{5} = \frac{2 \,:8}{5 \,:8} = \frac{16}{40}$ and $\frac{3}{8} = \frac{3 \,:8}{5 \,:8} = \frac{15}{40}$

When comparing the numerators of the equivalent fractions you can see that $\frac{16}{40}$ is greater than $\frac{15}{40}$. We say $\frac{2}{5} > \frac{3}{8}$ because $\frac{16}{40} > \frac{15}{40}$. $\frac{2}{5}$ is $\frac{1}{40}$ greater than $\frac{3}{8}$.

B. Which fraction is greater, $\frac{3}{5}$ or $\frac{13}{20}$? By how much?

C. Which fraction is smaller, $\frac{5}{9}$ or $\frac{4}{7}$? By how much?

PROBLEM 1

Determine whether the pairs of fractions are equal. If they are not equal, determine which is greater.

a. $\frac{3}{5}$ and $\frac{12}{50}$ b. $\frac{3}{4}$ and $\frac{7}{8}$ c. $\frac{5}{7}$ and $\frac{7}{10}$

There may be times when you must order more than two fractions. You may use the same common denominator method.

EXAMPLE

Lorenz has four different wrench sizes given by their diameter measures: $\frac{5}{8}$ inches, $\frac{11}{16}$ inches, $\frac{3}{4}$ inch and $\frac{1}{2}$ inch. Determine the order of the wrenches from largest to smallest.

SOLUTION

We note that all the wrenches are larger than $\frac{1}{2}$ except for the $\frac{1}{2}$ inch wrench. Therefore, $\frac{1}{2}$ is the smallest fraction. We must compare the remaining three fractions, $\frac{5}{8}$, $\frac{11}{16}$, and $\frac{3}{4}$. You may wish to use the fraction strip and see that $\frac{5}{8}$ is less than $\frac{3}{4}$. To determine where $\frac{11}{16}$ is situated, you may use benchmarks and notice that $\frac{5}{8}$ is $\frac{1}{8}$ larger than $\frac{1}{2}$ while $\frac{11}{16}$ is $\frac{3}{16}$ greater than $\frac{1}{2}$. $\frac{3}{4}$

PROBLEM 2

Picture frame: blue
Bracelet: yellow
Bow: green

EXERCISES

1. a. 1
   b. Answers may vary. An example is $\frac{8}{9}$
   c. $-\frac{1}{2}$

2. $\frac{2}{7}$ is closer to zero. One way to see this is to use the fraction strip. Another is to use the area model

is $\frac{1}{4}$ larger than $\frac{1}{2}$. So the order of the fractions from largest to smallest is $\frac{3}{4}$, $\frac{11}{16}$, $\frac{5}{8}$, then $\frac{1}{2}$.

Another approach is to use common denominators for all four fractions. Using 16, the least common multiple of 8, 16, 4, and 2, we can rewrite each of the fractions equivalently as follows:

$\frac{5}{8} = \frac{10}{16}$, $\frac{11}{16}$, $\frac{3}{4} = \frac{12}{16}$, and $\frac{1}{2} = \frac{8}{16}$. The order of the fractions can be made by comparing the numerators. From largest to smallest, we have

$\frac{12}{16} = \frac{3}{4} > \frac{11}{16} > \frac{10}{16} = \frac{5}{8} > \frac{4}{8} = \frac{1}{2}$.

PROBLEM 2

Kassandra has 3 pieces of different length ribbons. She has $\frac{7}{8}$ meter of blue ribbon, $\frac{5}{6}$ meter of green ribbon, and $\frac{2}{3}$ meter of yellow ribbon. She needs to use the longest ribbon for a picture frame and the shortest ribbon for a bracelet. The remaining ribbon is going to be used for a bow for a gift. What color is the ribbon that needs to be used for the picture frame? Bracelet? Bow?

EXPLORATION

Draw a number line and locate 0 and -1.

a. Use the number line to locate points with coordinates $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{3}$.

b. Determine which of the three numbers, $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{3}$ is greatest.

c. Determine which of the three numbers, $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{3}$ is smallest.

EXERCISES

1. a. Is $\frac{7}{8}$ closer to 0, $\frac{1}{2}$ or 1? Use a number line to prove your answer.
   b. Find a fraction that is even closer to 1 than $\frac{7}{8}$.
   c. Is $-\frac{3}{8}$ closer to 0, $-\frac{1}{2}$, or -1? Use a number line to prove your answer.
2. Which is closer to zero $\frac{2}{7}$ or $\frac{3}{8}$? Explain how you arrived at your conclusion. You may use a number line to prove your answer.

4. $\frac{3}{4}$ is closest to one cup.

5. $\frac{1}{2}$ is the next smallest size.

6. a. $\frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{4}$ b. $\frac{1}{4}, \frac{3}{5}, \frac{5}{8}, \frac{7}{10}$ c. $\frac{1}{16}, \frac{4}{32}, \frac{3}{8}, \frac{3}{4}$

d. $-\frac{4}{6}, -\frac{1}{2}, \frac{1}{4}, \frac{3}{8}$

7. Answers may vary.

8. a. $\frac{2}{3}$ c. $\frac{3}{4}$ e. $\frac{4}{5}$ g. $\frac{2}{7}$ i. $\frac{4}{5}$

b. $\frac{3}{4}$ d. $\frac{2}{3}$ f. $\frac{3}{10}$ h. $\frac{3}{5}$ j. $\frac{4}{5}$

9. $\frac{6}{7}$ is greater than $\frac{4}{5}$ by $\frac{2}{35}$.

11. a. $>$ b. $<$

c. $>$ d. $>$

e. $>$ f. $<$

10. For some reason, the skill and habit of simplifying fractions has disappeared. Many times using the simplified form is easier and reveals relationships hidden by larger numbers. Unsimplified fractions encourage students to use calculators when number sense is easier and faster.

3. The Miller team completed their project in $\frac{2}{3}$ of the time allowed. The Kealing team completed their project in $\frac{3}{8}$ of the time allowed. Determine which team finished their project first.

   Miller: 

   Kealing: 

Explain how you know which team finished first.

4. Ms. Lara is making iced tea and wants to add one cup of sugar to one gallon of unsweetened tea. She can only find measuring cups of $\frac{3}{4}$, $\frac{2}{3}$ and $\frac{1}{2}$. Which measuring cup is closest to one cup?

5. Mr. Reyna wants to find the correct size wrench for a bolt in order to change the oil in his car. The $\frac{9}{16}$ inch wrench is too large. He wants the next smaller size. Which should wrench should Mr. Reyna use, the $\frac{5}{8}$ inch or $\frac{1}{2}$ inch wrench?

6. Plot the numbers below on a number line. Then list the fractions in order from least to greatest:

   a. $\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{2}{5}$ b. $\frac{1}{4}, \frac{3}{5}, \frac{5}{8}, \frac{7}{10}$

   c. $\frac{3}{8}, \frac{3}{4}, \frac{1}{16}, \frac{4}{32}$ d. $-\frac{4}{6}, \frac{3}{8}, -\frac{1}{2}, \frac{1}{4}$

7. Use the Fraction Chart to discover and represent three fractions that are greater than $\frac{1}{4}$ and less than $\frac{1}{2}$. Explain why each fraction is between $\frac{1}{4}$ and $\frac{1}{2}$ without using the chart.

8. Write the equivalent fraction in simplest form for each of the following fractions.

   a. $\frac{4}{6}$ c. $\frac{12}{16}$ e. $\frac{16}{20}$ g. $\frac{6}{21}$ i. $\frac{24}{30}$

   b. $\frac{9}{12}$ d. $\frac{12}{18}$ f. $\frac{6}{20}$ h. $\frac{18}{30}$ j. $\frac{16}{30}$

9. Compare the two fractions $\frac{4}{5}$ and $\frac{6}{7}$ by plotting them on a number line. Which fraction is greater, $\frac{4}{5}$ or $\frac{6}{7}$? By how much? Explain your reasoning.

13. a. closest to 1 b. closest to $\frac{1}{2}$ c. closest to 0 d. closest to $\frac{1}{2}$

14. Evan is correct because $\frac{2}{7} < \frac{1}{3}$.

15. $\frac{14}{16}$ will create a larger bow.

16. Answer: B, A, D, C

10. Why is it important to know how to simplify fractions?

11. Use the different models or methods learned in this section to compare the following fractions. Use <, > or = between the two fractions.

a. $\frac{7}{8}$ **>** $\frac{5}{6}$

b. $\frac{8}{9}$ **<** $\frac{9}{10}$

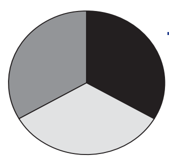
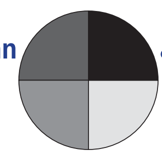
c. $\frac{2}{3}$ **>** $\frac{5}{8}$

d. $\frac{4}{7}$ **>** $\frac{5}{9}$

e. $\frac{1}{12}$ $-\frac{1}{3}$

f. $-\frac{3}{4}$ $-\frac{6}{16}$

12. Tianna has $\frac{1}{3}$ of a large pizza left. Desiree has $\frac{1}{4}$ of a large pizza left. Who has more pizza left? (Assume the black shaded region represents uneaten pizza.)

Tianna:  **Tia** Desiree: **nn**  **a**

13. Use a number line to determine which benchmark (0, $\frac{1}{2}$, 1) each fraction is closest to.

a. $\frac{11}{15}$ b. $\frac{3}{8}$ c. $\frac{0}{5}$ d. $\frac{9}{16}$

14. Evan ate $\frac{2}{7}$ of an apple pie. Pattie ate $\frac{1}{3}$ of the same pie. Evan says he ate less. Is Evan right or wrong? Explain your answer.

15. Lorianne bought two ribbons to make bows. One ribbon was $\frac{14}{16}$ of a yard and the other was $\frac{6}{8}$ of a yard. Which ribbon will create a larger bow?

16. Mr. Trevino's science class measured the distance four toy cars traveled on a yard stick. They found the distances:

a. Car A went $\frac{5}{6}$ of a yard
b. Car B went $\frac{11}{12}$ of a yard
c. Car C went $\frac{4}{9}$ of a yard
d. Card D went $\frac{18}{36}$ of a yard.

List the car distances in order from greatest to least. **C, B, A, D**

17. a. equal b. $\frac{3}{5} < \frac{25}{100}$ c. $\frac{9}{16} > \frac{8}{15}$

18. Celia, Rolinda, Melissa, Sarah

20.	Spiral Review (6.1D)	$2^3 \bullet 3 \bullet 13$

21.	Spiral Review (7.2E)	14

Ingenuity:

Since we want the fraction to be as great as possible, but the numerator must be less than the denominator, this means that we are trying to make a fraction that is as close to 1 as possible. So we should aim to make the numerator as close to the denominator as we can. Our strategy will be to consider the possible values of the hundreds digit of the denominator.

If the denominator begins with 2, then the numerator must begin with 1. The closest values we can get for the numerator and denominator are 165 and 234.

If the denominator begins with 3, then the closest values we can get for the numerator and denominator are 265 and 314.

If the denominator begins with 4, then the closest values we can get for the numerator and denominator are 365 and 412.

If the denominator begins with 5, then the closest values we can get for the numerator and denominator are 463 and 512.

If the denominator begins with 6, then the closest values we can get for the numerator and denominator are 543 and 612.

The only one of these that gives us a fraction that is greater than 9/10 is 463/512; we know that 463/512 is greater than 9/10 because 1 - 463/512 = 49/512, which is a bit less than 1/10. So the best possible fraction is 463/512.

Investigation:

In each case, the last fraction is obtained by adding the numerators and denominators of the first two fractions. We will see the significance of this shortly. When working out parts (a) through (d), most students will compare fractions using the techniques discussed in this section. In the process of doing this, they may discover a shortcut

a. 1/2, 3/5, 2/3 b. 1/5, 2/9, 1/4 c. 1/6, 3/13, 2/7 d. 5/7, 8/11, 3/4

17. Compare the two fractions. You may use a number line. Determine if the pair of fractions are equal to each other. If the two fractions are not equal, determine which is greater.

a. $\frac{1}{4}$ and $\frac{25}{100}$ b. $\frac{3}{5}$ and $\frac{13}{20}$ c. $\frac{9}{16}$ and $\frac{8}{15}$

18. The math team was asked to pose for a picture. Rolinda is 5 $\frac{2}{3}$ feet tall, Melissa is 5 $\frac{4}{7}$ feet tall, Sarah is 4 $\frac{7}{8}$ feet tall and Celia is 5 $\frac{5}{6}$ feet tall. The photographer has them line up from tallest to shortest. In what order should they line up? **Celia, Rolinda, Melissa, Sarah**

Spiral Review:

19. In a competition, Ray, Josh, Tyler, and Aaron were asked to draw toothpicks. Ray's toothpick was $\frac{12}{18}$ inches, Josh's was $\frac{2}{7}$ inches , Tyler's was $\frac{1}{5}$ inch., and Aaron's was $\frac{6}{7}$ inches long. Whose toothpick was closest to a whole toothpick? Explain your answer.

20. What is the prime factorization of 312?

21. What is the value of the expression $(4 + 4)^2 \div 8 + 3 \cdot 2$?**A**

22. **Ingenuity:**

Suppose we use the digits 1, 2, 3, 4, 5, and 6 to make the numerator and denominator of a fraction. Every digit must be used exactly once, the numerator and denominator must have three digits each, and the numerator must be less than the denominator. What is the greatest fraction we can make?

23. **Investigation:**

In parts (a) through (d), three fractions are given. Write the fractions in order from least to greatest. In each part, note the relationship between the last fraction and the first two.

a. $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}$

b. $\frac{1}{4}, \frac{1}{5}, \frac{2}{9}$

c. $\frac{1}{6}, \frac{2}{7}, \frac{3}{13}$

d. $\frac{3}{4}, \frac{5}{7}, \frac{8}{11}$

e. What do you notice in parts (a) through (d)?

e. In each case, the third fraction falls between the other two. Students may observe that this is what is happening, and wonder why this is always the case. One good model that helps us understand what is going on here is to consider a quarterback who plays two offensive possessions in a game of football. Using (d) as an example, suppose a quarterback completes 3 of 4 passes on one possession, and then completes 5 of 7 on the next. Then on the two possessions combined, the quarterback completed 8 of 11 passes. We can check that 5/7 is a bit less than 3/4; therefore, the quarterback was a little more accurate on the first possession than on the second. We know that the quarterback's overall completion percentage is greater than his completion percentage on the second possession (his first possession brought his completion percentage up), but less than his completion percentage on the first possession (his second possession brought the percentage down). Thus 8/11 is between 5/7 and 3/4.

Note that many of your students may not be football fans (or if they are, they may not be accustomed to thinking about completion percentages); however, most metaphors involving mixtures can be adapted to fit this problem.

(f) Using what you have discovered, give ten fractions that lie between 4/7 and 3/5.

Many different answers are possible. To get one fraction between 4/7 and 3/5, we can add the numerators and denominators to get 7/12. We can then do the same operation with 4/7 and 7/12 to get 11/19, which is between 4/7 and 7/12. We can repeat this process indefinitely; one possible answer is 7/12, 11/19, 15/26, 19/33, 23/40, 27/47, 31/54, 35/61, 39/68, 43/75. Note that these fractions are in decreasing order.

f. Using what you have discovered, give ten fractions that lie between $\frac{4}{7}$ and $\frac{3}{5}$.

Section 4.3 - Unit and Mixed Fractions

Big Idea:
Comparing and converting improper and mixed fractions

Key Objectives:
- Understand the relationship between improper and mixed fractions.
- Learn the definition of unit and proper fractions.
- Use fractions within the context of real-life situations.

Materials:

Grid paper, Ruler, Measuring cups in $\frac{1}{4}$ and 1 cup sizes.

Pedagogical/Orchestration:
This section involves connecting improper fractions to their mixed fraction equivalents through division. The computational practice in this section is important to understanding and being able to compute using fractions.

- Beginning of lesson foreshadows multiplication of fractions.
- Improper fractions are greater than 1, and proper fractions are less than 1.
- In this section, there should be some direct instructions on how to convert from mixed fraction to improper and vice versa.
- Connection to Measurement: ask students to find 2¼ inches on a ruler and so forth. See "Ruler Activity".

Activities:
"Basketball Double-Teaming", "It All Measures Up", "Ruler Activity"

Exercises:
Foreshadowing mixed fraction division in Exercises 12,13.

Internet Resources:
Jeopardy game to review proper and improper fractions: http://www.quia.com/cb/186132.html

Vocabulary:
unit fraction, proper fraction, improper fraction, mixed fraction, mixed number

TEKS:
6.1(A,B); 6.2(A,B); 6.8(D); 6.12(A,B); 6.13(A,B)

New 6.2(A,D, E, F)

Launch for Section 4.3:

In this Launch, the teacher can elect to visually demonstrate the concept of flexibly using mixed and improper fractions by bringing in sugar, a $\frac{1}{4}$ measuring cup and a 2 cup liquid measuring cup. Tell the class, "You are going to bake cookies and the recipe calls for 1 $\frac{3}{4}$ cups of sugar. You look all through the kitchen and the only measuring cup you can find is a $\frac{1}{4}$ measuring cup. How could you use that cup to measure the required amount of sugar? Let students discuss and explain to others in the class that 4 of the $\frac{1}{4}$ measuring cups will give one whole cup, and 3 of the $\frac{1}{4}$ cups will give the $\frac{3}{4}$ cups. This is 7 all together of the $\frac{1}{4}$ cups. Let students know that this is actually called $\frac{7}{4}$ of a cup and is known as an improper fraction because it is larger than 1 whole. Visually demonstrate by scooping 7 of the $\frac{1}{4}$ cups into the sugar and pouring them into the 2 cup liquid measuring cup. It will fill to the 1 $\frac{3}{4}$ line. Therefore $1\frac{3}{4} = \frac{7}{4}$. Tell your students, "The focus for today will be converting between improper fractions and mixed numbers just as we did with the amount of sugar in the cookie recipe."

Basketball Double-Teaming

Objective: Students will convert mixed numbers to improper fractions and improper fractions to mixed numbers. Teacher can also use this activity for adding / subtracting fractions, prime factorization, and writing equivalent forms for fractions, decimals, percents.

Materials:
Trash can
Soft basketball

Activity Instructions:
The teacher chooses two teams: Team Archimedes and Team Einstein.

1. Students are placed in 2 lines per team. One player line is the "coach" and the other player line is the "player". The "coach" can only offer assistance and tutorials to the "player". They MAY NOT solve the problem or give the answer.
2. Both teams will choose their first two players and coaches to go to the board to work a math problem made up by the teacher. (For this particular section of the book, math problems should be related to converting mixed numbers to improper fractions and vice versa.) "Players" and "Coaches" will work through the problem given, hoping to solve the problem correctly before the next team.
3. When a team completes the problem correctly, the "player" earns a point and goes to the free throw line. If player makes the shot, the team earns an additional 2 points.
4. If the winning "player" misses the 2 point free throw, then the opponent gets to attempt 2 shots from the free throw line if they also answered the problem correctly.
5. Players from both teams go to the end of the line but to opposite lines to switch roles so that the "coach" becomes the "player" and vice versa.

It All Measures Up

Objective: Students will recognize and convert improper fractions to mixed numbers and mixed numbers to improper fractions.

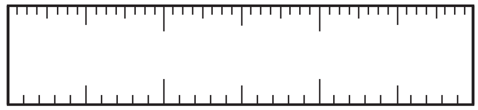
Materials:
Measuring tape
Pencils
Butcher paper

Activity Instruction:

1. Students will pair up and trace each others body silhouette on butcher paper.
2. Partners use the tape measure to find each other's heights from the silhouette and record all the heights, expressing the answers in the following manner: Ex. 5 feet 6 inches, 5 ½ feet, 66 inches, $^{66}/_{12}$ foot
3. Teacher will record class heights on the board or on butcher paper and discuss mixed numbers and improper fractions.

Extensions:
-Order the heights from tallest to shortest
-Convert to decimal form

Ruler Activity



Objective: The students will create a scale, practice fractional measurement, and compare mixed fractions with improper fractions.

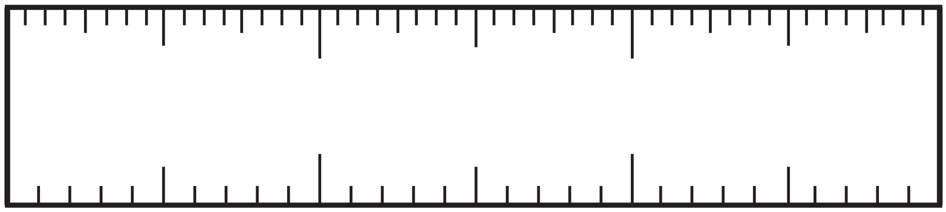
Materials:
Ruler Activity Handout

Activity Instructions:
Students will scale a ruler two different ways and compare measurements by following the directions on the Ruler Activity Handout.

Answer Key:

1. Each inch on the top scale is divided into sixteen pieces. We should call these sixteenths.
2. Each inch on the bottom scale is divided into 10 pieces. We should call these tenths.
3. The top can be labeled in sixteenths and as an option can also show fourths, eighths and halves that are equivalent to the sixteenths. The bottom scale can be labeled in tenths.
4. 11/4 = 2¾
5. 36
6. 18
7. There are many: 1½, 24/16, 12/8…
8. 17/10 = 1 7/10 = 1.7
9. 14/10 is 1/40 greater than 11/8. 14/10 = 1 4/10 and 11/8 = 1.
10. 1 3/16
11. The top scale would be divided into 8ths, and the bottom scale into 5ths.

Ruler Activity Handout



Use the above ruler for the following activities.

1. The ruler represents 3 inches. Label the marks representing 1 inch, 2 inches, and 3 inches.
2. How many equal pieces is the top scale divided into? What should we call these pieces?
3. How many equal pieces is the bottom scale divided into? What should we call these pieces?
4. Label the tick marks for each scale.
5. Which tick mark would be equal to a length of $\frac{11}{4}$? Label this mark with an A. What is the mixed fraction name for this length?
6. How many $\frac{1}{16}$ inches are in $2\frac{1}{4}$ inches?
7. How many $\frac{1}{8}$ inches are in $2\frac{1}{4}$ inches?
8. Write two other ways to say $\frac{3}{2}$ inches.
9. Which tick mark on the bottom scale would be equal to a length of $\frac{17}{10}$? Label this mark with a B. What is the mixed fraction name for this length? What is the decimal name for this length?
10. Comparing the two scales, which number is greater: $\frac{11}{8}$ inches or $\frac{14}{10}$ inches? What are the mixed fraction names for these lengths?
11. How much greater is $2\frac{1}{4}$ compared to $1\frac{1}{16}$?
12. If you had been told the above ruler was 6 inches long, how would that change the scale? In other words, how would that have changed the way you marked the tick marks?

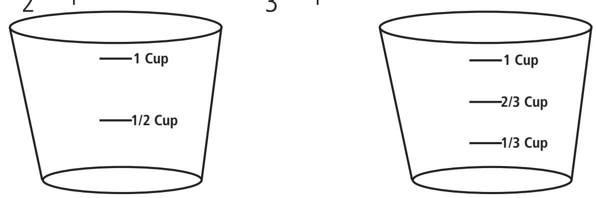
Some answers could $\frac{4}{3}$, $1\frac{1}{3}$ watch out for $\frac{4}{4}$ or $\frac{3}{4}$ and other possible misconceptions.

Help students visualize fraction chart, measuring cup, circle. What it means to be greater than a whole.

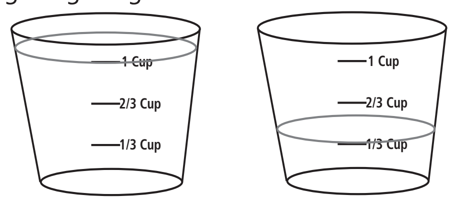
Students should be reminded of the customary conversions of mass and capacity.

SECTION 4.3 UNIT FRACTIONS, MIXED FRACTIONS, PROPER AND IMPROPER FRACTIONS

Recall on the fraction chart we have $\frac{1}{2} + \frac{1}{2} = 1$, $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$. Notice that we have two $\frac{1}{2}$ equals 1 and three $\frac{1}{3}$ equals 1.



What would $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$ equal?



A **unit fraction** always has 1 in the numerator and the denominator is a positive integer. For example, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and so on are all examples of unit fractions.

Now extend the addition of unit fractions to make another connection to multiplication. You have seen several models that represent the fraction $\frac{3}{5}$. In the area model, $\frac{3}{5}$ represents three $\frac{1}{5}$'s of a whole. This means $\frac{3}{5}$ is the sum of $\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$. In the frog model, this is the same as taking 3 jumps of length $\frac{1}{5}$. That is, $\frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{1}{5} \cdot 3 = \frac{3}{5}$. This understanding can be extended to all fractions. For example, the fraction $\frac{5}{9}$ is the same as the sum of 5 copies of $\frac{1}{9}$:

$$\frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{9} \cdot 5 = \frac{5}{9}.$$

Write the sum of 8 copies of $\frac{1}{5}$. Using the frog model, what is the result of 8 jumps of length $\frac{1}{5}$ each?

Using math with recipes can be fun not only for learning but for eating, too! Usually recipes involve quantities of ingredients and cooking directions. Here is a chocolate chip cookie recipe:

- $2\frac{1}{4}$ cups flour
- $\frac{3}{4}$ cup sugar
- $\frac{3}{4}$ cup brown sugar
- 12 oz chocolate chips
- 1 tsp baking soda
- 1 tsp salt
- 1 tsp vanilla

This makes approximately 6 dozen cookies.

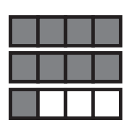
Each of the sugar quantities is $\frac{3}{4}$ of a cup. This quantity is called a **proper fraction** because it is a number less than 1.

Look at the first ingredient in our recipe, "two and one-fourth cups." Remember, "and" means adding two cups to one-fourth of a cup. Quantities like $2\frac{1}{4}$ are called **mixed fractions** or **mixed numbers** because they consist of an integer like 2, in addition to a fractional part that is less than a whole like $\frac{1}{4}$. It is customary to write the fractional part in simplified form. The mixed fraction $2\frac{1}{4}$ is actually the sum $2 + \frac{1}{4}$. The rest of the recipe contains both fractional parts of cups or teaspoons and numbers of ounces and dozens.

Look at the mixed fraction $2\frac{1}{4}$. If you have only a quarter-cup measure, describe how you can measure the correct amount with the quarter cup.

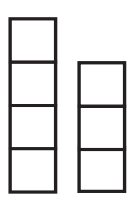
Did you find $2\frac{1}{4}$ equivalent to $\frac{9}{4}$? In fact, what you have found are two ways to write the same quantity: as a mixed fraction, $2\frac{1}{4}$, and as an **improper fraction**, $\frac{9}{4}$. How would you describe improper fractions? Why do you think they are called improper?

$\frac{9}{4}$



$2\frac{1}{4}$

If we have seven quarters, we can think of each quarter as $\frac{1}{4}$ of one dollar. Then we have $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. Because each of the four quarters equals one dollar, you have 1 dollar and 3 more quarters or $1\frac{3}{4}$ is equal to $\frac{7}{4}$.



PROBLEM 2

Have a discussion with many examples to illustrate the differences between proper and improper fractions. You could have discussion start in small groups and then share with the whole class.

PROBLEM 3

Depending on the context, a mixed fraction may be more useful if you wanted to know how many 6-seat cars are needed to transport 15 people. While improper fractions may be more useful when we perform multiplication and division of fractions.

PROBLEM 4

6 whole pizzas should be ordered. There will be $\frac{1}{3}$ of a pizza left over.

Show students different ways to rewrite improper fractions as mixed fractions. You may wish to use techniques to reinforce the division of the numerator by the denominator:

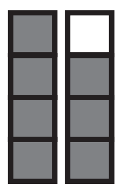
1. division idea with dividend / divisor
2. TiBo (top in bottom out)
3. Upside triangle topples over

PROBLEM 1

Use a model and repeated addition sentences to find the improper fraction for each of the following mixed fractions:

a. $1\frac{7}{8}$ b. $3\frac{3}{4}$

Another way to think about $\frac{7}{4}$ is to view this fraction as a division problem, 7 divided by 4. If we divide 7 by 4, we have a quotient of 1 with a remainder of 3. Using the area model, we group 4 of the 7 into a rectangle of dimension 4 by 1 because 4 is the divisor. The remaining 3 fill up $\frac{3}{4}$.

$\frac{7}{4}$  $1\frac{3}{4}$

$$\frac{7}{4} = 7 \div 4 = 1\frac{3}{4}$$

PROBLEM 2

Use division to write the improper fraction $\frac{7}{3}$ as a mixed fraction.

PROBLEM 3

State the difference between proper and improper fractions. What is the advantage of using an improper fractions or using a mixed number?

PROBLEM 4

Each student in Ms. Milligan's class is to receive $\frac{1}{3}$ of a pizza for lunch. There are 17 students in her class. How many whole pizzas should Ms. Milligan order? Will there be any portion of a pizza left for Ms. Milligan? If so, how much will be left for her?

EXPLORATION

Use the Internet to find a recipe for chocolate chip cookies from another country. Analyze the differences you see.

Interestingly enough, in the United States most recipes are written as fractions. Why do you think American recipes are usually written in fractional form?

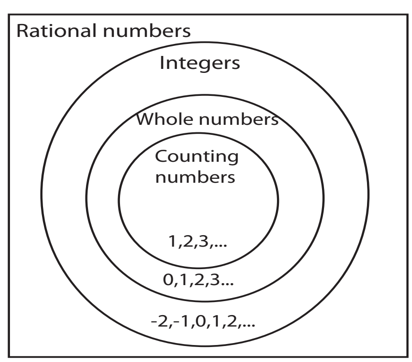
Many times recipes are either too large or too small for our purposes. If you double the recipe, how much sugar will you need? Write the sugar quantity in both mixed fraction and improper fraction form. Also write the chocolate chip quantity in the doubled recipe in terms of pounds.

a. A recipe for pancakes calls for $1\frac{3}{4}$ cups of flour. Locate this point on the number line. Describe the equivalent improper form for the mixed fraction. What does the numerator represent?

b. Jack has three identical pans of brownies and decides to divide each pan into 12 equal pieces. How many brownies pieces does he have in all? Because Jack was very hungry, he ate 2 of the pieces. If you assume each brownie pan represents 1 or a whole, express the amount of brownies that remains in terms of the whole and pieces.

c. If he takes half of the uneaten brownies to a party, what quantity will he take? Using the area model, draw the brownies quantities he will take and leave. Be sure to include the fact that each pan is divided into 12 pieces.

We now have fractions that include natural numbers, whole numbers, and integers. The fractions we are considering have a numerator and denominator that are integers, except the denominator cannot be 0. The set of fractions of this form $\frac{a}{b}$, with b not zero and a and b integers are called rational numbers. A Venn Diagram shows how these set are related



EXERCISES

1. a. $\frac{5}{4} = 1\frac{1}{4}$ c. $\frac{5}{3} = 1\frac{5}{3}$

   b. $\frac{7}{5} = 1\frac{2}{5}$ d. $\frac{6}{2} = 3$

2. $\frac{9}{18} = 1\frac{1}{8}$, $\frac{18}{8} = 2\frac{1}{4}$, $\frac{12}{8} = 1\frac{1}{2}$;16/8 = 2; 27/8 = 3 3/8; 11/8 = 1 3/8; 20/8 = 2 ½.

3. $\frac{17}{4} = 4\frac{1}{4}$. Students may use the number line model as in exercise 4 or they may use the fact that every fraction is a division problem and for example ask themselves - how many times does 5 go into 12? 2 times with a remainder of 2 and thus $2\frac{2}{5}$.

PROBLEM 5

For the Venn Diagram on p. 176, find the following:

a. What is a whole number that is not a counting number? Where would this be in the Venn Diagram?

b. What is an integer that is not a whole number? Where would this be in the Venn Diagram?

c. What is a rational number that is not an integer? Where would this be in the Venn Diagram?

d. Are all integers rational numbers? Explain.

e. Are all counting numbers integers? Explain.

EXERCISES

1. Rewrite these sums as an improper fraction and as a mixed fraction:
   a. $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=$
   b. $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=$
   c. $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=$
   d. $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=$

2. Convert each of these improper fractions to a mixed fraction. Sketch a number line from 0 to 4 with eighths marked and locate each mixed fraction.

$$\frac{9}{8}, \frac{18}{8}, \frac{12}{8}, \frac{16}{8}, \frac{27}{8}, \frac{11}{8}, \frac{20}{8}$$

3. Write the improper fraction $\frac{17}{4}$ as a mixed fraction. Explain the process you used.

4. a. 2 1/3 b. 2 ¼ c. -1 1/5 d. 4 e. -1 ½ f. 6 ½ g. 2 ½
h. 3 3/10 i. 5 2/5 j. -1 1/3 k. 3 ¼ l. -3

.

6. $2\frac{3}{4} = \frac{11}{4}$. Answers may vary as to what model students use. They may use the number line model or a modified multiplication model that reverses the process they used in Exercise 3. Using a modified multiplication model students will compute common denominators and add fractions. Multiplying 2 and 4 to get 8 and then adding 3 which is 11, the numerator of $\frac{11}{4}$.

4. Convert each improper fraction to a mixed fraction.
   Simplify any mixed fractions that are not already in lowest terms.

a. $\frac{7}{3}$	e. $-\frac{12}{8}$	i. $\frac{27}{5}$
b. $\frac{9}{4}$	f. $\frac{19}{3}$	j. $-\frac{24}{18}$
c. $-\frac{6}{5}$	g. $\frac{15}{6}$	k. $\frac{39}{12}$
d. $\frac{24}{6}$	h. $\frac{33}{10}$	l. $-\frac{45}{15}$

5. Let W = {whole numbers}, N = {counting number}, Z = {integers}, Q = {rational numbers}.

   The curly brackets { } enclose the elements in a set. Set notation is a shorter way of listing the elements. For example, {1,2,3} is the set containing the three integers 1, 2, and 3. Another way of describing this set is {x|x is a positive integer less than 4}. This notation means "the set consisting of x such that x is a positive integer less than 4." The vertical line | is read as "such that."

   a. Draw a Venn Diagram of the four sets, labeling each set.

   b. Find at least one number that is in each set that is not in the subset it contains.

   Notice that each of these sets W, N, Z, Q is a proper subset of the next set in the list, meaning that the set that follows a set in this list always has at least one new member.

6. Convert the mixed fraction $2\frac{3}{4}$ to an improper fraction. Explain the process you used.

7. a. 19/5 b. 19/8 c. -33/8 d. 27/5 e. 65/6 f. -48/5
g. 83/12 h. 120/9

8. 14

9. $2\frac{2}{3} > \frac{15}{6}$. Student explanations may vary.

10. $\frac{27}{8} > 3\frac{1}{4}$. Student explanations may vary.

11. Aunt Bea can serve 15 people. Foreshadows division of a mixed number by a fraction. Students will probably use the linear model and skip count to compute the number of servings.

12. Mr. Kellerman had 13 candy bars

13. The divisions are equal, so if $A = 1$, each division is equal to $\frac{1}{2}$. If $B = 1$, each division is equal to $\frac{1}{4}$, and so on.

	A	B	C	D	E	F	G
a.	1	2	$2\frac{1}{2}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
b.	$\frac{1}{2}$	1	$1\frac{1}{4}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
c.	$\frac{2}{5}$	$\frac{4}{5}$	1	$1\frac{2}{5}$	$1\frac{3}{5}$	$1\frac{4}{5}$	2
d.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$

7. Convert each of these mixed fractions to an improper fraction.

a. $3\frac{4}{5}$ e. $10\frac{5}{6}$

b. $2\frac{3}{8}$ f. $-9\frac{3}{5}$

c. $-4\frac{1}{8}$ g. $6\frac{11}{12}$

d. $5\frac{2}{5}$ h. $13\frac{3}{9}$

8. How many one-eighth cups of flour are in $1\frac{3}{4}$ cups of flour?

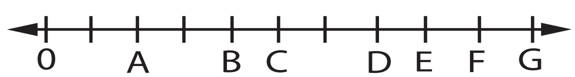
9. Which is greater: $2\frac{2}{3}$ or $\frac{15}{6}$? Explain your reasoning.

10. Which is greater: $\frac{27}{8}$ or $3\frac{1}{4}$? Explain.

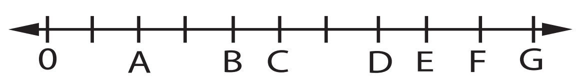
11. Aunt Bea is making pudding for dessert. She wants to serve everyone $\frac{1}{4}$ of a cup. If she makes $3\frac{3}{4}$ cups, how many people can she serve?

12. Mr. Kellerman had a box of candy bars. He cut them into fourths to give each student an equal amount of candy. If he distributed $\frac{52}{4}$, how many whole candy bars did he have in the box?

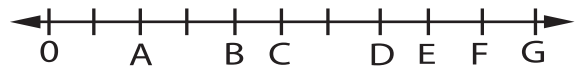
13. For each of the following questions, make a copy of the picture below and use it as a linear model for a fraction bar. Remember to make a separate fraction bar for each: a - d. Zero has already been placed for you.



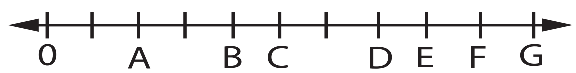
a. If the point **A** represents the number 1 on the number line, what numbers represent the other points? Write the values above the point on the number line. You may answer in mixed fractions or improper fractions.



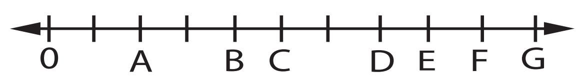
b. If the point **B** represents the number 1 on the number line, what numbers represent the other points? Write the values above the point on the number line. You may answer in mixed fractions or improper fractions.



c. If the point **C** represents the number 1 on the number line, what numbers represent the other points? Write the values above the point on the number line. You may answer in mixed fractions or improper fractions.



d. If the point **E** represents the number 1 on the number line, what numbers represent the other points? Write the values above the point on the number line. You may answer in mixed fractions or improper fractions.



14. Place zero in the center of this number line. Number by halves from -2 to 2.

Place the following numbers on the number line. (Hint: Consider simplifying your answers before placing them on the number line.)

a. $\frac{12}{8}$	c. $1\frac{5}{6}$	e. $\frac{24}{12}$
b. $-\frac{5}{4}$	d. $\frac{9}{12}$	f. $-\frac{6}{8}$

15.	Spiral Review (6.1E)	LCM (10, 15, 20) = 2 • 2 • 3 • 5 = 60

16.	Spiral Review (7.1A)	d. $-\frac{24}{32}$ and $-\frac{1}{4}$

17. **Ingenuity:**

(a) Students may simply verify that 3 1/7 is not the same as 31/7; for example, they may find that 3 1/7 is equal to 15/7, not 31/7. This is fine, though it does not directly explain why Dania's method is incorrect. One way to explain why Dania's method is incorrect is to notice that by putting the whole number part on the left of the numerator, Dania is adding (whole number part) x 10 / (denominator) to the fractional part. The value of the expression (whole number part) x 10 / (denominator) will usually be different from the whole number part, and thus the method will usually give a wrong answer.

(b) In part (a), we found that, when the fraction has a one-digit numerator, Dania's method only works if (whole number part) x 10 / (denominator) is equal to the whole number part. This happens to be true if the denominator of the fraction is 10. For example, if the mixed fraction is 3 1/10, then Dania's method works, because it yields the correct improper fraction 31/10. Similarly, we have 4 3/10 = 43/10.

Most students will not yet be able to explain what is going on here, but they may be able to find some examples of mixed fractions for which Dania's method works. If this happens, you may find it worthwhile to make a class list of mixed fractions for which Dania's method works, and see if students can spot the pattern and make a conjecture.

Spiral Review:

15. At a dance recital, a prize was given to the person sitting in the chair numbered with the least common multiple of 10, 15, and 20. Find the number of the prize winning chair.

16. The fraction $-\frac{3}{8}$ is found between which pair of fractions on a number line?

a. $-\frac{21}{22}$ and $-\frac{8}{16}$ c. $-\frac{19}{32}$ and $-\frac{9}{16}$

b. $-\frac{24}{32}$ and $-\frac{10}{16}$ d. $-\frac{24}{32}$ and $-\frac{1}{4}$

17. **Ingenuity:**

While doing a homework assignment on mixed numbers, Dania thought that she discovered a new way to rewrite a mixed fraction as an improper fraction. She moved the whole number part of the mixed fraction into the numerator, as shown in the examples below:

$$3\frac{1}{7} \rightarrow \frac{31}{7}$$
$$4\frac{3}{5} \rightarrow \frac{43}{5}$$

a. Explain why Dania's method is incorrect.

b. Are there any mixed fractions for which Dania's method would give the correct improper fraction? That is, are there any mixed fractions that we can convert to improper fractions simply by moving the whole number part into the numerator as shown above?

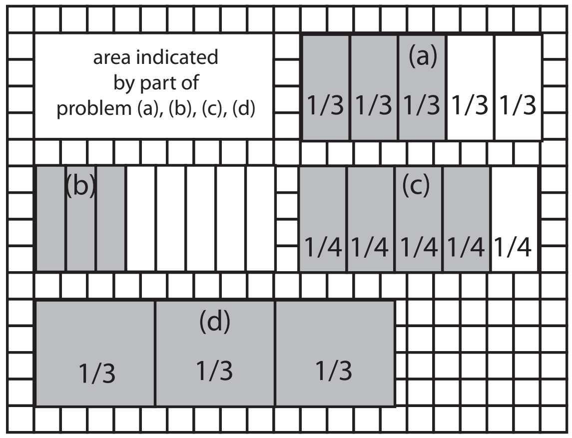
18. **Investigation:**

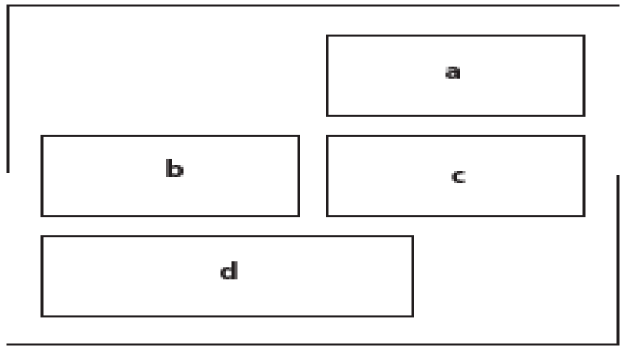
Copy the rectangle to the right for each problem and suppose that its area is given in each of the problems below. Determine and shade an area of 1 square unit for each of the rectangles.

a. $1\frac{2}{3}$ c. $\frac{5}{4}$

b. $2\frac{2}{3}$ d. $\frac{2}{3}$

18. In this exercise, "copy" need not be taken literally. To work this exercise, use grid paper, setting each unit square equal to the unit fraction that corresponds with each section. For instance, in part (a), set each unit square equal to $\frac{1}{3}$. Then draw the first rectangle using 2 squares for $\frac{2}{3}$. Because $1 = \frac{3}{3}$, extend the rectangle another square to get $\frac{3}{3} = 1$. Shade the whole extended rectangle. The idea is that the numerator represents the number of divisions present, so in $\frac{2}{3}$, divide the rectangle into halves. Then find 3 of those divisions. To do this, you will have to make another half.




a
b
c
d

Section 4.4 - Addition and Subtraction of Fractions

Big Idea:

Developing addition and subtraction of fractions with common and uncommon denominators.

Key Objectives:

- Use area and linear models to visualize addition and subtraction of fractions and to develop a method for adding and subtracting fractions.
- Discover the advantage of using the LCM for the common denominator.
- Solve real-life application problems using addition and subtraction of fractions.

Materials:

Grid paper, Picture of ruler for demonstration

Pedagogical/Orchestration:

- This section reviews adding and subtracting fractions using the area model. Some of your students may find grid paper to be useful. For instance, in Example 2, a 2 by 3 rectangle on a grid is chosen as the whole and used to help students add $\frac{1}{2}$ and $\frac{1}{3}$.
- Do not shy away from the fractions that have variables in the denominators found in the exercise. Dealing with variables in the denominator is actually easier in many cases than dealing with numbers, and can help the students develop a strategy for finding the least common denominator.

Activity:

"Spinning for LCD", "Pattern Block Fun"

Exercises:

Exercise 7 connects to measurement.

Internet Resources:

Jeopardy game to review adding and subtracting fractions: http://www.quia.com/cb/62195.html

Vocabulary:

common denominator, least common denominator

TEKS:

6.1(B,E,F); 6.2(A,B,D); 6.11(B,C); 6.12(A) New: 6.2(F); 6.3(E); 6.5(C)

Launch for Section 4.4:

Tell your students that George and Martha are baking cookies for their class, and need 8 eggs to do so. George has $\frac{1}{4}$ of a carton of eggs and Martha has $\frac{1}{3}$ of a carton of eggs. A carton holds a dozen eggs. Ask the students, "Do George and Martha have enough eggs to bake the cookies?" Have students draw a picture that will demonstrate this problem. Ask them, "How can you use fractions to solve this problem?" Let students share their various strategies and make sure to highlight any strategies that involve drawing a picture so that $\frac{1}{4}$ of a carton is represented as 3 out of 12 eggs, and $\frac{1}{3}$ of a carton is represented as 4 out of 12 eggs. George and Martha need 8 eggs but only have 7. Ask students to think about what strategies they used that help make adding $\frac{1}{4}$ to $\frac{1}{3}$ easier. Tell them these strategies will be used throughout this lesson in learning how to add and subtract fractions.

Spinning for LCD

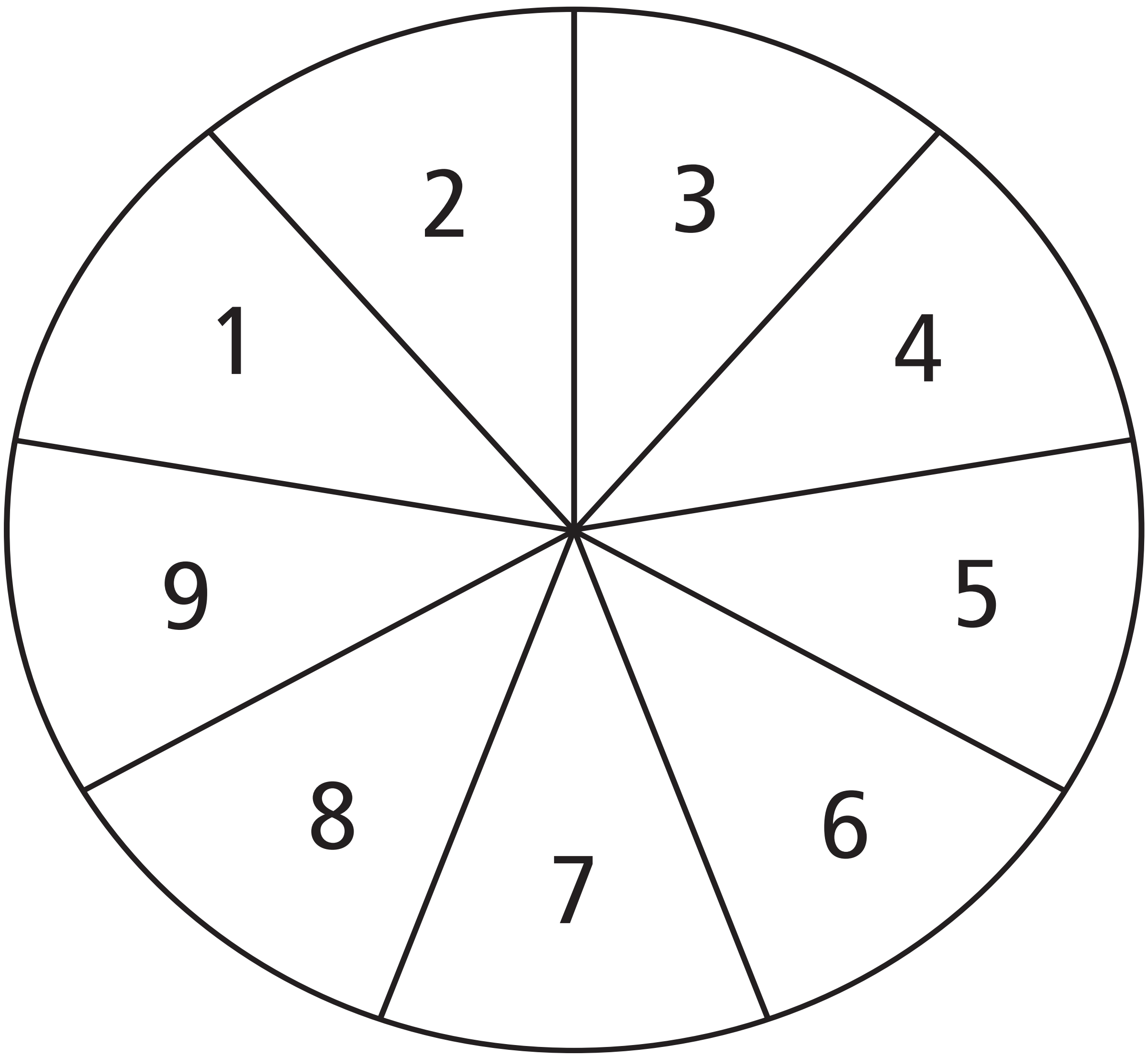
Objective: Students will be able to reinforce their flexibility with numbers by figuring out LCD's of two or more denominators.

Materials:
Number spinner (1 to 9)
Challenge spinner
Paper clip any size

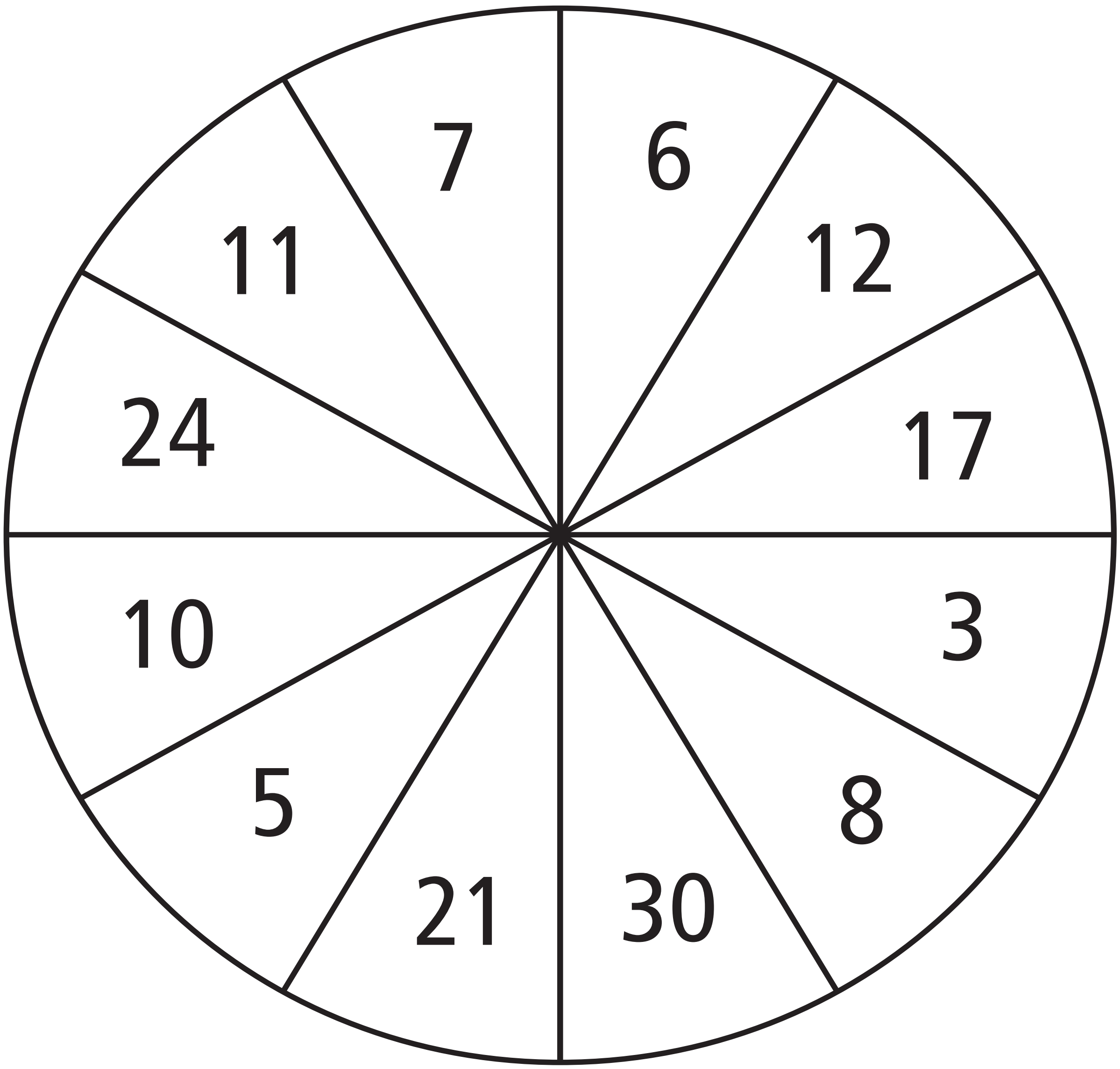
Activity Instructions:
Each student spins a number spinner as many times as needed to find 2 or 3 different denominators. Students will start with one-digit numbers. As students become more confident, they may use the challenge spinner. Students then, write these numbers down.

After the denominators are generated, each student finds the LCD of them. To figure out LCD, students may list common multiples & circle the LCM or LCD; use prime factorization (take the product of each prime raised to its larger exponent; Ex. the LCD is LCM (6,9) = $2 \times 3^2 = 18$); or any other strategy that makes sense to them.

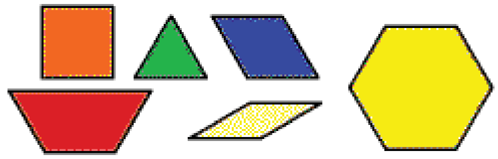
Number Spinner



Challenge Spinner



Pattern Block Fun



Objective: Students will use visuals with pattern blocks to reinforce addition and subtraction with fractions.

Materials:
Pattern Block Fun worksheet
Pattern Blocks (optional)

Activity Instructions:
A brief explanation about the four pattern block shapes may be necessary before the students begin this activity. If you have pattern blocks in your classroom, pull them out and let the students hold them, play with them and discuss the relationships between the shapes. If you don't have pattern blocks in your classroom, you can find examples of the blocks on the internet and copy them for your students to see. After a brief discussion about the pattern blocks, your students are ready to begin the activity.

Make a copy of the Pattern Block Fun worksheet for each student. Ask your students to follow the directions on the worksheet to complete the activity. This activity will work well if the students are grouped together, but it can also be done individually.

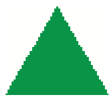
Answers:

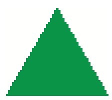
1. 1/6
2. 1/3
3. ½
4. 2/3
5. 1/9
6. ¼
7. 4
8. 3/7
9. 1
10.

Name ______________________________

Pattern Block Fun!

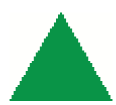
Directions: Look at the diagrams below and see if you can figure out the fractional patterns to answer each question.

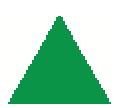
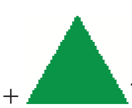
1. If = 1, then  = _____________.

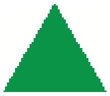
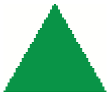
2. If  = 1, then  = ______________.

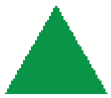
3. If = 1, then  = _____________.

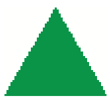
4. If  = 1, then  = _________________.

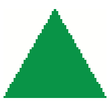
5. If +  = 1, what is  ? ___________

6. If +  = 1, what is  + ? ____________

7. If  +  = 1, what is +  ? ______________

8. If +  = 1, what is  ? __________________

9. If -  = 1, what is  +  ? _________

10. If  +  = 2/3, what is 1? ______________________

Draw the shape here

Notice that we don't warn kids to think about whether the denominators have to be the same. Make sure that they discover that you can only directly add the numerators when the denominators are the same.

Have your students explain in their own words what this rule is actually saying in terms of the models from above. For example, we added the numerators 1 and 2 because they tell us how many pieces we are combining. The denominator tells us the size of the pieces, which in this case is fifths, $\frac{1}{5}$. When we add, the size of the pieces remain the same. So the denominator remains 5. To combine fractions with the same denominators, add the numerators, and the result has the same denominator as the fractions being added. The Rule expresses this fact algebraically. It is important to remember that in order to use the Rule, the fractions must have the same denominator or the "pieces" must be the same size. If the fractions have the same denominator, combine the fractions by adding the numerators with the same denominator.

SECTION 4.4 ADDITION AND SUBTRACTION OF FRACTIONS

Adding 1 foot to 2 feet equals 3 feet. Combining 1 apple with 2 apples gives 3 apples. In each case, both numbers and units are important. Given these two examples, it seems reasonable to say that the sum of 1 fifth and 2 fifths is 3 fifths. More precisely, in Chapter 2, the linear skip counting model demonstrated that $\frac{3}{5}$ is $\frac{1}{5} + \frac{1}{5} + \frac{1}{5}$. Using skip counting, it is easy to see that

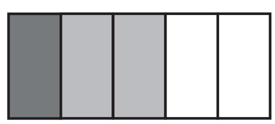
$$\frac{2}{5} + \frac{1}{5} = \left(\frac{1}{5} + \frac{1}{5}\right) + \frac{1}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}.$$

In general, for each positive integer m and n, the fraction $\frac{m}{n}$ is the sum of m unit fractions of the form $\frac{1}{n}$. For the rest of this chapter, assume all possible denominators are positive integers.

PROBLEM 1

Compute the sum of $\frac{3}{8}$ and $\frac{2}{8}$. Explain your answer.

Now look at the area model. How is the sum $\frac{1}{5} + \frac{2}{5}$ computed using the area model? Use a candy bar model. Betsy had $\frac{1}{5}$ of a candy bar, and her friend had $\frac{2}{5}$ of a candy bar like Betsy's.



Together, they have $\frac{3}{5}$ of a candy bar. Express this as $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$.

Write rules to generalize the previous discussion of adding fractions.

The sum of two fractions with like denominators, $\frac{a}{d}$ and $\frac{b}{d}$, is given by

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$$

PROBLEM 2

a. $\frac{9}{12} = \frac{3}{4}$ b. $1\frac{1}{5}$ c. 2

PROBLEM 3

(a). $\frac{3}{6} = \frac{1}{2}$

(b). $\frac{5}{10} = \frac{1}{2}$

(c). Make sure to note to students that $\frac{0}{8}$ should be written as 0.

PROBLEM 4

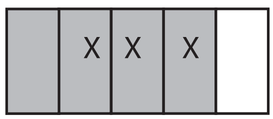
$$\frac{{}^\wedge 7 - 4h}{9} = \frac{3}{9} = \frac{1}{3}$$

PROBLEM 2

Find the sum and put in simplest form or if possible as a mixed fraction. Use model to illustrate these.

a. $\frac{4}{12} + \frac{5}{12}$ b. $\frac{3}{5} + \frac{3}{5}$ c. $\frac{2}{3} + \frac{2}{3} + \frac{2}{3}$

The same principle applies when subtracting fractions.



So, $\frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

PROBLEM 3

Find the difference and put in simplest form. Use models to illustrate each. Remember the order of operations.

a. $\frac{5}{6} - \frac{2}{6}$ b. $\frac{7}{10} - \frac{2}{10}$ c. $\frac{5}{8} - \frac{3}{8} - \frac{2}{8}$

PROBLEM 4

Compute $\frac{7}{9} - \frac{4}{9}$ and explain how to obtain the answer.

Describe how to subtract fractions with like denominators. Find the difference for $\frac{m}{n} - \frac{k}{n}$? How does your method compare to the addition rule above?

EXAMPLE 1

If you eat $\frac{2}{3}$ of a whole candy bar, how much of the candy bar is left? How can you use subtraction of fractions to answer this question?

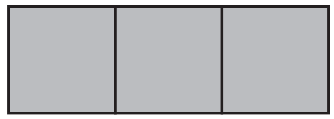
PROBLEM 5

Students might struggle with the 2. The 2 can be written as $1 + \frac{4}{4}$ or $\frac{8}{4}$. The first way will lead to the answer $1\frac{3}{4}$ and the second to the answer $\frac{7}{4}$.

SOLUTION

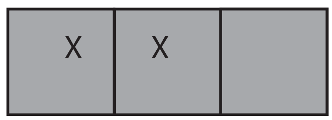
Using mathematical fractions, the problem looks like this: $1 - \frac{2}{3}$. In order to perform this calculation, begin by drawing a picture of a candy bar and divide it into three pieces. First convert 1 into the fraction $\frac{3}{3}$.

$1 = \frac{3}{3}$



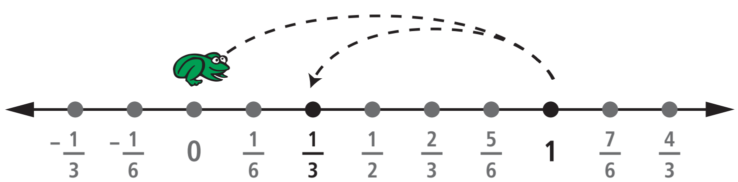
What happens when you subtract $\frac{2}{3}$ of the candy bar? X-out the portions that are subtracted, and the difference is

$1 - \frac{2}{3} = \frac{1}{3}$



Write this as $1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}$.

Another way to think of this subtraction uses the frog model. Like the model for subtracting integers, the frog hops 1 unit to the right and then hops backwards a distance of $\frac{2}{3}$ to land on the number $\frac{1}{3}$. This model represents $1 - \frac{2}{3}$.



PROBLEM 5

Compute the difference $2 - \frac{1}{4}$ and illustrate the process with either the area model or the linear model.

Now that you've explored adding and subtracting fractions with like denominators, explore finding the sum of an integer and a fraction, like the addition problem $2 + \frac{3}{5}$. Applying a similar process for this sum as you did for the last sum, there are several ways to combine these two quantities:

$$2 + \frac{3}{5} = 1 + 1 + \frac{3}{5} = \frac{5}{5} + \frac{5}{5} + \frac{3}{5} = \frac{13}{5}, \text{ or}$$

$$2 + \frac{3}{5} = 2\frac{3}{5}.$$

The first method trades one addition problem for another addition problem and results in an answer that is an improper fraction. The second way results in the mixed fraction $2\frac{3}{5}$. In Section 4.3, you learned how to convert mixed fraction to improper fractions and vice versa. Now you see that a mixed fraction can also be thought of as a sum. Which form is better, $2\frac{3}{5}$ or $\frac{13}{5}$? Explain your reasoning.

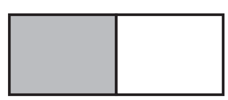
EXAMPLE 2

Explore how to use the ideas just learned to compute the sum of two fractions when the denominators are not the same.

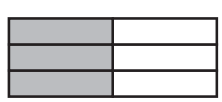
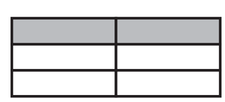
Use the area model to compute the sum $\frac{1}{2} + \frac{1}{3}$.

SOLUTION

Begin by looking at a visual representation.

$\frac{1}{2} =$  $\frac{1}{3} =$ 

Is it possible to combine the shaded amounts? Earlier, you discovered that in comparing the fractions $\frac{1}{2}$ and $\frac{1}{3}$, it was helpful to find equivalent fractions for both $\frac{1}{2}$ and $\frac{1}{3}$ to determine which is greater. Modify the picture above to display equivalent divisions of the whole.

$\frac{1}{2} =$  $= \frac{3}{6}$ $\frac{1}{3} =$  $= \frac{2}{6}$

EXPLORATION

The computation should be: $\frac{1}{3} + \frac{1}{4} = \frac{1(4)}{3(4)} + \frac{1(3)}{4(3)} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$.

The general process: $\frac{1}{a} + \frac{1}{b} = \frac{1(b)}{a(b)} + \frac{1(a)}{b(b)} = \frac{b}{ab} + \frac{a}{ab} = \frac{b+a}{ab}$.

Teacher, encourage your students to derive a general process for adding all fractions, a general $\frac{a}{b} + \frac{c}{d}$.

The process: $\frac{a}{b} + \frac{c}{d} = \frac{a(d)}{b(d)} + \frac{c(b)}{d(b)} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad+cb}{bd}$.

The rule is $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$.

PROBLEM 6

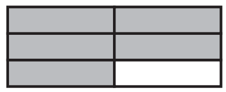
Some possible common denominators are 24, 36 and 12. They are all common multiples of 6 and 4. We could use $4 \cdot 6 = 24$ as the common denominator: $\frac{1}{6} + \frac{1}{4} = \frac{1(4)}{6(4)} + \frac{1(6)}{4(6)} = \frac{4}{24} + \frac{6}{24} = \frac{10}{24}$. Watch for any student who sees a better strategy than just multiplying the two denominators to get a "common denominator." By using 12, we get $\frac{1}{6} + \frac{1}{4} = \frac{1(2)}{6(2)} + \frac{1(3)}{4(3)} = \frac{2}{12} + \frac{3}{12} = \frac{5}{12}$, which is equivalent to $\frac{10}{24}$. Don't bring up the idea of LCD or LCM yet, but members of the class might make the connection to LCM. Let them work the next examples before formalizing the idea of LCD.

Teachers, don't worry if kids don't see this right away. Do not force the issue about least common denominators and adding fractions, this is covered more thoroughly in the text later.

a. 6 b. 48 c. 45 d. 35 e. 8 f. 36

To do this, divide the first model horizontally to represent $\frac{1}{2}$ as 3 parts out of 6 parts. Then, divide the second model vertically to represent $\frac{1}{3}$ as 2 parts out of 6 parts. It is easy to see from the model that $\frac{1}{2} = \frac{3}{6}$ and $\frac{1}{3} = \frac{2}{6}$ and that $\frac{3}{6}$ is greater than $\frac{2}{6}$.

More importantly, it is also easy to see how to add the two fractions in their equivalent forms.



Using the rule for adding fractions with like denominators, the sum is

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

In order to add the fractions, find a common-sized piece so that the two fractions can be written with the same or **common denominator**.

Using the equivalent fractions property, transform the two fractions to fractions with the same denominator

$$\frac{1}{2} = \frac{1 \times 3}{2 \times 3} = \frac{3}{6} \text{ and } \frac{1}{3} = \frac{1 \times 2}{3 \times 2} = \frac{2}{6}$$

The most important thing to remember when adding fractions is to ensure that you have a common denominator.

EXPLORATION

Compute the sum $\frac{1}{3} + \frac{1}{4}$ by first using the area model and then the equivalent fractions property to convert the fractions into equivalent fractions with like denominators.

Find the pattern to add the fractions $\frac{1}{a}$ and $\frac{1}{b}$ when a and b are not the same number and show the process.

PROBLEM 7

Have the class members share their work. Reflect with the class about each of the common multiples or common denominators they used. Students who make lists of multiples of denominators will use the LCM while others will just multiply the two denominators. Compare the two methods on the board and talk about the advantages or disadvantages for each. Using the LCM usually leads to an answer that is already in simplified form. The other method, often leads to an answer that needs to be simplified. After students see the advantage of using the LCM, define Least Common Denominator.

a. $\frac{7}{36}$ b. $\frac{19}{24}$ c. $\frac{31}{36}$

PROBLEM 6

Find three common denominators for the fractions $\frac{1}{6}$ and $\frac{1}{4}$. Write each fraction in equivalent forms using the three denominators. What do you notice about these common denominators? Which denominator would be the best choice for computing the sum $\frac{1}{6} + \frac{1}{4}$? Why?

Look at all the denominators you created in Problem 6. Do they have common factors? What is the relationship between every one of the common denominators and the original denominators? Which common denominator did you not have to simplify? Now combine the discoveries about common denominators in Problem 6 with those about common multiples from Section 3.5.

Let's review finding least common multiples.

Find the least common multiple of:

a. 3 and 6 c. 9 and 15 e. 2 and 8
b. 12 and 48 d. 5 and 7 f. 9, 6 and 12

When we add and subtract fractions, having a common denominator is very useful. In order to add $\frac{1}{3} + \frac{1}{6}$, use the equivalent fraction, $\frac{2}{6}$ for $\frac{1}{3}$. The restatement of the problem $\frac{1}{3} + \frac{1}{6}$ to $\frac{2}{6} + \frac{1}{6}$ makes finding the sum of $\frac{3}{6}$ easier to determine.

PROBLEM 7

To find the following sums: (1) find a common multiple for both denominators, (2) use it to find equivalent fractions for each fraction, (3) compute their sum and (4) simplify your answer, if necessary.

a. $\frac{1}{9} + \frac{1}{12}$ b. $\frac{3}{8} + \frac{5}{12}$ c. $\frac{7}{12} + \frac{5}{18}$

DEFINITION 4.1: LEAST COMMON DENOMINATOR
The **least common denominator** of the fractions $\frac{p}{n}$ and $\frac{k}{m}$ is the least common multiple of n and m.

EXERCISES

1. a. LCD = 35, $\frac{21}{35}$ and $\frac{30}{35}$
   LCD = 30, $\frac{36}{30}$, $\frac{9}{30}$, $\frac{4}{30}$
   b. LCD = 12, $\frac{6}{12}$, $\frac{9}{12}$ and $\frac{10}{12}$
   c. LCD = 24, $\frac{8}{24}$, $\frac{8}{24}$, $\frac{6}{24}$
   d. LCD = 16, $\frac{10}{16}$, $\frac{12}{16}$, $\frac{9}{16}$
   e. LCD = 36, $\frac{27}{36}$, $\frac{20}{36}$
   f. LCD = 9, $\frac{5}{9}$, $\frac{6}{9}$
   g.

2. a. $\frac{6}{8} = \frac{3}{4}$ b. $\frac{5}{10} = \frac{1}{2}$ c. $\frac{8}{5}$ d. $\frac{4}{7}$ e. 9/9 = 1 f. $\frac{4}{3}$ g. $3\frac{6}{8} = 3\frac{3}{4}$
   h. 30/3 = 10 i. 7/a j. 5/m k. $\frac{4}{x}$ l. 0/y = 0

   Observe how students deal with parts (k) and (l) to check if they understand the process. You might need to review what a variable is when working these problems.

3. a. $1\frac{4}{5}$ b. $\frac{3}{8}$ c. $\frac{7}{12}$ d. $\frac{5}{8}$ e. $\frac{5}{8}$
   f. $\frac{1}{20}$ g. $-\frac{1}{24}$ h. $\frac{1}{14}$ i. $\frac{1}{20}$ j. $\frac{5}{56}$
   k. $\frac{8}{15}$ l. $\frac{4}{9}$ m. $\frac{59}{56}$ n. $\frac{y+x}{xy}$ o. 5x

   Note that in part (n) the LCM or LCD is the product of the denominators. Why?

4. a. She plans to give away $\frac{3}{4}$. b. She has left $\frac{1}{4}$

In adding or subtracting fractions, the LCM of the denominators produces the least common denominator or LCD. Using the LCD has the advantage of working with smaller numbers.

EXPLORATION: MODELING FRACTION ADDITION

1. Draw a linear model to show the sum of $\frac{1}{2}$ and $\frac{1}{8}$. Explain what denominator you used and why.

2. Use an area model, for example a rectangle, to show the sum of $\frac{1}{2}$ and $\frac{1}{8}$.
3. Use another area model, for example a circle, to show the sum of $\frac{1}{2}$ and $\frac{1}{8}$.

EXERCISES

1. Find the least common denominator for the fractions. Write each equivalent fraction using the least common denominator.
   a. $\frac{3}{5}$ and $\frac{6}{7}$
   b. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}$
   c. $\frac{4}{12}, \frac{1}{3}, \frac{2}{8}$
   d. $\frac{5}{8}, \frac{3}{4}$ and $\frac{9}{16}$
   e. $\frac{3}{4}$ and $\frac{5}{9}$
   f. $\frac{2}{3}$ and $\frac{5}{9}$
   g. $\frac{6}{5}, \frac{3}{10}, \frac{2}{15}$
2. Add or subtract the following fractions. Write your answers in simplest form.
   a. $\frac{2}{8} + \frac{4}{8}$
   b. $\frac{3}{10} + \frac{2}{10}$
   c. $\frac{2}{5} + \frac{6}{5}$
   d. $1 - \frac{3}{7}$
   e. $\frac{4}{9} + \frac{5}{9}$
   f. $\frac{9}{3} - \frac{5}{3}$
   g. $4 - \frac{2}{8}$
   h. $\frac{16}{3} + \frac{14}{3}$
   i. $\frac{3}{a} + \frac{4}{a}$
   j. $\frac{8}{m} - \frac{3}{m}$
   k. $\frac{2}{x} + \frac{2}{x}$
   l. $\frac{5}{y} - \frac{5}{y}$
3. Compute the sums or differences. Write your answers in simplest form.
   a. $\frac{2}{3} + \frac{3}{5}$
   b. $\frac{5}{8} - \frac{1}{4}$
   c. $\frac{1}{4} + \frac{1}{3}$
   d. $\frac{6}{16} + \frac{2}{8}$
   e. $\frac{7}{8} - \frac{1}{4}$
   f. $\frac{1}{4} - \frac{1}{5}$
   g. $\frac{3}{8} - \frac{5}{12}$
   h. $\frac{6}{21} - \frac{3}{14}$
   i. $\frac{1}{10} - \frac{1}{20}$
   j. $\frac{3}{8} - \frac{2}{7}$
   k. $\frac{7}{10} - \frac{1}{6}$
   l. $\frac{4}{18} + \frac{2}{9}$
   m. $\frac{3}{7} + \frac{5}{8}$
   n. $\frac{1}{m} + \frac{1}{n}$
   o. $2x + 3x$

5. $\frac{19}{24}$

6. Paul is off by $\frac{2}{3} - \frac{3}{5} = \frac{10}{15} - \frac{9}{15} = \frac{1}{15}$. Mark is off by $\frac{7}{10} - \frac{2}{3} = \frac{21}{30} - \frac{20}{30} = \frac{1}{30}$. Since $\frac{1}{30}$ is less than $\frac{1}{15}$, then Rene's estimate is closer than Fred's estimate.

7. a. Students' models will differ. b. $\frac{3}{16}$ c. $\frac{3}{8}$

d. $\frac{7}{16}$

Observe whether students use subtraction or a strategy of counting from one point to the next. Reflect on both methods.

8. a. $1\frac{1}{8}$ b. $1\frac{3}{10}$ c. $1\frac{1}{6}$ d. $1\frac{7}{24}$ e. $\frac{69}{40} = 1\frac{29}{40}$

f. $\frac{75}{72} = 1\frac{1}{24}$

9. a. $\frac{4 + 9}{6} = \frac{13}{6} = 2\frac{1}{6}$, which is $\frac{2^2 : 3^2}{2 : 3}$ b. $\frac{16 + 25}{20} = \frac{41}{20} = 2\frac{1}{20}$, which is $\frac{4^2 + 5^2}{4 : 5}$

c. $\frac{9 + 49}{21} = \frac{58}{21} = 2\frac{16}{21}$, which is $\frac{3^2 : 7^2}{3 : 7}$

10. Spiral Review (6.1A)

a.

4. Julie made a large rectangular cake. She decided to give $\frac{1}{4}$ of it to her neighbor, $\frac{3}{8}$ to her mom, and $\frac{1}{8}$ to her sister as shown in the model below.

   a. Use the model to determine how much of the cake she plans to give away. Write an equation to show how this problem is represented mathematically.

   b. How much of the cake is left for her family?

5. To make a certain shade of purple paint, Jennifer must mix $\frac{2}{3}$ of a gallon of blue paint with $\frac{1}{8}$ of a gallon of red paint. How much paint will she make?

6. Sam is sponsoring a contest to see what fraction of the jellybeans in his jar are red. Paul estimates that $\frac{3}{5}$ of them are red while Mark estimates the amount to be $\frac{7}{10}$. Sam knows the answer is $\frac{2}{3}$. Whose estimate was closest to the actual fraction? How much closer is the winner to the amount?

7. Create a number line to resemble the section of the ruler between 0 and 1.

   a. Place and label the following points on the number line you created: $A = \frac{1}{8}$ $B = \frac{5}{16}$ $C = \frac{1}{2}$ $D = \frac{15}{20}$

   b. Using your model, find the distance from point A to point B.

   c. Using your model, find the distance from point A to point C.

   d. Using your model, find the distance from point B to point D.

8. Compute and simplify. Express as a mixed fraction, if needed:

   a. $\frac{3}{8} + \frac{1}{4} + \frac{1}{2}$ d. $\frac{3}{8} + \frac{2}{3} + \frac{3}{12}$

   b. $\frac{2}{4} + \frac{1}{5} + \frac{6}{10}$ e. $\frac{3}{5} + \frac{3}{8} + \frac{3}{4}$

   c. $\frac{1}{3} + \frac{3}{9} + \frac{2}{4}$ f. $\frac{4}{12} + \frac{3}{8} + \frac{3}{9}$

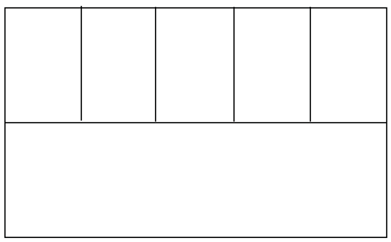
9. Compute the following sums. Express each as a simplified mixed fraction.

   a. $\frac{2}{3} + \frac{3}{2}$ b. $\frac{4}{5} + \frac{5}{4}$ c. $\frac{3}{7} + \frac{7}{3}$

11. Spiral Review (6.12A)
d.

Ingenuity:

12. Since the family ate one half of the cake, let's draw a picture of a cake and divide it into halves. Beau's sister, mother, and father all ate equal pieces of the cake, and Beau had a larger share that was equal to two pieces of cake. So we can treat the part of the cake that Beau's family ate as a total of five pieces of cake, all of which make up one half of the cake:



If we extend the vertical dividing lines down into the bottom half of the cake, we see that each piece was 1/10 of the cake. Since Beau ate two of these pieces, he had 2/10, or 1/5, of the cake.

Investigation:

13. TE: This Investigation foreshadows Section 4.5, in which we discuss how to add and subtract mixed fractions.

(a) Draw a picture that represents this situation.

(b) Using the picture or another method of your choice, figure out how many pounds of flour Candice now has. Express your answer as a mixed fraction.

TE: Candice now has 1 5/8 pounds of flour from her own house, and 2 1/2 pounds from Carl's. We can divide these amounts of flour into the whole number parts and fractional parts. The sum of the whole number parts is 3 pounds of flour. The sum of the fractional parts is 5/8 + 1/2 = 5/8 + 4/8 = 9/8 = 1 1/8. So Candice has a total of 3 + 1 1/8 = 4 1/8 pounds of flour.

Spiral Review:

10. Which statement about the mixed number $1\frac{1}{4}$ is true?

    a. $1\frac{3}{10} > 1\frac{1}{4}$
    b. $2 < 1\frac{1}{4}$
    c. $1\frac{1}{4} > 1\frac{3}{10}$
    d. $1\frac{1}{4} < 1\frac{1}{10}$

11. If Mr. Jones drives at a constant speed of 60 miles per hour, which method can be used to find the number of hours it will take him to drive 300 miles?

    a. Add 60 and 300
    b. Subtract 60 from 300
    c. Multiply 300 by 60
    d. Divide 300 by 60

12. **Ingenuity:**

    Beau's family baked him a cake for his birthday. After Beau blew out the candles, his sister, mother, and father all ate equal-sized pieces of the cake. Since it was Beau's birthday, he got to have a piece twice as big as what each of the others had. All together, Beau and his family ate one half of the cake. What fraction of the cake did Beau eat?

13. **Investigation:**

    Candice is baking a huge batch of cookies for a bake sale. She has $1\frac{5}{8}$ pounds of flour in her house. She decides this is not enough, so she borrows an additional $2\frac{1}{2}$ pounds of flour from Carl, her neighbor. Candice wants to know how many pounds of flour she has now.

    a. Draw a picture that represents this situation.
    b. Using the picture or another method of your choice, figure out how many pounds of flour Candice now has. Express your answer as a mixed fraction.

Section 4.5 - Common Denominators and Mixed Fractions

Big Idea:
Adding and subtracting mixed fractions, Regrouping in addition and subtraction of mixed fractions.

Key Objectives:
- Use models to add or subtract mixed fractions.
- Develop an algorithm for computing least common denominator.
- Use least common denominator to add or subtract mixed fractions.

Materials:
No extra materials needed

Pedagogical/Orchestration:
This section carefully shows the difference between common denominator and LEAST common denominator. The section also considers alternate approaching to adding mixed fractions: as mixed numbers and possible regrouping or as improper fractions. The students should have a chance to gain facility and understanding of both methods.

Activity:
"Addition Math Maze". Good for estimation as well as adding and subtracting fractions.

Vocabulary:
least common denominator (LCD), improper fractions, vertical addition, like parts, regroup, mixed numbers or fractions

TEKS:
6.2(B); 6.11(B); New: 6.3(E)

Launch for Section 4.5:

Write $1\frac{1}{a} + 1\frac{1}{b}$ on the board. Tell your students that we do not know what a and b are exactly but we know that they are positive integers greater than 1. Tell your students that the values to be added are called mixed fractions or mixed numbers because they represent the sum of a whole number and a fraction. In fact the mixed fractions can be rewritten like this: $1 + \frac{1}{a} + 1 + \frac{1}{b}$. Ask students to estimate what $1 + \frac{1}{a} + 1 + \frac{1}{b}$ equals. What two whole numbers would the sum be between? Let students think about this for awhile. If they struggle, suggest replacing a and b with integers such as 2 and 4, etc. The answer is that the sum is greater than 2 but less than or equal to 3. The students may even be able to tell you that the sum equals $2 + \frac{a+b}{ab}$. Go as far as you can with the students and then let them know that today they will be adding and subtracting mixed fractions and developing strategies for finding the least common denominator.

Alternate Launch for Section 4.5:

Discuss a situation in which the class has 5 pizzas. The class eats 2 3/4 of the pizza. How much of the pizzas remain. Draw pictures and bring in the circular representation. Use the picture and record the numerical process that goes with the process proposed by the students in the class.

One way to think about the problem is to break up the whole part from the fractional part. 5 pizzas can be thought of as 4 whole pizzas and 1 more pizza but cut into fourths. This is similar to the idea of having \$5 in five single bills or having \$4 in single bills and four quarters. Both are equal to \$5. The idea of decomposing numbers will be important in other settings including measurement and time.

Supplementary Lesson: Subtracting Fractions with Regrouping

To rename a whole number into a mixed number you borrow "one" from the whole number and rename it as a fraction.

Example: 7

Step 1: Borrow $7 - 1 = 6$

Step 2: Rename 7 as 6 + 1 ("1" is renamed to $\frac{4}{4}$)

Step 3: Regroup 7 as $6 + \frac{4}{4}$

$= 6\frac{4}{4}$

1) $9 = 8\frac{\ }{3}$	2) $4 = 3\frac{\ }{4}$	3) $7 = 6\frac{\ }{8}$
4) $9 - 2\frac{5}{8}$	5) $4 - 1\frac{2}{3}$	6) $7 - 6\frac{3}{4}$
7) $9\frac{3}{8} - 2\frac{5}{8}$	8) $4\frac{1}{3} - 1\frac{2}{3}$	9) $7\frac{1}{4} - 6\frac{3}{4}$
10) $8\frac{1}{9} - 2\frac{5}{6}$	11) $9\frac{3}{10} - 4\frac{1}{2}$	12) $4\frac{5}{6} - 3\frac{8}{9}$

Name______________________ Date ___________________________

Addition Math Maze

Directions: Using the **Addition** operation only, find the trail of numbers in each maze. Start with the number at the "Start" position and end with the number in the "Gray" square. You may only move vertically or horizontally.

1

	7 4/5	2 2/5	3 1/3
	6 2/3	5 1/5	4 1/5
Start	11 1/3	8 2/3	2 1/3
			37 7/15

2

	5 1/24	9 1/3	1 7/24
	4 2/3	3 ¾	7 1/6
Start	8 5/6	6 ½	2 7/8
			27 7/24

3

	1 2/5	9 7/10	6 1/5
	7 9/10	5 3/5	1 1/4
Start	2 ½	4 ¾	3 7/10
			20 19/20

4

	3 1/3	7 5/9	5 3/4
	6 4/9	9 ½	2 1/4
Start	4 2/3	8 1/9	1 3/7
			35 5/28

Addition Math Maze

Answer Key:

1. Start: 11 1/3 + 8 2/3 + 5 1/5 + 2 2/5 + 3 1/3 + 4 1/5 + 2 1/3 = 37 7/15

2. Start: 8 5/6 + 4 2/3 + 3 ¾ + 7 1/6 + 2 7/8 = 27 7/24

3. Start: 2 ½ + 7 9/10 + 5 3/5 + 1 ¼ + 3 7/10 = 20 19/20

Start: 4 2/3 + 6 4/9 + 3 1/3 + 7 5/9 + 9 ½ + 2 1/4 + 1 3/7 = 35 5/28

SECTION 4.5 COMMON DENOMINATORS AND MIXED FRACTIONS

In Section 4.4, the discussion of adding two fractions involved finding a common denominator. In many problems, you probably used the least common denominator (LCD), the LCM of the given denominators. We will discuss the advantages of using the LCD and then develop a systematic approach to computing the LCD.

In the Exploration in Section 4.4 you discovered a rule for adding two fractions with unknown and unlike denominators:

$$\frac{1}{a} + \frac{1}{b} = \frac{1 \cdot b}{a \cdot b} + \frac{1 \cdot a}{b \cdot a} = \frac{b}{a \cdot b} + \frac{a}{a \cdot b} = \frac{b + a}{a \cdot b}$$

Use this pattern to compute the sum in the following example.

EXAMPLE 1

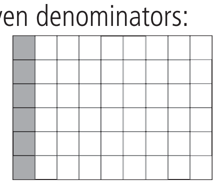
Compute: $\frac{1}{6} + \frac{1}{9}$

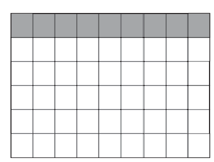
SOLUTION

Common Denominator Method:

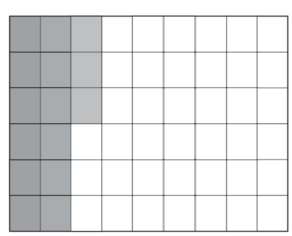
As in the rule above, you can create a common denominator for each sum by multiplying the two given denominators:

$$\frac{1}{6} + \frac{1}{9} = \frac{1 \cdot 9}{6 \cdot 9} + \frac{1 \cdot 6}{9 \cdot 6}$$





$$= \frac{9 + 6}{6 \cdot 9} = \frac{15}{54}$$



PROBLEM 1

Make sure your students use the prime factorization for the first few examples and some of the exercises. The main point is to choose the right factors in multiplying the numerator and denominator so that they get an equivalent fraction with the LCD. The process is shown for part (a) and is similar for parts (b) and (c).

$$
\begin{aligned}
\frac{1}{40} + \frac{1}{50} \quad &= \frac{1}{2^3 \cdot 5} + \frac{1}{2 \cdot 5^2} \\
&= \frac{1(5)}{2^3 \cdot 5(5)} + \frac{1(2^2)}{2 \cdot 5^2(2^2)} = \frac{5}{(40)(5)} + \frac{4}{50(4)} \\
&= \frac{5}{2^3 \cdot 5^2} + \frac{2^2}{2^3 \cdot 5^2} = \frac{5}{200} + \frac{4}{200} \\
&= \frac{5 + 2^2}{2^3 \cdot 5^2} = \frac{5+4}{200} \\
&= \frac{9}{200}
\end{aligned}
$$

Are these fractions simplified? Do the numerator and denominator have any common factors? The numerator and denominator can be factored into primes. A factor of 3 is common to both the numerator and the denominator.

$$\frac{15}{54} = \frac{3 \cdot 5}{2 \cdot 3 \cdot 3 \cdot 3} = \frac{5}{18}$$

Notice that this approach did not involve finding the LCD.

LCD Method:

Another approach is to first find the LCD of the fractions in each sum or, equivalently, the LCM of the denominators. A method for finding the LCD is to list the multiples of each denominator until the LCM is found. An alternate strategy is using the prime factorization. Look at the prime factorizations of the denominators of the fractions: $6 = 2 \cdot 3$, $9 = 3^2$. Remember the rule for finding the LCM of two numbers from their prime factorizations: take the product of each prime raised to its larger exponent. So, the LCDs are $\text{LCM}(6, 9) = 2 \cdot 3^2 = 18$.

Now, in computing the sum $\frac{1}{6} + \frac{1}{9}$, multiply the numerator and denominator of each fraction by a factor that will make the denominator the LCD:

$$\frac{1}{6} + \frac{1}{9} = \frac{1}{2 \cdot 3} + \frac{1}{3^2}$$

$$= \frac{1 \cdot 3}{(2 \cdot 3) \cdot 3} + \frac{1 \cdot 2}{3^2 \cdot 2}$$

$$= \frac{3 + 2}{2 \cdot 3^2} = \frac{5}{18}$$

Notice that the final answer is already simplified.

PROBLEM 1

Use the process just developed to compute the LCD for the fractions and then compute the sum:

a. $\frac{1}{40} + \frac{1}{50}$ b. $\frac{3}{8} + \frac{5}{12}$ c. $\frac{7}{10} + \frac{4}{9}$

Teachers, formulating the written procedure is a group activity that acts as a final check for understanding and a way for students to help each other and themselves when they discuss the procedure.

EXPLORATION 1

Have the class work on this problem and look for 2 approaches: (1) Converting the mixed fractions to improper fractions, add them and then convert them to a mixed fraction or (2) Add 3 + 5 and $\frac{1}{6} + \frac{1}{4}$. Then combine to form a mixed fraction. Reflect on the advantage of both methods. In the class discussion, look for a general approach for adding mixed fractions.

The general rule for the correct form in an answer is to leave the answer in its original form, in this case a mixed fraction.

Make sure that your students discover that when adding mixed numbers, adding just the whole number gives a good estimate of the result. Silvia has at least 3 + 5 = 8 pounds of sugar, and no more than 4 + 6 = 10 pounds.

EXAMPLE 2

$10\frac{11}{35}$

Formulate a written procedure that describes the process of:

- Finding the LCD for any two fractions
- Rewriting fractions equivalently using the LCD
- Computing sums and differences of two fractions.

EXPLORATION 1

Silvia is baking six sheet cakes for a party. The recipe she is using calls for $3\frac{1}{6}$ pounds of refined sugar and $5\frac{1}{4}$ pounds of unrefined sugar. First use the linear model to give an estimate of how much sugar Silvia needs. Then compute how many pounds of sugar Silvia needs. Explain your process for both the estimation and the calculation. Can you use the same process to add other mixed numbers?

Activity: Recipe Project

Bring a recipe from home. Make sure there are at least 8 ingredients. At least 4 of the ingredients should contain fractions and 2 should be mixed numbers. Be sure to indicate serving size.

1. Rewrite the recipe so that it would serve twice the number of people as the original recipe.
2. Rewrite the recipe so that it would serve half the number of people as the original recipe.

EXAMPLE 2

Compute the sum $6\frac{3}{5} + 3\frac{5}{7}$.

SOLUTION

There are at least three ways to compute this sum.

1. Improper Fractions:

You may wish to remind students what the Commutative and Associative Properties of Addition are.

One approach is to treat this as an ordinary fraction addition problem by converting from mixed to improper fractions and back again. First, convert the mixed fractions to improper fractions:

$$6\frac{3}{5} = \frac{6 \cdot 5}{1 \cdot 5} + \frac{3}{5} = \frac{33}{5} \quad \text{and} \quad 3\frac{5}{7} = \frac{3 \cdot 7}{1 \cdot 7} + \frac{5}{7} = \frac{26}{7}$$

Then, find the LCD and compute the sum. Note that in this case the denominators are relatively prime, so the LCD is their product.

$$\begin{aligned} \frac{33}{5} + \frac{26}{7} &= \frac{33 \cdot 7}{5 \cdot 7} + \frac{26 \cdot 5}{7 \cdot 5} \\ &= \frac{231}{35} + \frac{130}{35} \\ &= \frac{361}{35} \end{aligned}$$

Finally, convert the improper fraction to a mixed fraction and simplify. Because the largest multiple of 35 less than 361 is 350, convert 361 to $35 \cdot 10 + 11 = 350 + 11$ or $361 \div 35$ is 10 with a remainder of 11.

$$\begin{aligned} \frac{361}{35} &= \frac{350 + 11}{35} \\ &= \frac{350}{35} + \frac{11}{35} \\ &= 10 + \frac{11}{35} \\ &= 10\frac{11}{35} \end{aligned}$$

2. Combining Like Parts:

The improper fractions approach can be cumbersome because it involves working with relatively large numbers. Another approach is to consider each mixed fraction as the sum of an integer and a proper fraction and regroup the whole parts together and the proper fractions together:

$$6\frac{3}{5} + 3\frac{5}{7} = \left(6 + \frac{3}{5}\right) + \left(3 + \frac{5}{7}\right) = (6 + 3) + \left(\frac{3}{5} + \frac{5}{7}\right)$$

The general rule for the correct form in an answer is to leave the answer in its original form: in this case, a mixed fraction.

Note that in each step, the whole problem is written in the column. The process of finding the LCD and rewriting the fractions in equivalent form with the LCD as their denominators is not included below. This work should be shown somewhere else.

After going through the example, reflect with the class on advantages and disadvantages of the horizontal and stacking methods.

Finding the difference uses the same techniques, so its method is the same unless it is impossible to subtract the fractional parts without altering the mixed numbers, as we will demonstrate.

This leads to the sum of proper fractions:

$$\frac{3}{5} + \frac{5}{7} = \frac{3 \cdot 7}{5 \cdot 7} + \frac{5 \cdot 5}{7 \cdot 5}$$

$$= \frac{21}{35} + \frac{25}{35}$$

$$= \frac{46}{35}$$

$$= 1\frac{11}{35}$$

Combining these results, the original sum is

$$6\frac{3}{5} + 3\frac{5}{7} = \left(6 + \frac{3}{5}\right) + \left(3 + \frac{5}{7}\right)$$

$$= (6 + 3) + \left(1 + \frac{11}{35}\right)$$

$$= 10 + \frac{11}{35}$$

$$= 10\frac{11}{35}$$

As you can see, in computing the sum of mixed fractions, it is often easier to separate the mixed fractions as whole parts and fractional parts, add each group and then combine these two partial sums.

3. Vertical Addition:

There is another way to organize and write this same process vertically:

$$\begin{array}{r} 6\frac{3}{5} \\ +\,3\frac{5}{7} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{r} 6 \\ +\,3 \\ \hline \end{array} \quad \begin{array}{r} \frac{3}{5} \\ +\,\frac{5}{7} \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{r} 6 \\ +\,3 \\ \hline 9 \end{array} \quad \begin{array}{r} \frac{21}{35} \\ +\,\frac{25}{35} \\ \hline \frac{46}{35} \end{array} \quad \begin{array}{l} = 9 + \left(1 + \frac{11}{35}\right) \\ = 10 + \frac{11}{35} \\ = 10\frac{11}{35} \end{array}$$

How would finding the difference between two mixed fractions be different?

EXAMPLE 3

Compute the following differences:

a. $8\frac{4}{5} - 5\frac{3}{10}$ b. $6\frac{3}{5} - 3\frac{5}{7}$ c. $4 - 2\frac{3}{5}$

SOLUTION

a. Use the vertical method from the previous example:

$$\begin{array}{r} 8\frac{4}{5} \\ -\ 5\frac{3}{10} \\ \hline \end{array} \rightarrow \begin{array}{rr} 8 & \frac{4}{5} \\ -\ 5 & -\ \frac{3}{10} \\ \hline \end{array} \rightarrow \begin{array}{rr} 8 & \frac{8}{10} \\ -\ 5 & -\ \frac{3}{10} \\ \hline 3 & \frac{5}{10} \end{array} \begin{array}{l} = 3 + \frac{5}{10} \\ = 3 + \frac{1 :5}{2 :5} \\ = 3\frac{1}{2} \end{array}$$

Notice that the fraction $\frac{5}{10}$ in the solution is simplified to its equivalent fraction $\frac{1}{2}$.

b. Again, use the vertical method. However, a complication arises when attempting to subtract $\frac{5}{7}$ from $\frac{3}{5}$ since $\frac{5}{7}$ is greater than $\frac{3}{5}$.

$$\begin{array}{r} 6\frac{3}{5} \\ -\ 3\frac{5}{7} \\ \hline \end{array} \rightarrow \begin{array}{rr} 6 & \frac{3}{5} \\ -\ 3 & -\ \frac{5}{7} \\ \hline \end{array} \rightarrow \begin{array}{rr} 6 & \frac{3 :7}{5 :7} \\ -\ 3 & -\ \frac{5 :5}{7 :5} \\ \hline \end{array} \rightarrow \begin{array}{rr} 6 & \frac{21}{35} \\ -\ 3 & -\ \frac{25}{35} \\ \hline \end{array}$$

To avoid the negative fraction, rename 6 as 5 + 1 and regroup 1, or $\frac{35}{35}$, with the fraction $\frac{21}{35}$.

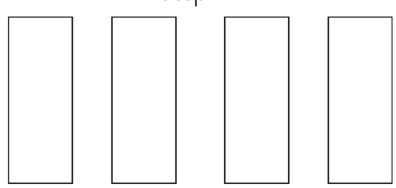
$$\begin{array}{rr} 5 & 1 + \frac{21}{35} \\ -\ 3 & -\ \frac{25}{35} \\ \hline \end{array} \rightarrow \begin{array}{rr} 5 & \frac{35}{35} + \frac{21}{35} \\ -\ 3 & -\ \frac{25}{35} \\ \hline \end{array} \rightarrow \begin{array}{rr} 5 & \frac{56}{35} \\ -\ 3 & -\ \frac{25}{35} \\ \hline 2 & \frac{31}{35} \end{array} \begin{array}{l} = 2 + \frac{31}{35} \\ = 2\frac{31}{35} \end{array}$$

c. Since you need to subtract $\frac{3}{5}$ you must regroup or borrow a whole from 4.

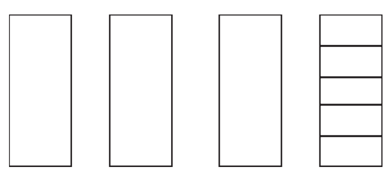
$$\begin{array}{r} 4 \\ -\,2\frac{3}{5} \\ \hline \end{array} \rightarrow \begin{array}{r} 3\frac{5}{5} \\ -\,2\frac{3}{5} \\ \hline \end{array} \rightarrow 1\frac{2}{5}$$

We can visualize the stages as follows where we start with Step 1 as shown below. The regroup 4 as $3 + 1 = 3 + \frac{5}{5}$ as shown in Step 2. Finally, subtract $2\frac{3}{5}$ as blacked out in Step 3 leaving $1\frac{2}{3}$.

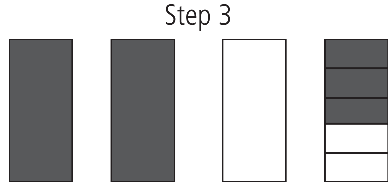
Step 1



Step 2



Step 3



1. a. $6\frac{4}{6} = 6\frac{2}{3}$
   b. $6 + 2 + \frac{2}{5} + \frac{2}{7} = 8 + \frac{14+10}{35} = 8 + \frac{24}{35} = 8\frac{24}{35}$
   c. $6\frac{3}{3} = 7$
   d. $8\frac{1}{8}$
   e. $12\frac{7}{12}$
   f. $9\frac{5}{12}$

2. Solutions for parts a - e are given, the rest of the parts work similarly.
   a. $3\frac{1}{30}$
   b. $1\frac{27}{40}$
   c. $2\frac{2}{3}$
   d. $3\frac{1}{2}$
   e. $5\frac{7}{8}$

3. $5\frac{3}{8} + 4\frac{1}{4} = 9 + \frac{3+2}{8} = 9\frac{5}{8}$

4. $1\frac{1}{8}$

5. $2\frac{2}{3}$

6. $4\frac{7}{8}$

7. $4\frac{23}{24}$

EXERCISES

1. Compute the following sums of mixed fractions using either the horizontal or vertical method. Show all the steps in the process. Simplify your answers if needed.

   a. $4\frac{1}{6} + 2\frac{3}{6}$ c. $2\frac{2}{3} + 4\frac{1}{3}$ e. $5\frac{3}{4} + 6\frac{5}{6}$

   b. $6\frac{2}{5} + 2\frac{2}{7}$ d. $3\frac{3}{8} + 4\frac{9}{12}$ f. $6\frac{6}{8} + 2\frac{2}{3}$

2. Compute the following differences of mixed fractions using the vertical method. Show all the steps in the process.

   a. $6\frac{1}{5} - 3\frac{1}{6}$ c. $6 - 3\frac{2}{6}$ e. $9\frac{3}{6} - 3\frac{5}{8}$

   b. $5\frac{3}{10} - 3\frac{5}{8}$ d. $7\frac{1}{8} - 3\frac{5}{8}$ f. $10\frac{4}{12} - 9\frac{7}{8}$

   For problems 3 - 9 below, write an expression that can be used to solve the problem. Then solve the problem and simplify if necessary.

3. Travis has $5\frac{3}{8}$ gallons of orange juice. Alex has $4\frac{1}{4}$ gallons of pineapple juice. How much fruit juice do the boys have together?

4. While training for a marathon, Joseph rode his bike $3\frac{5}{8}$ miles on Monday and $4\frac{3}{4}$ miles on Tuesday. How much farther did he ride on Tuesday than on Monday?

5. Noah is painting a large wall at the gym. He pours $2\frac{2}{6}$ gallons from a 5 gallon container. How much paint is left in the container?

6. Phillip cut $\frac{3}{8}$ of a foot off a piece of PVC pipe that was $5\frac{1}{4}$ feet long. How much pipe does he have left?

7. Frankie ate $2\frac{1}{3}$ servings of carrots, $1\frac{3}{4}$ servings of corn, $\frac{7}{8}$ servings of green beans, and $2\frac{2}{5}$ servings of bananas this weekend. How many servings of vegetables did Frankie eat this weekend?

8. During a recent stormy day it rained $10\frac{2}{10}$ inches in Austin and $7\frac{1}{2}$ inches in San Marcos. How much more rain fell in Austin than San Marcos?

8. $10\frac{2}{10} - 7\frac{1}{2} = 2\frac{7}{10}$ inches of rain.

9. Class A has 24 students of which $\frac{1}{3}$ are male. Since $\frac{1}{3} = \frac{8}{24}$, the number of males in class A is 8. Class B has 30 students of which $\frac{2}{3}$ are male. Since $\frac{2}{3} = \frac{20}{30}$, the number of males in class B is 20. The total number of males in both classes is 54. The total number of students in both classes is 52. The fraction of females in the PE class is $\frac{29}{52}$. The fraction of males in the PE class is 28 out of 54 . This is a subtle problem and is often missed by high school students.

10. a. $8\frac{3}{3}$ b. $3\frac{4}{4}$ c. $6\frac{8}{8}$

11. a. $6\frac{3}{8}$ b. $2\frac{1}{3}$ c. $\frac{1}{4}$

12. a. $6\frac{3}{4}$ b. $2\frac{2}{3}$ c. $\frac{1}{2}$

13. a. $5\frac{5}{8}$ b. $4\frac{4}{5}$ c. $\frac{17}{18}$

Spiral Review (6.1 E): 14. 72

Spiral Review (6.1 E): 15. $\frac{3}{4}$

Ingenuity

16. TE: We need to find the sum of the following mixed fractions:

1/4 + 2/4 + 3/4 + ... + 2 1/4 + 2 2/4 + 2 3/4

We could go from left to right and add the fractions two at a time, but there is a better way. Notice that if we put the shortest stripe and the longest stripe together, we get a 3-meter stripe. If we do the same with the second-shortest stripe and the second-longest stripe, we get another 3-meter stripe. We can continue pairing up all of the stripes in this manner, except for the 1 1/2-meter stripe in the middle, which does not have another 1 1/2-meter stripe that we can pair it with. When we finish this process, we have five 3-meter stripes, and one 1 1/2-meter stripe. The total of these lengths is 16 1/2 meters.

9. There are 2 fifth grade classes at Midway Elementary. Class A has $\frac{1}{3}$ males and Class B has $\frac{2}{3}$ males. Both classes have PE together. Class A has a total of 24 students and Class B has 30 students. Use what you've learned about fractions to find what fraction of the combined PE class is male.

10. Practice renaming whole numbers as mixed numbers:

a. $9 = 8\frac{}{3}$ b. $4 = 3\frac{}{4}$ c. $7 = 6\frac{}{8}$

11. Practice subtracting mixed numbers from whole numbers:

a. $9 - 2\frac{5}{8}$ b. $4 - 1\frac{2}{3}$ c. $7 - 6\frac{3}{4}$

12. Practice subtracting mixed numbers with like denominators:

a. $9\frac{3}{8} - 2\frac{5}{8}$ b. $4\frac{1}{3} - 1\frac{2}{3}$ c. $7\frac{1}{4} - 6\frac{3}{4}$

13. Practice subtracting mixed numbers with unlike denominators:

a. $8\frac{1}{9} - 2\frac{5}{6}$ b. $9\frac{3}{10} - 4\frac{1}{2}$ c. $4\frac{5}{6} - 3\frac{8}{9}$

Spiral Review:

14. What is the least common multiple that Pat can use to add three fractions with denominators of 3, 8, and 9?

15. Marco's homeroom teacher ordered 3 pizzas (pepperoni, cheese, and sausage) as a reward for their hard work. Each pizza was divided into 8 pieces. The students ate 5 pieces of pepperoni, 6 pieces of cheese and 7 pieces of sausage. What portion of the pizza was not eaten?

16. **Ingenuity:**

Doug Bob, a farmer, plans to paint gold stripes on one side of his tool shed. The side of his tool shed is triangular, as shown below, and is 3 meters tall at its highest point. Doug Bob plans to paint eleven stripes. The first stripe will be $\frac{1}{4}$ meter tall, and each successive stripe will be $\frac{1}{4}$ meter taller than the previous one, with the last stripe being $2\frac{3}{4}$ meters tall. What is the total length of the stripes Doug Bob will paint?

Investigation

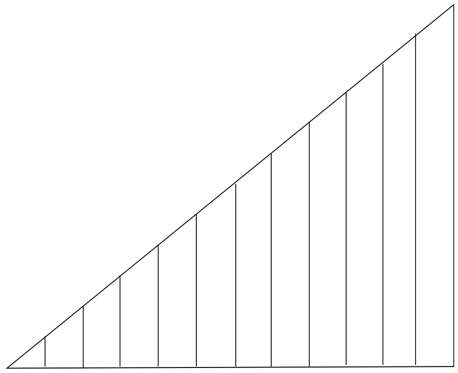
17. (a) 2/3 + 3/2 (b) 3/5 + 5/3 (c) 4/7 + 7/4 (d) 5/6 + 6/5
    TE: 2 1/6 TE: 2 4/15 TE: 2 9/28 TE: 2 1/30
    (e) 5/8 + 8/5 (f) 5/9 + 9/5 (g) 11/13 + 13/11 (h) 25/36 + 36/25
    TE: 2 9/40 TE: 2 16/45 TE: 2 4/143 TE: 2 121/900
    In each case, we are adding the fractions a/b and b/a, where a and b are positive integers. Every time, our answer is equal to the mixed fraction 2 (a-b)^2 / ab. Students may wonder why this is the case. Although the algebraic explanation is a bit beyond the scope of this course, we include it here. If we try writing the mixed fraction 2 (a-b)^2 / ab as an improper fraction, we get

$2\ (a-b)^2 / ab = 2ab / ab + (a-b)^2 / ab$
$= (2ab + a^2 - 2ab + b^2) / ab$
$= (a^2 + b^2) / ab$
$= a^2 / ab + b^2 / ab$
$= a/b + b/a,$

which are the two fractions we were adding together. Note that if a and b are very far apart, then (a-b)^2 may be greater than ab, in which case the fractional part (a-b)^2 / ab will be improper. In these cases, if we write the sum as a mixed fraction, we will actually get a whole number part greater than 2.

Challenge

18. 5



17. **Investigation:**

Add the following pairs of fractions, and in each case, write your answer as a mixed fraction. See if you can find a pattern in your answers.

a. $\frac{2}{3} + \frac{3}{2}$ b. $\frac{3}{5} + \frac{5}{3}$ c. $\frac{4}{7} + \frac{7}{4}$ d. $\frac{5}{6} + \frac{6}{5}$

e. $\frac{5}{8} + \frac{8}{5}$ f. $\frac{5}{9} + \frac{9}{5}$ g. $\frac{11}{13} + \frac{13}{11}$ h. $\frac{25}{36} + \frac{36}{25}$

18. **Challenge:**

How many pairs of positive integers $(a,\ b)$ with $a \le b$ satisfy

$\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$?

1. Answers may vary, but some equivalent fractions include:

a. $\frac{4}{10}$

b. $\frac{3}{5}$

c. $\frac{4}{6}$

d. $\frac{3}{12}$

e. $\frac{2}{3}$

2. Ordered from least to greatest:

e. $\frac{2}{8} = \frac{1}{4}$

c. $\frac{2}{6} = \frac{1}{3}$

b. $\frac{3}{8}$

d. $\frac{4}{10} = \frac{2}{5}$

a. $\frac{5}{10} = \frac{1}{2}$

f. $\frac{4}{6} = \frac{2}{3}$

3. Ordered from greatest to least:

Triangles: $\frac{8}{21}$ Circles: $\frac{7}{21} = \frac{1}{3}$ Squares: $\frac{6}{21}$

4. Answers may vary, but an example is: $\frac{7}{10}, \frac{3}{4}, \frac{4}{5}$

5. a. $\frac{4}{5}$ b. $\frac{9}{13}$ c. $\frac{1}{3}$ d. $\frac{16}{17}$ e. $\frac{1}{4}$ f. $\frac{6}{17}$

6. a. $\frac{4}{5}$ b. they are equal c. $\frac{5}{8}$ d. $\frac{4}{10}$ e. $\frac{8}{9}$ f. $\frac{6}{7}$

REVIEW PROBLEMS

1. Find an equivalent fraction for each of the following:

   a. $\frac{2}{5}$

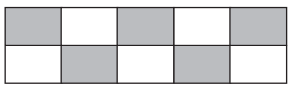
   b. $\frac{6}{10}$

   c. $\frac{2}{3}$

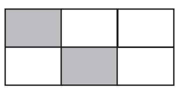
   d. $\frac{1}{4}$

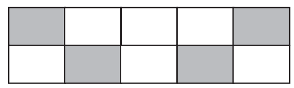
   e. $\frac{8}{12}$

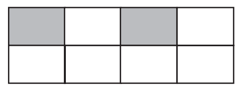
2. Label the fractional parts of the following and order from least to greatest:

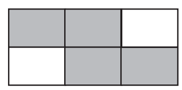
   a. 

   b. 

   c. 

   d. 

   e. 

   f. 

3. What fractions of the total number of these figures are squares, triangles, and circles? Order those fractions from greatest to least.

   △○□△□○△○△○△○△□○△□○□△□

4. Using the number line below, give three fractions that are greater than $\frac{1}{2}$ and less than $\frac{4}{5}$.

5. Rewrite each of the following fractions in simplest form.

   a. $\frac{20}{25}$

   b. $\frac{45}{65}$

   c. $\frac{12}{36}$

   d. $\frac{16}{17}$

   e. $\frac{17}{68}$

   f. $\frac{18}{51}$

6. Answer the following using <. >, or =

   a. $\frac{4}{5}$ ____ $\frac{4}{9}$

   b. $\frac{4}{6}$ ____ $\frac{10}{15}$

   c. $\frac{6}{10}$ ____ $\frac{5}{8}$

   d. $\frac{3}{8}$ ____ $\frac{4}{10}$

   e. $\frac{8}{9}$ ____ $\frac{7}{8}$

   f. $\frac{36}{42}$ ____ $\frac{6}{7}$

7. Isabel worked on her homework for $\frac{20}{60}$ of an hour = $\frac{1}{3}$ of an hour.

8. I ate the most pizza: $\frac{1}{2} = \frac{3}{6}$. My friend ate $\frac{1}{3} = \frac{2}{6}$. There is $\frac{1}{6}$ of the pizza left over.

9. 30 minutes

10. 20 days

11. a. $\frac{8}{5} = 1\frac{3}{5}$ b. $\frac{20}{8} = 2\frac{4}{8} = 2\frac{1}{2}$ c. $\frac{25}{10} = 2\frac{5}{10} = 2\frac{1}{2}$

12. a. $1\frac{5}{7}$ c. $9\frac{1}{6}$

b. $5\frac{3}{4}$ d. $3\frac{1}{3}$

13. a. $\frac{11}{3}$ c. $\frac{65}{8}$

b. $\frac{29}{5}$ d. $\frac{77}{6}$

Create a model for each of the following situations. Then answer each question.

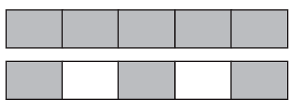
7. If Isabel worked on her math homework for 20 minutes, what fraction of an hour did she work on her homework? Write this fraction in its simplest form.

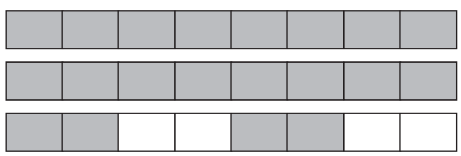
8. Your friend eats $\frac{1}{3}$ of a pizza and you eat $\frac{1}{2}$ of the same pizza. Who ate the most pizza? How much of the pizza is left over?

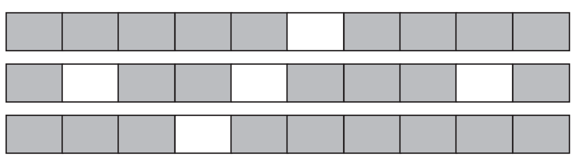
9. Joseph practiced the piano for $\frac{1}{4}$ of an hour and then cleaned the instrument for $\frac{1}{4}$ of an hour. How much time did he spend doing both activities?

10. The month of April has 30 days. If Pam worked $\frac{2}{3}$ of the number of days in April, how many days would that be?

11. Write the following shaded areas as an improper fraction and a mixed number.

a. 

b. 

c. 

12. Convert each improper fraction to a mixed number.

a. $\frac{12}{7}$

c. $\frac{55}{6}$

b. $\frac{23}{4}$

d. $\frac{10}{3}$

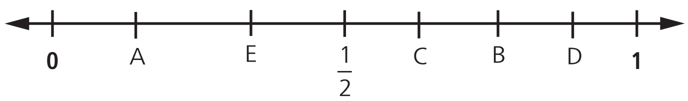
13. Convert each mixed number to an improper fraction.

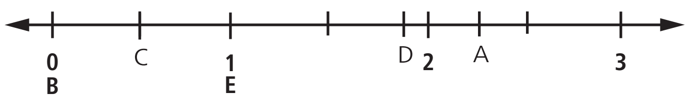
a. $3\frac{2}{3}$

c. $8\frac{1}{8}$

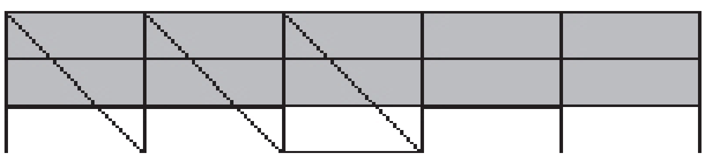
b. $5\frac{4}{5}$

d. $12\frac{5}{6}$

14. 

15. 

16. Use 15 as the least common denominator because 15 is the LCM of 3 and 5.



17. a. $\frac{29}{35}$ b. $\frac{9}{8} = 1\frac{1}{8}$ c. $\frac{35}{24} = 1\frac{11}{24}$ d. 7

e. $\frac{19}{12} = 1\frac{7}{12}$ f. $\frac{x - y}{p}$

14. Create a number line showing the following benchmark fractions: 0, $\frac{1}{2}$, and 1. Insert the approximate location of each given fraction by placing a point at the best location and naming it with the given letters.
A = $\frac{2}{11}$ B = $\frac{3}{4}$ C = $\frac{5}{9}$ D = $\frac{7}{8}$ E = $\frac{3}{10}$

15. Create a number line showing the following benchmark fractions by halves from 0 to 3. Insert the approximate location of each given fraction by placing a point at the best location and naming it with the given letters.
A = $2\frac{1}{4}$ B = $\frac{0}{8}$ C = $\frac{3}{6}$ D = $\frac{15}{8}$ E = $\frac{7}{7}$

16. What common denominator can you use to add $\frac{2}{3}$ and $\frac{3}{5}$. Explain how you decided and show it visually as well.

17. Perform the following operations. Write your answer in simplest form and as a mixed fraction, if appropriate.

a. $\frac{2}{5} + \frac{3}{7}$

b. $1\frac{1}{2} - \frac{3}{8}$

c. $\frac{4 : 5}{6} + \frac{5 : 3}{8}$

d. $1\frac{1}{4} - 5\frac{3}{4}$

e. $4\frac{1}{3} - 2\frac{3}{4}$

f. $\frac{x}{p} - \frac{y}{p}$

MATH EXPLORATIONS

Decimal and Percent Representations

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Section 5.1 develops the decimal concepts through a review of place values and transferring students' understanding of integers on the number line to now include decimals. Covered in this section are concepts of comparing, ordering, rounding decimal numbers. Students are also asked to divide a larger number into a smaller number with quotients that are now decimal numbers. Decimal addition and subtraction are presented in section 5.2 with an emphasis on an awareness of place value. Multiplication and division are also covered in this section motivated by the area model. Section 5.3 makes a connection between fractional and decimal representation of rational numbers and how students can convert from one form to the other. Decimal division is reinforced in the process of converting fractions to decimals with a connection made between the fractional notation as a way of representing the division operation of the numerator divided by the denominator. A discussion about repeating decimals that result from certain fractional representations is also included. Section 5.4 incorporates the use of percents as another way of representing rational numbers. The rational numbers are also given visual representations including 10 x 10 grid, fraction strips, and number lines. A project in the appendix can be used to introduce matters of financial literacy.

Section 5.1 - Constructing Decimals

Big Ideas:
- Develop decimal concepts.

Key Objectives:
- Review place value.
- Transfer integer understanding of the number line to decimals.
- Compare and order decimals.
- Round decimal numbers.
- Divide whole numbers into decimal numbers.

Pedagogical/Orchestration:
- Exploration is good practice on constructing number lines for decimal values.
- Use "Shopping Trip" activity at end of section.
- Paragraph just before Exercises is foreshadowing scientific notation.

Materials:
Number line handout from the CD, Graph paper, Rulers

Activity:
"Place Value Die", "Shopping Trip" and "High/Low"

Internet Resources:
Rags to riches game to review rounding and naming decimals: http://www.quia.com/rr/157757.html

Vocabulary:
denomination, place value, estimate, tenths, hundredths, thousandths

TEKS:
6.1(A,B,C); 6.11(A,B,C,D); 6.12(A,B); 6.13(A,B)

New: 6.4(G) 6.2(D,E,F)

Launch for Section 5.1:

Ask your students if they know the definition of a digit. Listen to responses and make sure that they understand that a digit is a numeral in the decimal number system. Ask the students, "How many digits are in this number system?" They have many misconceptions about this, so make sure they understand there are exactly 10 digits in our number system: 0-9. Ask students, "Why do you think that 10 digits were chosen for our number system base? Why not 8, or 11? (You can even show them how counting in a Base 8 system would look: 1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,20…) The answer to the question of why a base 10 system is of course because of our 8 fingers and 2 thumbs also known as "digits." Let students know that even the root of the word decimal, "decem," means 10 in Latin. Ask them what other words they know have this root and have something to do with the number 10. (decade, decimeter, even December which at one time was the 10th month of the year in the old Roman calendar.) Remind students, "Because of place value and powers of 10, we are able to represent any number you want using only these few 10 digits. (This is easier than the Mayan culture which had a vigesimal system with 20 symbols in its number system – presumably including toes with the fingers and thumbs.) After that, you might discuss what they know about place value, both before and after the decimal point. Finally, let students know that today is all about working with decimal numbers and understanding how the place value works.

Place Value Die

Objective: The student will be able to identify place value of numbers with place values from the thousands to the ten-thousandths. Materials: Place value chart 10-sided die

Materials:
Notebook Paper

Activity Instructions:

1. Students will work in small groups of 3 or 4.
2. Each student in their working group will roll the die 4 times to get 4 numbers which they may place in any order they wish. The fifth roll of the die will determine where the decimal point will be placed, according to the following guide:
    - Rolling 0 or 1 = decimal point goes all the way to the right of the 4th digit.
    - Rolling 2 or 3 = decimal point goes between the 3rd and 4th digits
    - Rolling 4 or 5 = decimal point goes between the 2nd and 3rd digits.
    - Rolling 6 or 7 = decimal point goes between the 1st and 2nd digits.
    - Rolling 8 or 9 = decimal point goes to the left of the 1st digit.

Digit Key:

1st	2nd	3rd	4th

3. In each group, students collect as many numbers as teacher chooses to assign them. Then, teacher asks students to share which student had the greatest number out of the entire class. The group with the greatest number out of the entire class gets a reward (teacher's choice).

Extension: Teacher may have students write out all numbers in word form.

Place Value Chart

PLACE VALUE CHART										
HUNDRED THOUSANDS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	ONES		TENTHS	HUNDREDTHS	THOUSANDTHS	TEN-THOUSANDTHS
100,000	10,000	1,000	100	10	1	.	0.1	0.01	0.001	0.0001

Shopping Trip

Objective: The students will answer questions about a shopping trip to reinforce and practice their skills with adding, subtracting, and estimating with decimals.

Materials:
Shopping Trip worksheet
Pencil
Calculator (for teacher only)

Activity Instructions:

1. Pass out one worksheet per student. Feel free to let the students work in groups or in partners, if they want.
2. While students are working on their worksheet, the teacher should be walking around the room and monitoring their progress. You might want to have a calculator in your hand while you are roaming. If you notice that a student has a wrong answer, have them check it on the calculator. Once they see that they have made a mistake, see if you can get them to discover where they went wrong.
3. Once all worksheets are completed, let students wander the room and compare answers with others. This is an excellent way to encourage students to defend their answer choices. If two students have answers that do not agree, then they should both discuss their methods and see if they can figure out who made the mistake and why.

Answer Key:
1. $10.74
2. $6.82
3. Answers will vary.
4. No, $20 is not enough. He would need $7.15 more to make all of those purchases.

Name ________________________________

Shopping Trip Worksheet

Jacob's daddy went grocery shopping this week at the local supermarket. Below is a list of some of the items that he was interested in buying. Use the table of items to answer the questions below about his shopping trip.

Corny Flakes Cereal	$4.59	Chocolate Chip Cookies	$3.45
Orange Juice	$4.65	Whole Milk	$2.70
Wheat Bread	$1.79	Peanut Butter	$2.89
Dozen Eggs	$0.99	Strawberry Ice Cream	$3.15
Hot Dogs	$1.35	Bagels	$1.59

1. If Jacob's daddy only bought cookies, cereal, and milk, how much money did he spend?

2. If Jacob's daddy only bought hot dogs, eggs, bagels and peanut butter, how much money did he spend?

3. If Jacob's daddy had a $20 bill, choose 5 items that he could buy. Then, tell me how much change he would have left after making this purchase.

4. If Jacob's daddy decided to buy everything in the chart, would $20 be enough to pay for it all? If so, how much extra money will he have left over? If not, how much more does he need?

High/Low

Objective: To reinforce skills learned in section 5.1 about comparing and ordering decimals.

Materials:
Copy of "High/Low" cards (preferably on cardstock)
Scratch Paper

Activity Instructions:

1. Announce that all cards represent decimal numbers between 0 and 10. Also, no decimal is extended beyond the hundredths place.
2. Ask students to separate into groups of 2 to 4. The students will decide on some strategy for deciding who goes first (roll a die, paper-rock-scissors, ...). Once it has been established who goes first, the game will continue in a clock-wise direction.
3. Shuffle the High/Low cards, then place the stack face-down in the center of the group.
4. The first person will draw one card and the person to the left will try to guess the number on that card. Each person will get exactly 10 guesses to try to guess the number on the card. As each student makes a guess, the person holding the card will either tell them that their guess is too high, too low or right on. After 10 guesses, if the number has not been established, the next person tries to guess the same number. If no one in the group guesses the card's number, the card is returned to the bottom of the pile, and the game continues with the next player.
5. Once a player has discovered the number, that player wins the card. The player keeps that card until the end of the game. The player with the most cards at the end of the game is the WINNER!

To modify this game to fit the needs of your class, just make your own deck of cards. For a less challenging game, make all of the decimal numbers to the tenths place. For a more challenging game, you extend the decimal numbers further than the hundredths place.

3.56	2.01	5.95	6.23
0.34	1.86	4.35	7.89
8.76	9.17	0.65	1.33
2.67	3.09	4.8	5.9

EXPLORATION:

Make sure for the difficult-to-draw number lines that you use the number lines on the CD that are created for this section.

DECIMAL AND PERCENT REPRESENTATIONS

5

SECTION 5.1 CONSTRUCTING DECIMALS

We have all been concerned about the cost of an item when we shop or go out to eat. For example, a cheeseburger might cost \$3.85 at a restaurant, a pair of jeans might cost \$18.97 at a store or the entrance fee to an amusement park might cost \$27.00. All of these prices use decimal notation, a common way of writing numbers that include parts that are less than 1. In the last example, \$27.00, we could have written \$27 instead. This means 2 tens and 7 ones. However, when we write \$27.00, the zeros to the right of 7 give us more information: that the price is exact and includes no cents. Just as with the integers, a place value is one-tenth the value of the digit to its immediate left. For example, in the number 43.26, the 3 is in the one's place which is one-tenth the ten's place occupied by the 4. Then 2 is in the tenth's place which is one-tenth the one's place. In an example with money, notice that the 8 in the cost of the \$3.85 cheeseburger is in the dime's place or the tenths-of-a-dollar place and 5 is in the penny's place or the hundredth's place. It takes 10 dimes to make a dollar and it takes 10 pennies to make a dime. Therefore it takes 100 pennies to make a dollar.

EXPLORATION: LOCATING DECIMAL NUMBERS ON A NUMBER LINE.

If we think of the number 1 on the number line as \$1.00, where would we locate half a dollar or \$0.50? Because there are 10 dimes in a dollar, where would \$0.10 be located on the number line? \$0.20? \$0.30? Can you locate \$0.01 or more simply 0.01 on the number line, knowing that there are 10 pennies in a dime?

We know when we write the number 0.30 that there is another way that this decimal can be written. Thirty hundredths can be written as 0.3. How could you show the two numbers 0.3 and 0.30 are really equivalent to each other on the number line?

Make sure your students discover the strategy for determining which number is located to the right of the other on the number line, i.e., 0.4 is to the right of 0.27 because 0.4 = 0.40 and 40 is greater than 27. As a general rule, write all numbers so they are exact to the same decimal place, adding zeros when necessary. Then start with the largest place value and order the digits, moving to the smaller place values. If the digits are equal, just go to the next smaller decimal place.

Connect idea of ordering decimals to alphabetizing words like cat, camp, and case. Note reading from left to right and ordering. Similarly order decimals.

Use a number line like the one below to find the locations of the following decimal numbers. Notice that 0 and 1 are labeled on the number line.

a. 0.4 c. 0.68

b. 0.27 d. 0.7

Discuss how you approximated the specific location. What strategy did you use to determine which number is greater than or less than another? In general, what strategy can you use to compare decimal numbers?

Numbers can be compared on the number line. The smaller number appears to the left of the larger number. Another strategy in comparing decimals is to write the numbers so that they have the same number of place value. Compare the digits from left to right until there is a difference in the digits in the same place value. For example, 0.2 and 0.27 can be written as 0.20 and 0.27. When we compare the hundredths place, clearly 27 hundredths is more than 20 hundredths because 27 is greater than 20.

For each pair of numbers, determine which is greater. Justify your answer using the number line.

a. 0.68 and 0.7 b. 0.34 and 0.339 c. 0.268 and 0.271

Remember what you have learned about decimals. You know that the number line is one tool that can help you locate decimals, order them, and compare them. The place value chart is another tool that can help you do this. Let's take a look at the next example to understand how the place value chart may be useful.

EXAMPLE 1

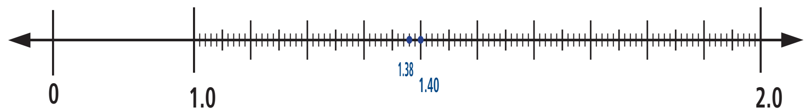
In a science project, Jeremy measured the distance two cars traveled from a common starting point. Car A traveled 1.38 m and Car B traveled 1.4 m. Which car traveled farther?

PROBLEM 1

a. $0.68 < 0.7$ b. $0.34 > 0.339$ c. $0.268 < 0.271$

d. $1.12 > 1.02$ e. $3.45 > 3.045$ f. $2.133 > 2.10$

SOLUTION



Since Car A is to the left of Car B on the number line, you can say that 1.38 < 1.4. Furthermore, by placing the lengths on the place value chart you can see that 38 hundredths is less than 40 hundredths.

If you place both numbers in a Place Value Chart you can clearly see that 1.38 is less than 1.40.

Place Value Chart

Tens	Ones	.	Tenths	Hundredths
	1	.	3	8
	1	.	4	0

PROBLEM 1

For each pair of numbers, determine which is greater. Justify your answer using a number line.

a. 0.68 and 0.7 b. 0.34 and 0.339 c. 0.268 and 0.271

d. 1. 12 and 1.02 e. 3.45 and 3.045 f. 2.133 and 2.10

Like whole numbers, decimals can also be rounded to a specified place value. Consider the number 18.625. If you were asked to round this to the nearest tenth, you begin by underlining the 6 in the tenths place, 18.$\underline{6}$25. Then, you look to the right of the underlined digit to determine if you should round up or down. Rounding up allows you to increase the underlined digit by 1 while rounding down means the underlined digit remains the same. In our example, the 2 means that the underlined 6 will round down or stay the same: 18.625 rounded to the tenth place is 18.6. An equivalent answer is 18.600 but the ending zeros are generally dropped when a decimal number ends in zeros.

Be sure that the students can read the decimal numbers correctly. Rather than reading a) "0.4" as "point 4" the students should be encouraged to read this as 4 tenths, b) "0.27" as "twenty seven one hundredths."

Rounding formally with decimals noting place value is important. Review whole number rounding process. 5+ up and 4 down. Include area model along with linear model. Focus on tenths and hundredths using linear models also weave in base ten area model as an alternative model. Use the place value chart. Explore locations of decimals on number line. Have students then articulate this process.

Be sure to explain zeros after significant digit and their effects on the number. For example 0.32 to nearest thousandths or 0.065 to nearest tenths.

PROBLEM 2

a. 127 b. 127.4 c. 127.40
d. 127.398

*Note: In part c, students may be confused with the rounding. Give extra, similar examples if needed.

PROBLEM 3

a. $719
b. $700

To round decimals:

1. Find the place value you want to round to. Call this the specified digit (The underlined place in our previous example above). Look at the digit to the right of it.

2. If the digit to the right is less than 5, do not change the specified digit and drop all digits to the right of it.

3. If the digit to the right is 5 or greater, add one to the specified digit and drop all digits to the right of it.

EXAMPLE 2

Round 21.0<u>9</u>5 to the nearest hundredth.

SOLUTION

Underline the 9, 21.0<u>9</u>5. Since the number to its right, 5, tells the 9 to round up, the 9 becomes 10. Remember that in place value you can only have one digit per place, so you write the 0 in the hundredths place value and carry the 1 to the tenths place. Therefore, 21.095 rounded to the nearest hundredth is 21.10.

PROBLEM 2

Round the following number 127.398359 to the specified place value:

a) nearest one's b) nearest tenths c) nearest hundredths
d) nearest thousandths

PROBLEM 3

The cost of a 50 inch flat screen television on sale is \$719.29.

a) Round the cost to the nearest dollar.

PROBLEM 4

$3.065 < 3.56 < 3.6 < 3.605 < 3.65$

If we follow the model of skip counting, we would say "how many jumps of 6 would reach 3?" This would be a $\frac{1}{2}$ of a jump. However, this process gets tricky if there are, say, 6 candy bars for \$1.80. This is an example of where we would switch the roles of 6 from the length of the jump to the number of jumps and the missing number becomes the length of the jump instead of the number of jumps.

This picture illustrates $x \cdot 6 = 3$ instead of $6 \cdot x = 3$ like the model normally suggests.

The reason why we can do this is because multiplication is commutative.

b) Round the cost to the nearest hundred dollars.

When alphabetizing two or more words, you ignore beginning letters that are alike until the first different letter to determine which word comes before another. For example, concentrate, concert, and cone are in alphabetical order. Can you see why? Similarly, when ordering numbers, compare the digits starting with the largest place value. Ignore the digits that are alike until the first place value that shows a difference. This process is often easiest to see if you make a vertical column of the numbers you are ordering.

PROBLEM 4

Write the following numbers in order from least to greatest.

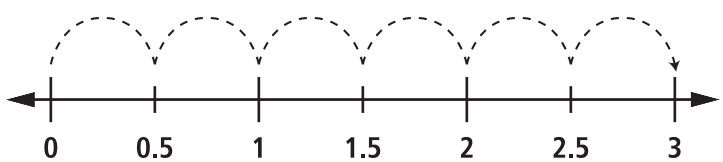
3.065, 3.6, 3.56, 3.605, 3.65

EXAMPLE 3

Sara spent \$3 on 6 chocolate bars. How much did each candy bar cost?

SOLUTION

This is a typical division problem where the cost of each candy bar is 3 ÷ 6. You might expect trouble because the divisor is greater than the dividend. Using the linear skip counting model, how long does each skip need to be to travel a distance of 3 units, or 3 dollars in this case, in 6 skips?



You can see that each jump is \$0.50 or half a dollar. This represents the fact that each candy bar costs \$0.50. If necessary, verify this using the calculator by computing 3 ÷ 6 or adding six skips 0.50 long. The long division method gives us the same result, because there is a decimal point before the 5. You might write the problem like step 1 on the next page. The divisor is greater than the dividend, so modify the long division process by placing a decimal point. Include two zeros

Notice that in problem (a), you must place a decimal point in the dividend and then add 0's as necessary to complete the long division until you get a remainder of 0. Remind students that in the traditional method, when dividing by a whole number they always line the quotient with the dividend and place the decimal point in the quotient above the decimal point in the dividend. Point out that this results in the same placement of the decimal points that the linear model predicts.

a. $\begin{array}{r} 0.20 \\ 5\overline{)1.00} \\ \underline{1.00} \\ 0.00 \end{array}$ b. $\begin{array}{r} 0.20 \\ 8\overline{)1.60} \\ \underline{1.60} \\ 0.00 \end{array}$ c. $\begin{array}{r} 0.30 \\ 4\overline{)1.20} \\ \underline{1.20} \\ 0.00 \end{array}$

In developing the area model with \$4 columns, we use the partitive model of division in which each column represents a distribution to each nephew, giving each a dollar. After 6 rounds, Mr. Garza has \$2 left over. We then distribute \$0.50 to each nephew on the last round.

Again, using the area model with \$4 columns, we find that he gives each child \$6 and has \$3.40 left over. He then distributes 85 cents to each child.

Answer: a. \$6.50 b. \$6.85

EXERCISES

1. a.) 0.3 b.) 0.1 c.) 4.5 d.) 3.0

to the right of the decimal point of the dividend because we are working with money. We know that \$3 = \$3.00 where \$0.00 represents no cents. Where does the decimal place appear in the quotient? Why does this make sense?

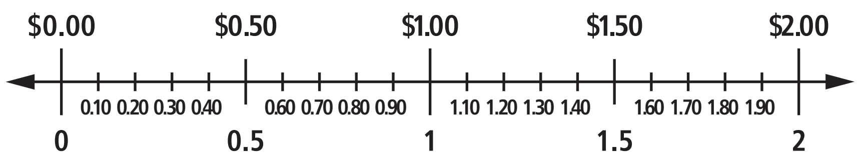
Step 1: $6\overline{)3}$

Step 2:

$$\begin{array}{r} 0.50 \\ 6\overline{)3.00} \\ \underline{-\ 3.00} \\ 0.00 \end{array}$$

Compute the following division problems by using an abbreviated number line from 0 to 2, like the one below. Find the quotient using the skip-counting method, then use the scaffolding method to verify your answer. Make sure the decimal point in the quotient makes sense in the context of the problem. Use the calculator to confirm your work, if necessary.

a. \$1 ÷ 5 b. \$1.60 ÷ 8 c. \$1.20 ÷ 4



When long division involves two-digit numbers, the skip-counting model becomes more difficult. We can also use the reverse of the area model to understand long division. At this point knowing the multiplication facts and how to use them makes life much simpler.

Mr. Garza has some money in his pocket that he intends to divide equally among his four nephews. Use the area model and the scaffolding model to compute how much each nephew receives if he has

a. \$26 in his pocket,

b. \$27.40 in his pocket.

EXERCISES

1. Round the following numbers to the nearest tenths:

   a. 0.25 b. 0.059 c. 4.531 d. 2.99

2. a.) 3.16 b.) 210.15 c.) 0.04

3. a.) 0.253 b.) 1.010 c.) 0.133

4. a.) 0.78, 0.8, 1 b.) 5.25, 5.3, 5 c.) 10.95, 10.9, 11

5.

Decimals	Words
0.07	seven hundredths
0.3	three tenths
0.90	Ninety hundredths
300.03	Three hundred and three hundredths
0.85	eighty-five hundredths
0.603	six hundred three thousandths
1000.0024	One thousand and twenty-four ten thousandths

6. Compare the following pairs of numbers using <, >. or =

a.	0.034 < 0.039	b.	1.21 = 1.210	c.	27.137 < 27.15
d.	0.835 < 0.836	e.	3.510 > 3.501	f.	61.35 = 61.350
g.	0.713 < 0.731	h.	3.63 < 3.6317		

7.
a. 0.035, 0.305, 0.350, 0.530
b. 25.0305, 25.1305, 25.1350, 25.1503

8. Terry walked more.
The ordered distances from least to greatest are: 2.09, 2.26, 2.73

9. The ordered scores are
87.3, 89.5, 90.01, 92.21, 93.4, 93.53

2. Round the following numbers to the nearest hundredths:

   a. 3.163 b. 210.152 c. 0.039

3. Round the following numbers to the nearest thousandths:

   a. 0.2531 b. 1.0099 c. 0.1327

4. Round the following numbers first to the nearest hundredths, then to the nearest tenths, and finally to the nearest ones:

   a. 0.78 b. 5.254 c. 10.949

5. Complete the table below:

Decimals	Words
0.07	
0.3	
	Ninety hundreths
	Three hundred and three hundredths
0.85	
0.603	
	One thousand and twenty-four ten thousandths

6. Compare the following pairs of numbers using <, >, or =

   a. 0.034 ____0.039 b. 1.21 ____1.210 c. 27.137____27.15

   d. 0.835 ____0.836 e. 3.510 ____3.501 f. 61.35____61.350

   g. 0.713 ____0.731 h. 3.63 ____3.6317

7. Order the decimals from the least to the greatest

   a. 0.305, 0.035, 0.350, 0.530

   b. 25.1305, 25.1350, 25.1503, 25.0305

8. Diann and Terry walked to exercise. If Diann walked 2.26 miles and Terry walked 2.73 miles, who walked more miles? Joan then joined them the next evening. She walked 2.09 miles. Write the distances in order from greatest to least.

9. Students in Ms. Martin's class scored well on their math benchmarks. Their scores were: 93.4, 89.5, 87.3, 92.21, 93.53, and 90.01. Order the scores from least to greatest.

10. a. 0.4 b. 0.2 c. 1.27

11. a., b., and c. Answers will vary.

12. Spiral Review (6.2 B)

$$4\frac{1}{3}+3\frac{1}{4}+2\frac{1}{3} - \frac{13}{3}+\frac{13}{4}+\frac{7}{3}$$
$$=\frac{52}{12}+\frac{39}{12}+\frac{28}{12}$$
$$=\frac{119}{12} - 9\frac{11}{12}$$

13. Spiral Review (7.2 E)

$8 + 7 \text{ x } 6^2 \div 2 =$

$8 + 7 \ \ 36 \div 2 =$

$8 + 252 \div 2 =$

$8 + 126 = 134$

14. **Ingenuity**

If we write a number that has a 1 in the ones place, then we know it will be less than 2.5. If we write a number with a 2 in the ones place, then the tenths place must be 4 or less, which means our number will still be less than 2.5. On the other hand, if we put a 3 or 4 in the ones place, then the number will be greater than 2.5, regardless of what we choose to put in the remaining positions.

There are six possible numbers with 3 in the ones place: 3.124, 3.142, 3.214, 3.241, 3.412, 3.421. Similarly there are six possible numbers with 4 in the ones place. Therefore, 12 of the numbers described are greater than 2.5.

This problem gives a good opportunity to check whether students understand how to compare decimal numbers which are expressed to different levels of precision. Students may be confused when confronted with numbers that have different number of digits after the decimal point and may think that a number with a greater number of digits shown after the decimal point (such as 2.341) is greater than a number with fewer digits shown after the decimal point (such as 2.5)

15. **Investigation**

a. $\frac{5}{10}$ are shaded.

b. Yes. We see here that $\frac{1}{2}$ is equivalent to $\frac{5}{10}$; as a decimal, this is 0.5.

c. 2 tenths are shaded, so $\frac{1}{5} = 0.2$

d. For this problem, we need to use a rectangle made of 100 pieces, such as a 10 x 10 square. If we shade $\frac{1}{4}$ of this rectangle, then we will have shaded 25 parts, or 25 hundredths. Therefore $\frac{1}{4} = 0.25$. Note that students may initially have some trouble thinking of 0.25 as “25 hundredths” rather than “2 tenths and 5 hundredths.” It may be worthwhile to work out that these are equivalent.

10. Divide each of the following.

    a. $2 \div 5$ b. $\$1.40 \div 7$ c. $15.24 \div 12$

11. In working each of the following exercises, be careful to scale your number line appropriately.

    a. Name two decimals between 0.1 and 0.3. Draw a number line and locate the four decimal numbers on it.

    b. Name four decimals between 0.29 and 0.307. Draw a number line and locate the six decimal numbers on it.

    c. Name two decimals between $^{-}0.11$ and $^{-}0.24$. Show whether $^{-}0.109$ is between $^{-}0.11$ and $^{-}0.24$. Draw a number line and locate the five decimal numbers on it.

Spiral Review

12. Joe is making a snack mix for a study group that is meeting at his house after school. He mixed $4\frac{1}{3}$ cups of cereal, $3\frac{1}{4}$ cups of nuts, and $2\frac{1}{3}$ cups of pretzels. How many total cups of ingredients are in his snack mix?

13. What is the value of the expression $8 + 7 \times 6^2 \div 2$?

14. **Ingenuity:**

    Consider all of the numbers that we can make by putting the digits 1, 2, 3, and 4 in some order, in the blanks below:

    _ . _ _ _

    How many of these numbers are greater than 2.5?

15. **Investigation:**

    Consider the following rectangle, which is divided into tenths:

    a. Copy the figure above, and shade $\frac{1}{2}$ of the figure. How many tenths are shaded?

    b. Can we use what we found in part a to write $\frac{1}{2}$ as a decimal?

    c. Copy the original figure again, and shade $\frac{1}{5}$ of it. How many tenths are shaded? Use this information to write $\frac{1}{5}$ as a decimal.

    d. How could we write $\frac{1}{4}$ as a decimal? (Hint: Consider using a rectangle which is divided up into pieces smaller than tenths.)

Section 5.2 - Operating with Decimals

Big Idea:
Adding, Subtracting, Multiplying, and Dividing Decimal numbers

Key Objectives:

- Add and subtract decimal numbers with an awareness of place value.
- Multiply and divide decimals with connection to the area and linear models.

Materials:
Paper for folding (sentence strips or adding machine tape) Grid paper, Fraction Chart from the CD

Pedagogical/Orchestration:

- Refer to Section 5.1 to review place value. The Place Value Chart will help students make a connection between decimals and fractions. Reinforce to students that when you say the name of a decimal, you are saying the fraction that it is equal to. For example: $0.751 = \frac{751}{1000}$; $0.75 = \frac{75}{100}$; $0.7 = \frac{7}{10}$.

Activities:
"Concentration Game", "Shopping Trip"

Exercises:
Exercise 2: Either a number line approach or subtraction approach can be used. You may observe student preference or have students do some using both approaches. Idea of distance is being used.

Vocabulary:

fraction, decimal

TEKS:
6.1(C); 6.2(A,B,D); New: 6.3(D,E)

Launch for Section 5.2:
Do Shopping Trip Activity (refer to section 5.1 for activity worksheet)

Concentration Game Activity:

Materials:

10 index cards (or more) with a fraction written on each)
Corresponding decimal representation on the other 10 cards

Activity Instructions:

1. Turn the cards face down and have students take turns trying to match the fraction with the decimal value.
2. The winner is the player who matches the most number of pairs correctly.

Linear Model

a.	0.26	c.	0.77
b.	0.85	d.	0.57

SECTION 5.2 OPERATING WITH DECIMALS

ADDITION AND SUBTRACTION OF DECIMALS

EXPLORATION 1

Betty is about to take a trip. She fills her car with gas for \$29.90 and buys a map for \$3.49, a drink for \$1.09 and a pack of gum for \$0.99. Estimate the cost of her purchase before taxes. Is \$40.00 enough to pay for the purchase, excluding tax?

If you calculated \$30.00 + \$3.50 + \$1.00 + \$1.00 to get \$35.50 you had a good estimate of her cost. To get the exact cost, however, you must add \$29.90 + \$3.49 + \$1.09 + \$0.99 or take the estimated cost and subtract the excess of 10¢ + 1¢ + 1¢ from the estimated cost, then add in 9¢ for the underestimation of \$1.09. You overestimated by 3¢ so you should subtract 3¢ from \$35.50 for the actual cost.

When you subtract 3¢ from \$35.50, you are really subtracting \$0.03 from \$35.50 to get \$35.47. As with addition, it is important to keep in mind the place value and subtract the hundredths from the hundredths, the tenths from the tenths, and so forth. You might have heard the phrase "line up the decimals." This vertical, stacking method assures that the place values also line up to do the calculation.

$$\begin{array}{r} 29.90 \\ 3.49 \\ 1.09 \\ +\ \ 0.99 \\ \hline 35.47 \end{array}$$

Linear Model:

How do you use the number line to add decimal numbers? Compute the sums in parts a & b using the number line. Compute the sums in parts c & d using the stacking method.

a. 0.2 + 0.06

b. 0.38 + 0.47

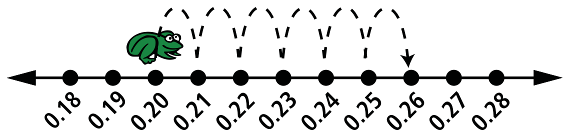
c. 0.23 + 0.54

d. 0.26 + 0.31

a. 0.16 b. 0.14

a. 1,500,000
b. 900,000
c. 8,900,000,000

For example, 0.2 + 0.06 = 0.20 + 0.06.



Explain how the number line can help to estimate a sum before you calculate the actual total.

How do you use the number line to subtract decimals? Compute the following differences using the number line, and then subtract using the traditional stacking method.

a. 0.63 – 0. 47 b. 0.2 – 0.06

There are times when we need to write really big numbers but want to avoid writing too many zeros. For example, we write 2 million instead of 2,000,000 or 30 million instead of 30,000,000. If we write 700 thousand, we mean 700,000. Finally, if 1 million is 1,000,000 then 0.5 million is half of 1,000,000 or 500,000.

a. What is another way of writing 1.5 million?
b. What is another way of writing .9 million?
c. Write 8.9 billion as a whole number.

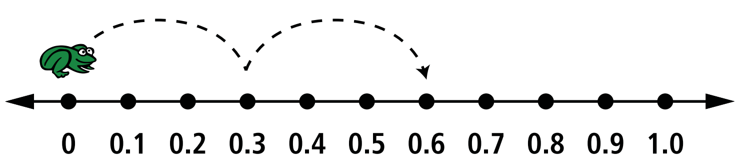
MULTIPLICATION OF DECIMALS

EXPLORATION 2

Use the linear model to show how to compute the following products:

a. $3 \cdot 2$
b. $(0.3) \cdot 2$
c. $(0.3) \cdot (.2)$

The first product is simply 2 jumps of length 3. The second product is 2 jumps of length 0.3, as shown below:



Revisit the area model and have students draw the (3)(2) model.

Have students observe that there are 100 small squares of dimensions 0.1 by 0.1. That would make each of its area 1/100 the area of the 1 by 1 square. So, each 0.1 by 0.1 square has area 0.01 square unit.

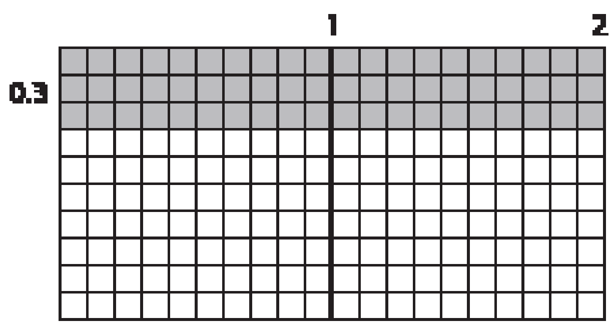
Two jumps of length 0.3 gives us the location of 0.6.

However, modeling the third product, (0.3)(0.2), is not clear. What do we mean by 0.2 jumps? For this product, the area model may be more helpful.

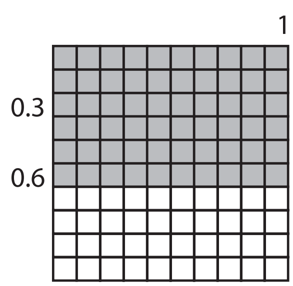
Using the area model for (3)(2), we find the area of a 3 by 2 rectangle. How do we use the area model for (0.3)(2)?

Draw a rectangle that has length 0.3 and width 2. When drawing this rectangle it is helpful to use grid paper and choose an appropriate scale. In this case, we need to measure both 0.3 and 2. Using a grid, assign each small square a length of 0.1

Outline a 1 by 1 square using the 0.1 grid. How many 0.1 by 0.1 small squares are in a 1 by 1 square? Now outline a 0.3 by 2 rectangle on the grid. With a picture of the rectangle and its dimensions, you can see that the area is made up of 60 small squares, each with an area of $0.1 \times 0.1 = 0.01$. So the product of $(0.3)(2) = (60)(0.01)$.

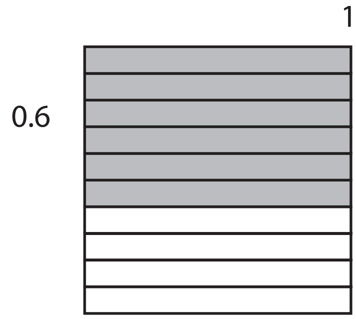


We can rearrange the model above to look like the following:

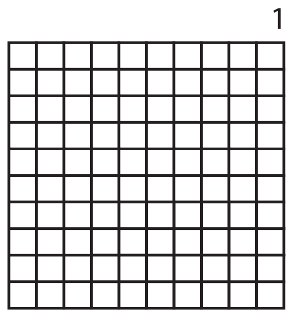


Thus, $(60)(0.01) = 0.60$ since there are 60 hundredths shaded above.

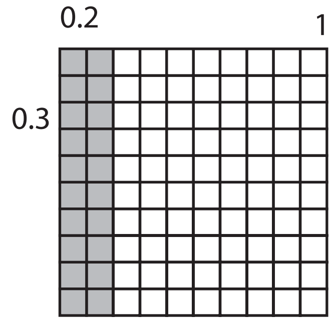
However, we can write 0.60 as 0.6, or 6 tenths, which is modeled below.



How do we compute the product (0.3)(0.2) using the area model? Consider the grid below. Each side of the square has length 1. Note that the length of each little square is 0.1. Use this grid to model the product (0.3)(0.2).



One way to show the product (0.3)(0.2) is to shade a rectangle within the grid that is 0.3 long (horizontally) and 0.2 wide (vertically).



The result is a small rectangle with 6 little squares. What is the area of each little square? Since the large square has area 1 and there are 100 little squares, the area of each little square is 0.01 or 1/100. So the area of 6 little squares is 0.06 or 6/100.

PROBLEM 1

What patterns do you notice?

a.	0.42	c.	0.24
b.	1	d.	1.26

PROBLEM 2

(3)(1) = 3	(0.3)(2) = 0.6	(0.6)(7) = 4.2
(3)(0.1) = 0.3	(0.3)(0.2) = 0.06	(0.6)(0.7) = 0.42
(3)(0.01) = 0.03	(0.3)(0.02) = 0.006	(0.6)(0.07) = 0.042
(3)(0.001) = 0.003	(0.3)(0.002) = 0.0006	(0.6)(0.007) = 0.0042
(0.3)(0.1) = 0.03	(0.03)(0.2) = 0.006	(0.06)(0.07) = 0.0042

Discuss why the decimal seems to move and why this might be the case. Some of this could be visualized on the number line. The important point is for students to see if the magnitude of the product is reasonable given the factors and their magnitudes. This discussion should precede any conclusion about a rule for how many decimal places to use in the product. This discussion can come up after problem 3.

PROBLEM 3

(2.4)(3.1) = 7.44	(562)(7) = 3934	(0.483)(27) = 13.041
(0.24)(3.1) = 0.744	(562)(0.7) = 393.4	(4.83)(27) = 130.41
(0.024)(3.1) = 0.0744	(56.2)(0.7) = 39.34	(48.3)(27) = 1304.1
(0.24)(0.31) = 0.0744	(5.62)(0.7) = 3.934	(4.83)(2.7) = 13.041
(24)(0.31) = 7.44	(0.562)(7) = 3.934	(4.83)(0.27) = 1.3041

PROBLEM 1

Compute the following products using the grid.

a. (0.6)(0.7) c. (0.3)(0.8)

b. (0.5)(2) d. (0.9)(1.4)

PROBLEM 2

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

(3)(1) =	(0.3)(2) =	(0.6)(7) =
(3)(0.1) =	(0.3)(0.2) =	(0.6)(0.7) =
(3)(0.01) =	(0.3)(0.02) =	(0.6)(0.07) =
(3)(0.001) =	(0.3)(0.002) =	(0.6)(0.007) =
(0.3)(0.1) =	(0.03)(0.2) =	(0.06)(0.07) =

What patterns do you notice?

PROBLEM 3

Compute the following groups of products. Look for patterns that can help us multiply decimal numbers. You may use a calculator.

(2.4)(3.1) =	(562)(7) =	(0.483)(27) =
(0.24)(3.1) =	(562)(0.7) =	(4.83)(27) =
(0.024)(3.1) =	(56.2)(0.7) =	(48.3)(27) =
(0.24)(0.31) =	(5.62)(0.7) =	(4.83)(2.7) =
(24)(0.31) =	(0.562)(7) =	(4.83)(0.27) =

a. What patterns do you notice?

b. How many decimal places does each factor have?

c. How many decimal places are in each product?

d. What is the connection between these two for each product?

Have a discussion about the placement of the decimal in the product and whether their answer seems reasonable with respect to the magnitude of the factors. The other part of the decimal multiplication is the opportunity to review whole number multiplication.

One observation you may have made is that the products have the same digits as if you were doing the whole number multiplication. Another important observation is that the number of decimal places in the product is equal to the sum of the number of decimal places in the factors.

For example,

.2	(1 decimal place)	2	(no decimal places)
x .3	(1 decimal place)	x .3	(1 decimal place)
.06	(2 decimal places)	.6	(1 decimal place)

This pattern is useful when we multiply numbers with many decimal places. The linear model and even the area model become more difficult to use, but the pattern continues to be useful.

For example,

.2	(1 decimal place)
X .03	(2 decimal places)
.006	(3 decimal places)

You may wish to use this method of multiplying the decimal numbers by first multiplying the factors as if they were whole numbers. After you find the whole number, include the decimal point in the correct place using the pattern we used above.

DIVISION OF DECIMALS

When you divide whole numbers, you may find the resulting quotient is no longer a whole number. Here is an example.

Lisa spent $4 on 8 identical candy bars. How much did each candy bar cost? Using a linear model, how long does each trip need to be in order to reach 4 using 8 total hops? You can see that each hop is $.50 or half a dollar. This represents the fact that each candy bar costs $0.50.

The long division method gives us the same result, because there is a decimal point before the 5. You could write the problem as we have below in step1.

Step 1: $8\overline{)4}$

The divisor is greater than the dividend, so modify the long division process as in step 2 below by placing a decimal point after the dividend. Include the two zeros in the dividend because we are working with money and we know that $4 = $4.00 where $0.00 represents zero cents.

PROBLEM 4

$0.30 or 30 cents each

EXERCISES

1.	a. 0.89	e. 1.4	i. 20.66
	b. 1.05	f. 2.53	j. 2.787
	c. 5.766	g. -0.229	k. 45.43
	d. 16.86	h. 22.277	l. 1.987

2. a. 0.34 and 0.37
   b. 0.892 and 0.908
   c. 0.098 and 0.11

3. between 327.6 grams and 324.4 grams

4. 6.755 meters

5. 1.361 grams

Step 2:

$$\begin{array}{r} 0.50 \\ 8\overline{)4.00} \\ -4.00 \\ \hline 0.00 \end{array}$$

PROBLEM 4

Adam bought 9 jawbreakers for $2.70. Use the long division process to determine the cost of each jawbreaker.

EXERCISES

1. Compute the following using the stacking method:

a. 0.53 + 0.36	e. 0.83 + 0.57	i. 18.9 + 1.76
b. 0.4 + 0.65	f. 4.6 – 2.07	j. 1.7 + 1.087
c. 2.406 + 3.36	g. 0.058 – 0.287	k. 16.43 + 29
d. 13.8 + 2.97 + 0.09	h. 3.59 + 0.087	l. 0.007 + 0.08 + 1.9

2. Determine which of the following pairs of numbers is closer together.
   a. 0.6 and 0.7 or 0.34 and 0.37
   b. 0.87 and 0.91 or 0.892 and 0.908
   c. 0.23 and 0.196 or 0.098 and 0.11

3. Maria is making homemade tortillas from her grandmother's recipe. The recipe calls for 326 grams of salt. While experimenting with the recipe, she discovers that if she increases or decreases the recipe by 1.6 grams of salt, the tortillas will taste the same. Between what amounts of salt will the taste of the tortillas stay the same?

4. Erica planted a young tree last spring. During the next few months, the tree grew 1.745 meters. The height of the tree was 8.5 meters after this growth spurt. How tall was the tree when she planted it?

5. Mr. Trevino is conducting two different science experiments. He needs 0.789 grams of sodium for Experiment A and 2.15 grams for both Experiment A and B. How many grams of sodium does he need for Experiment B?

6. increase of 3.2º

7. 1.45 feet for third bow

8. 4.53 miles

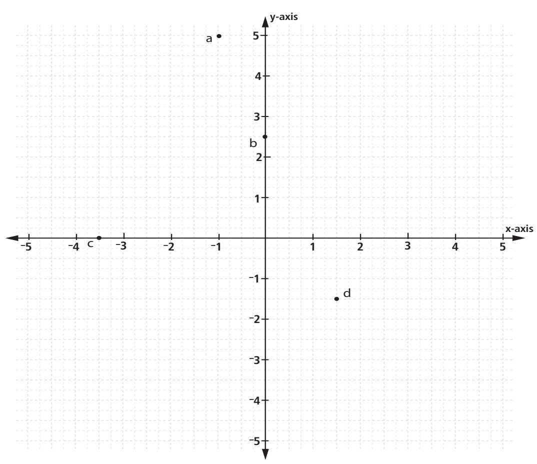
9. 109.98 on video games, no, Evan does not have enough money for the cell phone.

10. a. 0.06 b. 0.3 c. 0.56 d. 0.36

11. a. 0.006 b. 2.52 c. 30.68 d. 10.18

12. a. 0.4 b. 0.25 c. 0.6 d. 0.4 e. 0.1

13. Spiral Review (7.7A)



6. Normal body temperature is 98.6 °F. One day Justin develops a fever and when the nurse takes his temperature, she finds it to be 101.8 °F. What was the increase to Justin's body temperature due to his fever?

7. Pattie bought a 5 foot roll of red ribbon. She needed to make two bows. One bow required 2.3 feet of ribbon. The second bow is smaller and took only 1.25 feet of ribbon. How much ribbon will Pattie have left over for a third ribbon?

8. Andrew lives 1.48 miles from Christian. Christian lives 3.05 miles from Emilio. How far will Andrew travel if he drives from his house, picks up Christian and then drives to Emilio's?

9. Evan had $239.57. He went to the store to buy some video games and a cell phone. He bought the first game for $59.99 and the second game for $49.99 (including taxes). How much money did he spend on the video games? If Evan wants to buy a cell phone for $129.99, does he have enough money left?

10. Compute the following products using the area grid.

a. (0.3)(0.2) c. (0.8)(0.7)

b. (0.6)(0.5) d. (0.4)(0.9)

11. Compute the following products using the pattern.

a. 0.03 x 0.2 c. 11.8 x 2.6

b. 3.6 x .7 d. 5.09 x 2

12. Compute the following quotients using the long division method.

a. $2 \div 5$ c. $4.20 \div 7$

b. $1 \div 4$ d. $3.6 \div 9$

e. $1.20 \div 12$

Spiral Review

13. Graph the following points on a coordinate grid.

a. (-1,5) c. $(-3\frac{1}{2}, 0)$

b. $(0, 2\frac{1}{2})$ d. $(1\frac{1}{2}, -1\frac{1}{2})$

14. Spiral Review (7.2B)
$\frac{29}{8}$ feet, $3\frac{5}{8}$ yards

15. **Ingenuity**

a. We have AX = 0.72 – 0.12 = 0.60, BX = 1.35 – 0.12 = 1.23, and CX = 2.88 – 0.12 = 2.76. So AX + BX + CX = 0.60 + 1.23 + 2.76 = 4.59.

b. We have AX = 2.49 – 0.72 = 1.77, BX = 2.49 – 1.35 = 1.14, and CX = 2.88 – 2.49 = 0.39. So AX + BX + CX = 1.77 + 1.14 + 0.39 = 3.30.

c. We have AX = 10 – 0.72 = 9.28, BX = 10 – 1.35 = 8.65, and CX = 10 – 2.88 = 7.12. So AX + BX + CX = 9.28 + 8.65 + 7.12 = 25.05.

d. A good strategy is to begin by thinking about the distances AX and CX, since the points A and C are the farthest apart. We know that no matter what point we choose X to be, the sum AX + CX must be at least the distance from A to C which is 2.88 – 0.72 = 2.16. (To see this, try choosing some different points X and draw the distances AX and CX on the number line.) As for BX, we can force this to be equal to zero by choosing X = B. So if we choose X = 1.35, we get AX + BX + CX = 0.63 + 0 + 1.53 = 2.16.

16. **Investigation**

a. At this early stage, students may go ahead and line the numbers up vertically in order to add; this is okay and should be encouraged if students can't come up with a better approach. However, we can find a quicker strategy by thinking about the problem in terms of money. Suppose Rich has $2.45, and his friend Sue has $0.98. We want to know how much they have altogether. Suppose Sue decides to sue rich, and makes Rich give her two cents. Then Rich has $2.43, and Sue has $1.00. The combined amount of money that have has not changed. It is now much easier to find the total, which is $2.43 + $1.00 = $3.43. So $2.45 + 0.98 = 3.43.

b. We have:

4.83 + 6.96 = 4.79 + 7.00 = 11.79

10.99 + 4.99 = 11.00 + 4.98 = 15.98

54.22 + 199.99 = 54.21 + 200.00 = 254.21

c. This time, suppose that Rich has $5.16 and Sue has $0.97. We want to know how much richer Rich is. We know the answer to the question would remain unchanged if Rich and Sue both found an additional $0.03. If this happened, Rich would have $5.19 and Sue would have $1.00. It is now clear that Rich is richer by $5.19 – $1.00 = $4.19. So $5.16 – $0.97 = $4.19.

d. We have:

8.24 – 2.95 = 8.29 – 3.00 = 5.29

14.41 – 8.92 = 14.49 – 9.00 = 5.49

266.17 – 79.99 = 266.18 – 80.00 = 186.18

14. Carol is making a border for her flower bed. She has a landscape timber that is $5\frac{7}{8}$ feet long. If she cuts off a piece that is $\frac{3}{4}$ yard long, represent the portion that is left of the original strip as a mixed fraction and an improper fraction.

15. **Ingenuity:**

    Suppose we draw a number line, and mark the points A = 0.72, B = 1.35, and C = 2.88. If X is a point on the number line, we use the symbols AX, BX, and CX to represent the distance from A to X, the distance from B to X, and the distance from C to X, respectively.

    a. If X = 0.12, what is the sum AX + BX + CX?

    b. If X = 2.49, what is the sum AX + BX + CX?

    c. If X = 10, what is the sum AX + BX + CX?

    d. If X can be any point on the number line, what is the smallest possible value of the sum AX + BX + CX?

16. **Investigation:**

    In this Investigation, we will explore what happens when we add and subtract decimals that are close to integers.

    a. What is the sum 2.45 + 0.98? Can you think of a way to find this without aligning the numbers vertically and using long addition? (Hint: It may be helpful to think of the problem in terms of money.)

    b. Computer the following sums:

       i. 4.83 + 6.96 ii. 10.99 + 4.99 iii. 54.22 + 199.99

    c. What is the difference 5.16 - 0.97? Can you think of a way to find this without aligning the numbers vertically and using long subtraction?

    d. Computer the following sums:

       i. 8.24 - 2.95 - 79.99 ii. 14.41 - 8.92 iii. 266.17 - 79.99

Section 5.3 - Numbers as Decimals and Fractions

Big Idea:
Represent numbers as fractions and as decimals

Key Objectives:
- Convert fractions to decimal form.
- Convert decimals to fractions.
- Understand the relationship between fractions of dollars and decimals representing cents.
- Become familiar with repeating decimals and their notations.

Materials:
Paper for folding (sentence strips or adding machine tape) Grid paper, Fraction Chart from the CD

Pedagogical/Orchestration:
- This section involves connecting fractions to their decimal equivalents through division. Students also get practice using the linear model for fractions through paper folding, and using the area model for fractions by shading regions on a grid. Students can be shown the Fraction Chart from the CD as it is an extension of the linear model that makes ordering common proper fractions visually very simple.
- Refer to Section 5.1 to review place value. The Place Value Chart will help students make a connection of decimals to fractions. Reinforce to students that when you say the name of a decimal, you are saying the fraction that it is equal to. For example: $0.751 = \frac{751}{1000}$; $0.75 = \frac{75}{100}$; $0.7 = \frac{7}{10}$.

Activities:
"Best Sugar Cookies"

Exercises:
Exercise 5: Remind students about repeating decimals from fractions. . Allow students to use guess and check with calculators to find the repeating decimal values, and remind them to look for patterns.

Vocabulary:
fraction, decimal, equivalent, rounding

TEKS:
6.1(A,B); 6.2(C,D,E) New: 6.4(G)

Launch for Section 5.3:
Tell your class that your class has 10 dimes of which you had contributed 3. How can you represent the amount you contributed? Discussion may include 3/10 in a fractional representation. The notation can be 0.30 as you would with money or 0.3. All these represent the same quantity.

The Best Sugar Cookies

Objective: To practice converting between mixed numbers, improper fractions, decimals and percents.

Materials:

Copy of sugar cookie recipe (either one per student, or one projected at the front of the room)

Notebook Paper

Pencil

Activity Instructions:

1. Make a copy of the sugar cookie recipe attached and have it available for all students to see. If you make a copy for each student, then each student can complete the chart on this page. If you make a copy and project it for the whole class to see, each student will make a personal copy of the chart.
2. Ask your students to tell you what type of numbers they see in the list.
3. Ask your students to convert all mixed numbers into improper fractions first.
4. Then ask your students to convert all fractions into decimals.

Ingredients:	Improper Fractions:	Decimals:
$1\frac{1}{2}$ cups sugar		
$\frac{2}{3}$ cups shortening or butter		
2 eggs		
2 tablespoons milk		
$\frac{1}{4}$ teaspoon vanilla extract		
$3\frac{1}{4}$ cups flour		
$2\frac{1}{2}$ teaspoons baking powder		
$\frac{1}{2}$ teaspoon salt		

The students should notice why 35 hundreds is the same as 3 tenths and 5 hundredths. An area model with 1 x 1 grid that is broken down into 100 squares can best illustrate how 3 tenths is really 30 hundredths and 5 more hundreds shows the .30 + .05 = .35

PROBLEM 1

a. $\frac{7}{10}$ b. $\frac{1}{100}$ c. $\frac{216}{1000}$ d. $\frac{903}{1000}$ e. $\frac{54}{10}$

SECTION 5.3 NUMBERS AS DECIMAL AND FRACTIONS

In the past, you have probably referred to one-half of a dollar as \$0.50 or 50 cents. One half is a fraction that is equal to 0.50, a decimal. We say that $\frac{1}{2}$ is **equivalent** to 0.50 in that they represent the same quantity though are written differently. In this section, we will review how a fraction can be represented as a decimal number and how some decimals can be represented as fractions.

If you buy four apples for a dollar, how much does each apple cost? Dividing \$1 by 4 yields, \$1÷4 = \$0.25, or 0.25 dollars. We can also say each apple costs a quarter or $\frac{1}{4}$ of a dollar. So $\frac{1}{4}$ and 0.25 are equal quantities or equivalent numbers.

Decimals can be read in terms of the place values that the digits occupy. For example, 0.47 is read "forty seven one hundredths." You know that the fractional representation $\frac{47}{100}$ is also read "forty seven one hundredths." You can also locate the numbers on the number line as the same point. The decimal form 0.47 and the fractional form $\frac{47}{100}$ actually represent the same value.

In converting a decimal to a fraction, we take advantage of the fact that we use the base ten system to write each decimal number. For example, the number 0.3 is called three tenths and so is equivalent to $\frac{3}{10}$. While the number 0.35 has 3 tenths and 5 hundredths, we usually read the number as 35 hundredths and is the same as the fractional form $\frac{35}{100}$.

PROBLEM 1

Write the fractional form of the following decimal numbers:

a. 0.7 b. 0.01 c. 0.216 d. 0.903 e. 5.4

Does the fraction $\frac{1}{5}$ have a decimal form? If we buy 5 bananas for \$1, we know that each banana costs \$1 ÷ 5 = \$0.20 or 20 cents. In other words, each banana costs $\frac{1}{5}$ of a dollar because it takes 5(\$0.20) to make a whole dollar. So $\frac{1}{5} = 1 \div 5 = 0.20$ or 20 hundredths. But the decimal 0.20, or twenty hundredths, has the same name as the fraction $\frac{20}{100}$. Does this mean the fraction $\frac{1}{5}$ is equal to $\frac{20}{100}$? These are equivalent fractions, so the decimal 0.20 and the fractions $\frac{1}{5}$ and $\frac{20}{100}$ are equal.

You can simultaneously make a large number line model on the board or on the floor using string that does not stretch and masking tape. But this is no substitute for a students' own number line model. Place the fractions above the line and decimals below the line. As you build the model, we want students to use their knowledge about converting fractions to decimals and decimals to fractions to build the number line.

1. Expect responses like half a dollar, 50 cents or $.50. Notice that some points will have multiple labels, like $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$. Remind your students that they learned that $\frac{1}{2} = 0.5$ in Section 8.1. Students may also use a piece of string because it is easier to fold into thirds and fifths. Have students make their number line with care because this number line is the beginning of a master number line that they will amend over the next couple of activities.

2. Teachers, make connections between decimals and their monetary equivalents. For example, 0.25 is equal to $0.25.

3. What is the decimal equivalent of $\frac{1}{8}$? ($\frac{1}{8} = \frac{1}{2}$ of 0.25 or 0.125) Use this method to locate, mark and label all the eighths on the number line. TE: The number line should now have the following points labeled: 0, 1, $\frac{1}{2}, \frac{2}{2}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}$ above the line and their decimal equivalents below the corresponding points on the number line.
4. Some possible comparisons the students might notice include:
   - The ruler and yard stick are divided into fractional amounts but their number line has labels for each fractional amount.
   - The yardstick, ruler and their own number line are using different scales, i.e., half of a yard is different than half of a foot or half of their own unit.
   - A meter stick would lend itself to a decimal notation.

The following property summarizes the relationship between division and the fractional notation that we just saw.

PROPERTY 5.1: FRACTIONS AND DIVISION

For any number m and nonzero number n the **fraction** $\frac{m}{n}$ is equivalent to the quotient $m \div n$.

LINEAR MODEL FOR FRACTIONS ACTIVITY

On a number line, each integer corresponds to a point. Recall that there are many other points between each pair of integers on the number line, and each of these points also corresponds to a number. We will now construct a number line from 0 to 1.

Materials: You will need a long strip of paper like a sentence strip or an 18-inch piece of adding machine paper.

1. Mark the left end point of the strip as 0 and the right end point as 1. You may fold this strip or use another strip to fold and transfer points to this master number line. Fold the strip end to end into two equal parts and mark the crease as the midpoint between 0 and 1. What fraction is this midpoint equivalent to? What decimal? Label the points on the number line as fractions above the line and as decimals below the number line.
2. Fold the strip again and use the creases to mark and label points on the master number line. Because the strip is now folded into 4 equal parts, we label the first point as $\frac{1}{4}$, which is equivalent to 0.25. Although the second point already has two labels, $\frac{1}{2}$ and 0.5 or 0.50, add the label $\frac{2}{4}$ above this point. The last crease will be labeled as $\frac{3}{4}$ or 0.75.
3. Repeat this method by folding the strip again into 8 equal parts, transferring the locations to the master number line and labeling the points with fractions and decimals. What is the decimal equivalent of $\frac{1}{8}$? Use this method to locate, mark and label all the eighths on the number line.
4. Compare the number line with a typical foot ruler or yardstick.

5. Talk to your students about how they can fold an unmeasured strip into fifths. Solicit ideas and lead them, if necessary, to realize that they will make four equally-spaced creases in their number line to do that. The next hard part is thirds, but having made fifths and applying the lessons there, this should be easier. Sixths and ninths will follow.

   Teachers, switch to the number line from the packet in the teacher's folder (or one that you printed from the CD) that has hundredths marked but not labeled. Transfer the fractions (from halves to sixteenths) from the folded number line to this number line. Continue locating other fractions below using this new number line.

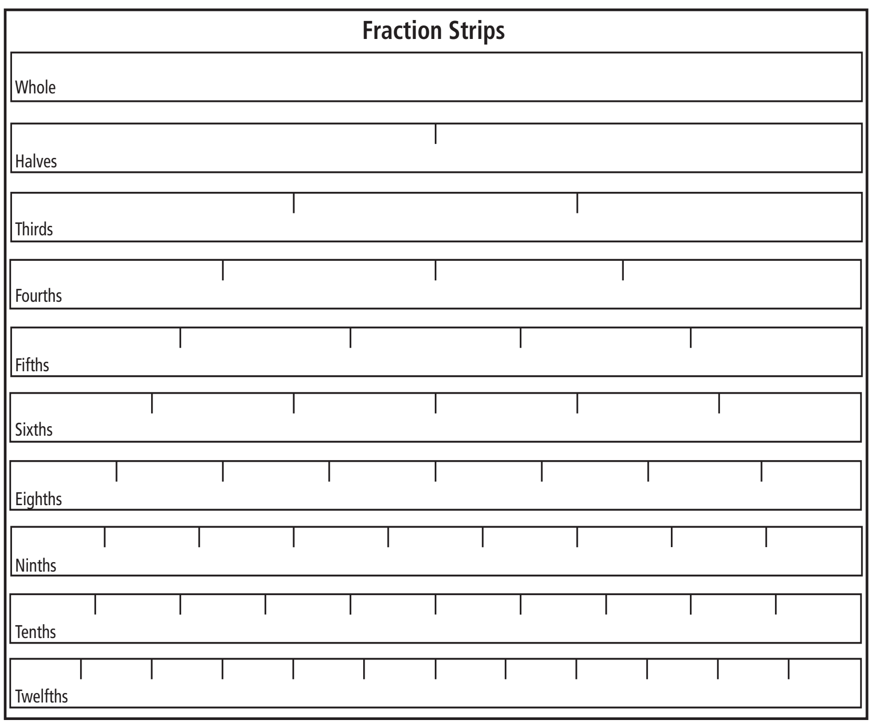
6. Talk about how to estimate the decimal form from what you know. If it is valuable, have some student with a calculator check the class' guesses. If necessary, remind students of the fraction-dollar relationship.

a.	b.	c.	d.
0.05	≈ 0.08	≈ 0.06	0.04
0.35	≈ 0.4	≈ 0.3	0.08
0.025	≈ 0.04	≈ 0.03	0.02
0.075	0.8	≈ 0.15	0.1

5. Label these fractions and their decimal equivalents on the number line.
   a. $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$
   b. $\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, \frac{10}{10}$
   c. $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$
   d. $\frac{1}{6}, \ldots, \frac{5}{6}, \frac{6}{6}$
   e. $\frac{1}{9}, \ldots, \frac{8}{9}, \frac{9}{9}$
6. Use your new number line to estimate the decimal form of the following fractions.

a.	b.	c.	d.
$\frac{1}{20}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{25}$
$\frac{7}{20}$	$\frac{5}{12}$	$\frac{5}{16}$	$\frac{2}{25}$
$\frac{1}{40}$	$\frac{1}{24}$	$\frac{1}{32}$	$\frac{1}{50}$
$\frac{3}{40}$	$\frac{4}{5}$	$\frac{5}{32}$	$\frac{5}{50}$

The rows of the Fraction Chart below represent the result of folding the whole into equal parts to represent fractions as we did in the Linear Model Activity. The computer that created it allows the folds to be very accurate. For example, using the Fraction Chart, determine which fraction is greater: $\frac{2}{5}$ or $\frac{3}{7}$.



a. 0.25, 0.5, 0.75

b. 0.2, 0.4, 0.6, 0.8

c. 0.125, 0.25, 0.375, 0.625

d. 0.33, 0.033, 0.11, 0.011

Any fraction with denominators of 3, 6, or 9 will repeat the same digit in the decimal form. There are others such as denominator of 7 that will repeat a set of decimals but it will require a few steps in the division. You may have the students investigate what $\frac{1}{7}$ is as a decimal representation. The 0.142857__________, will repeat. This may be a good reminder of decimal division.

Convert the following fractions to decimal form. You may verify your answer by dividing with a calculator, if necessary.

a. $\frac{1}{4}$	b. $\frac{1}{5}$	c. $\frac{1}{8}$	d. $\frac{1}{3}$
$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{8}$	$\frac{1}{30}$
$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{8}$	$\frac{1}{9}$
	$\frac{4}{5}$	$\frac{5}{8}$	$\frac{1}{90}$

Now we ask, "What decimal is equivalent to $\frac{1}{3}$?" We could also ask what is $\frac{1}{3}$ of a dollar? We can use our new rule to see that $\frac{1}{3}$ is equivalent to the quotient of $1 \div 3$. The quotient is a repeating decimal, 0.33333... which can be written as $0.\overline{3}$. The bar over the 3 tells us that the digit 3 repeats without end. Previously, we discovered that this quotient is 1 ' 3 = 0.3333... = $0.\overline{3}$. So $\frac{1}{3}$ of a dollar is $\$0.3\overline{3}$, and we cannot practically divide \$1 into 3 equal parts with our present set of coins, so we often approximate to \$ 0.33 There are other fractions that equal repeating decimals such as:

2 ' 3 = 0.6666... = $0.\overline{6}$ • 0.67
1 ' 6 = 0.1666... = $0.1\overline{6}$ • 0.167

Find other fractions that have repeating decimals.

PROBLEM:
Complete the missing parts in the table below.

Fractions	Decimals	Words
	0.5	Five tenths
	0.35	
$\frac{3}{4}$		
		Six and 3 one hundredths
$8\frac{3}{12}$		

EXERCISES

1. $\frac{3}{4} > \frac{1}{3} > 0.20 > 0.125$
2. $0.01 < 0.1 < \frac{1}{5} < \frac{2}{8}$
3. 6.100, 6.2, 7.001, 7.321, 7.4
4. First, monitor to see whether your students use the unit fraction's decimal equivalent or whether they use division each time. Encourage best practices here. Look for simplified forms or denominators that convert to powers of 10. As a last resort, use division. Heads up on part c.

a.		b.		c.		d.	
	.45		.52		0.33...		0.75
	.45		.52		.1		0.75
					.05		0.75
					.025		0.75

All fractions in parts a, b, and d are equivalent fractions.

5. Have students reflect on any patterns they see, e.g., they are decreasing, some are multiples of previous numbers. There is no general pattern between all of these fractions.

$\frac{1}{2} = 0.5$	$\frac{1}{3} = 0.333...$	$\frac{1}{4} = 0.25$	$\frac{1}{5} = 0.2$
$\frac{1}{6} = 0.1666...$	$\frac{1}{7} = 0.142857142857...$	$\frac{1}{8} = 0.125$	$\frac{1}{9} = 0.111...$
$\frac{1}{10} = 0.1$	$\frac{1}{11} = 0.0909...$	$\frac{1}{12} = 0.08333...$	

6. Encourage students to make connections between a fraction in hundredths and the decimal form of a number. Students can also extend what they know about fractions to find an equivalent form in tenths and thousandths.

Simplest Form	Fraction in Hundredths	Decimal Form
$\frac{3}{5}$	$\frac{60}{100}$	0.6
$\frac{7}{10}$	$\frac{70}{100}$	0.7
$\frac{9}{20}$	$\frac{45}{100}$	0.45
$\frac{1}{8}$	$\frac{12\frac{1}{2}}{100}$ or $\frac{12.5}{100}$ *	0.125
$\frac{7}{25}$	$\frac{28}{100}$	0.28

*Though typically we write $\frac{125}{1000}$

EXERCISES

1. Order from largest to smallest: 0.20, $\frac{1}{3}$, $\frac{3}{4}$, 0.125
2. Order from smallest to largest: $\frac{2}{8}$, $\frac{1}{5}$, 0.01, 0.1
3. Following are final running times of 5 runners: 7.321, 7.4, 7.001, 6.2, 6.100.

   Write the times in order from fastest to slowest.
4. Convert each fraction to an equivalent decimal. (Hint: use what you have learned about simplifying fractions before converting to decimal form.) Explain the pattern you notice in each set of fractions.

   a. $\frac{9}{20}$ $\frac{45}{100}$

   b. $\frac{13}{25}$ $\frac{52}{100}$

   c. $\frac{1}{3}$ $\frac{1}{10}$ $\frac{1}{20}$ $\frac{1}{40}$

   d. $\frac{9}{12}$ $\frac{18}{24}$ $\frac{15}{20}$ $\frac{24}{32}$
5. Convert each unit fraction to an equivalent decimal. Write the decimals in a vertical list so that you can compare them.

   $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{7}$ $\frac{1}{8}$ $\frac{1}{9}$ $\frac{1}{10}$ $\frac{1}{11}$ $\frac{1}{12}$
6. Complete the table..

Simplest Form	Fractions in Tenths, Hundredths or Thousandths	Decimal Form
$\frac{3}{5}$		
$\frac{7}{10}$		
$\frac{9}{20}$		
$\frac{1}{8}$		
$\frac{7}{25}$		

7. Have your students read the correct name for each decimal. For instance, 0.375 is "three hundred seventy-five thousandths," not "point three seven five." Correctly naming a decimal reveals its fractional equivalent of 375/1000. Also encourage the following insights: 0.375 is 0.125 less than 0.5; 0.375 is half way between 0.25 and 0.5; and 0.375 is 0.125 more than 0.25.

a. $\frac{3}{5}$	e. $\frac{1}{4}$	i. $\frac{12}{25}$
b. $\frac{7}{10}$	f. $\frac{1}{8}$	j. $\frac{24}{25}$
c. $\frac{4}{25}$	g. $\frac{7}{25}$	k. $\frac{3}{40}$
d. $\frac{1}{20}$	h. $\frac{11}{20}$	l. $\frac{5}{8}$

8. Explanations will vary.
   a. 14.25 dollars
   b. $\frac{1}{3}$ lb.
   c. $\frac{9}{12}$ of the pizza.

9. Expect your students to use common denominators and equivalent fractions to justify their choices. They might also use decimals and convert them to fractions.

10. Spiral Review (6.1A)

    1.55 cm, 1.56 cm, 1.59 cm

    Answers may vary

11. Spiral Review (7.2B)

    Sergio jumped a total of $79\frac{1}{2}$ feet. The difference between the longest and shortest jumps is $3\frac{1}{4}$ feet.

7. Convert each decimal to an equivalent fraction. Simplify if needed.

a. 0.6	e. 0.25	i. 0.48
b. 0.7	f. 0.125	j. 0.96
c. 0.16	g. 0.28	k. 0.075
d. 0.05	h. 0.55	l. 0.625

8. Determine whether a decimal or fraction representation is more appropriate in the following situation. Explain your answer.
   a. Renee orders a meal at a restaurant. What best represents the cost of the meal, \$14.25 or \$ $14\frac{1}{4}$?
   b. Billy orders a hamburger at a fast food restaurant. What best represents the weight of the hamburger, $\frac{1}{3}$ lb or 0.333… lbs?
   c. A large pizza is divided into 12 slices. Danny and Marie eat 9 slices from a large supreme pizza. What best represents the portion of the pizza eaten, 0.75 or $\frac{9}{12}$?

9. Use the Fraction Chart to discover and represent three fractions that are greater than $\frac{1}{4}$ and less than $\frac{1}{2}$. Explain how you can tell that a fraction is between $\frac{1}{4}$ and $\frac{1}{2}$ without using a Fraction Chart.

10. Micah measured the diameter of the button on his shirt. The diameter was between 1.5 and 1.6 centimeters. Name three possible lengths of the diameter of his button.

11. The table below shows the distance Sergio jumped in the long jump contest at the school's track meet. What was the total distance Sergio jumped? What was the difference between the longest jump and the shortest jump?

Jump	Distance Jumped (feet)
1	$20\frac{1}{4}$ ft.
2	$18\frac{1}{2}$ ft.
3	19 ft.
4	$21\frac{3}{4}$ ft

Ingenuity

12. Notice that 0.7727272727... is equal to 0.2727272727... + 0.5. We know that $0.5 = \frac{1}{2}$, so 0.7727272727... $= \frac{3}{11} + \frac{1}{2} = \frac{6}{22} + \frac{11}{22} = \frac{17}{22}$.

Investigation

13.

a. Each son will get 3 tenths, with 1 tenth left over
b. The remaining 1 tenth gets divided into 10 hundredths. Each son will get 3 hundredths, with 1 hundredth left over.
c. The remaining 1 hundredth gets divided into 10 thousandths, and divides these up evenly. How many thousandths will each son get? How many will be left over?
d. The pattern continues, with each son getting 3 pieces at each place value. So each son gets 3 tenths, 3 hundredths, 3 thousandths, and so on.
e. We know that 1 licorice stick, divided by 3, gives 3 tenths, 3 hundredths, 3 thousandths, and so on. so $\frac{1}{3} =$ 0.3333...
f. We could imagine Janalyn dividing up another licorice stick, this time amont nine sons. This time, each son gets 1 piece at each place value, and we have $\frac{1}{9} = 0.1111...$

12. **Ingenuity:**

    The decimal 0.2727272727... is equivalent to the fraction $\frac{3}{11}$. What is the fraction equivalent of the decimal 0.7727272727...?

13. **Investigation:**

    Janalyn has a piece of licorice that she wants to divide equally among her three sons Cody, Ross, and Trent.

    a. Suppose Janalyn divides the licorice stick into tenths. How many tenths will each son get? How many tenths will be left over?
    b. Since she doesn't want to waste any licorice, Janalyn then divides any tenths that remain into hundredths will each son get? How many will be left over?
    c. Janalyn then divides any remaining hundredths into thousandths, and divides these up evenly. How thousandths will each son get? How many will be left over?
    d. What will happen if Janalyn continues this process forever? How much licorice will each son get?
    e. How can we use our findings from parts (a) through (d) to find a decimal equivalent for $\frac{1}{3}$?
    f. How could we find a decimal equivalent for $\frac{1}{9}$?

Section 5.4 - Fractions, Decimals, and Percents

Big Idea:
Understanding the relationship between fractions, decimals, and percents

Key Objectives:

- Convert between fractions, decimals, and percents.
- Compare and order fractions, decimals, and percents.
- Find percent of a number.
- Represent rational numbers using visual models.
- Use benchmarks to estimate rational numbers.
- Financial literacy

Materials:
10 by 10 grid for each student for Launch activity, Index cards for ordering activity, Number lines.

Pedagogical/Orchestration:

- Encourage all your students who do not understand percents to make the following analogous relationship: decimals are to percents as dollars are to cents. 1.25 is to 125% just as $1.25 is to 125¢.

Activities:
"Lost in Space"

Index Card Activity to compare/order fractions, decimals, and percents

Hands on Banking Activities such as checking accounts, balancing a bank account

Internet Resources:
Jeopardy Game: Fractions, Decimals, and Percent Conversions
http://www.quia.com/cb/34887.html
http://www.handsonbanking.org/en/instructional-resources.html

An excellent financial literacy resource for teachers is available on the Wells Fargo website. The teen version is appropriate for middle school.
www.handsonbanking.org

Vocabulary:
percent

TEKS:

6.1 (A,B,D); 6.2(D); 6.3(A,B); 6.11(A,B,C) New: 6.4(F,G); 6.5(B,C); 6.14(A, B, C, D, E, F)

Launch for Section 5.4:

Give each student a 10 by 10 square grid. Tell them to start at the top left corner and shade only the squares that are within both the first 5 rows and first 5 columns of the grid. (The students will end up shading a 5 by 5 square made up of 25 unit squares and located in the top left quadrant of the 10 by 10 grid.) Ask students for a way to describe the portion of the big square that is shaded. Responses may include $\frac{1}{4}$ of the whole or $\frac{25}{100}$. If no one mentions percents, then remind students that percent means per 100 and ask if they can come up with the percent equivalent. On a piece of chart paper, write the title "Important Benchmarks" and write the following: $\frac{1}{4} = 0.25 = 25\%$. Ask the students for other benchmark values they know and add them to the list. Students having trouble can do more shading with the 10 by 10 grids. Tell your students, "We will be adding to our list of important benchmarks throughout the week. Today we will be exploring how fractions relate to decimals and percents and it is helpful to keep these benchmarks in mind."

Lost in Space Activity:

Materials:

Spaceship Worksheet - 1 per team (from Math Explorations 6th Grade CD)
1/2 sheet black poster board - 1 per team (from Math Explorations 6th Grade CD)
Map pencils or crayons
Scissors
Glue

Activity Instructions:

Students should be divided into teams of 3 or 4.

1. Teams will cut out all of the aliens (30 in all) and spaceships (10 in all) and then determine the alien species (equivalent forms).
2. After cutting out all 10 spaceships, students will glue the spaceships on the black 1/2 sheet of poster board. Encourage them to figure out a way to evenly space the ships to make their posters look neat and organized.
3. Next, students will group the alien species according to equivalent forms and glue them on a spaceship (an example of a finished spaceship can be found on the Math Explorations 6th Grade CD).
4. After the team has finished grouping the aliens and attaching them on the poster, students will brainstorm to give each group of aliens a planet name. The name of each planet should be mathematical. Students will label the space ship with the planet name.

Spaceship Worksheet

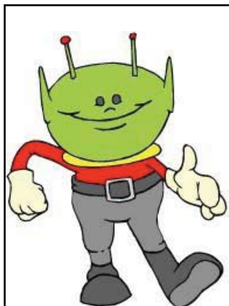

$^{1}/_{4}$


0.75


$^{3}/_{50}$

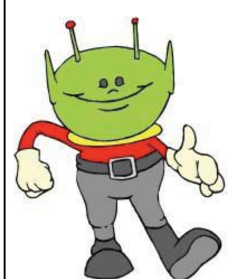

0.625


50%


$^{7}/_{10}$


$^{18}/_{100}$


$^{2}/_{5}$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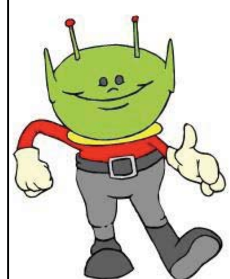
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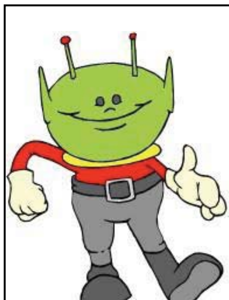
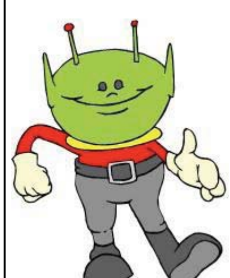
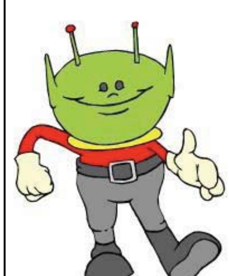

20%


0.55


$^{1}/_{2}$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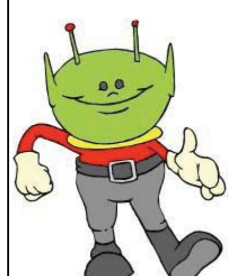

75%


0.4


$^{5}/_{8}$


$^{11}/_{20}$


6%


0.25


18%


40%


0.5

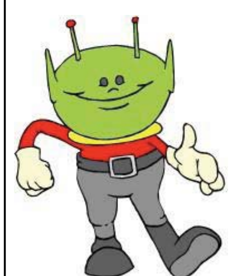

70%


55%

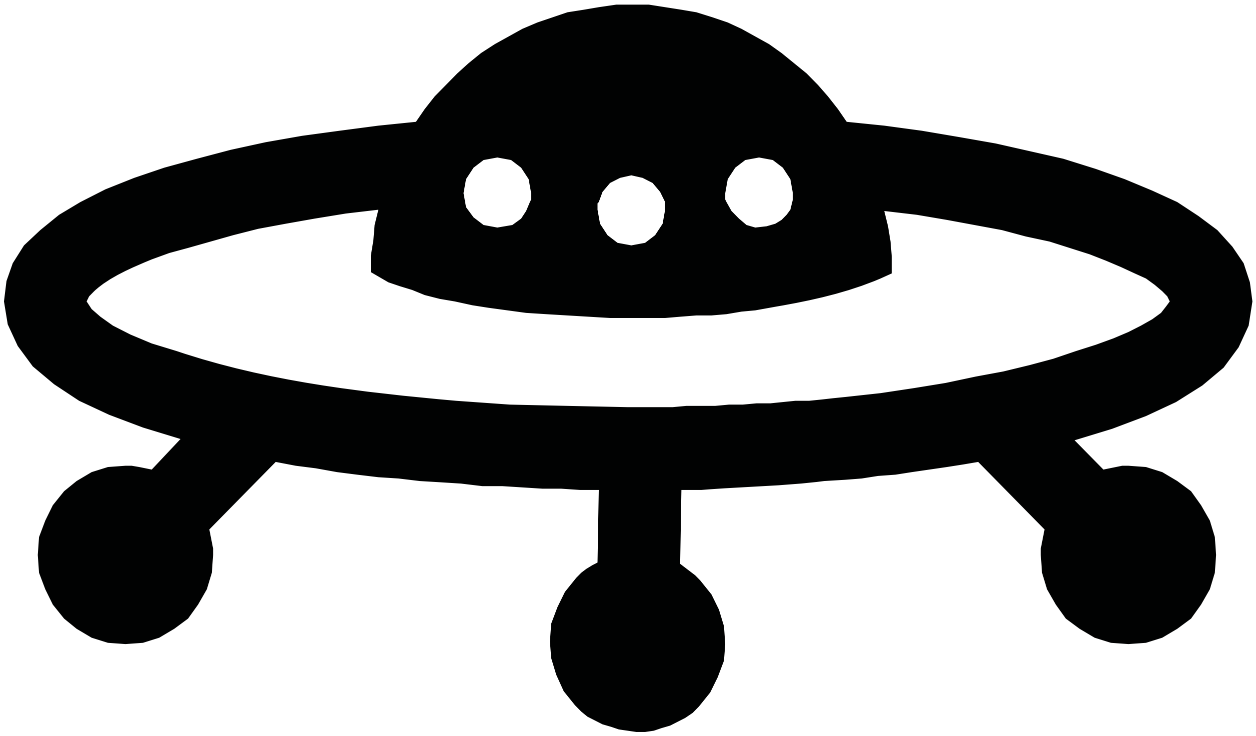

62.5%

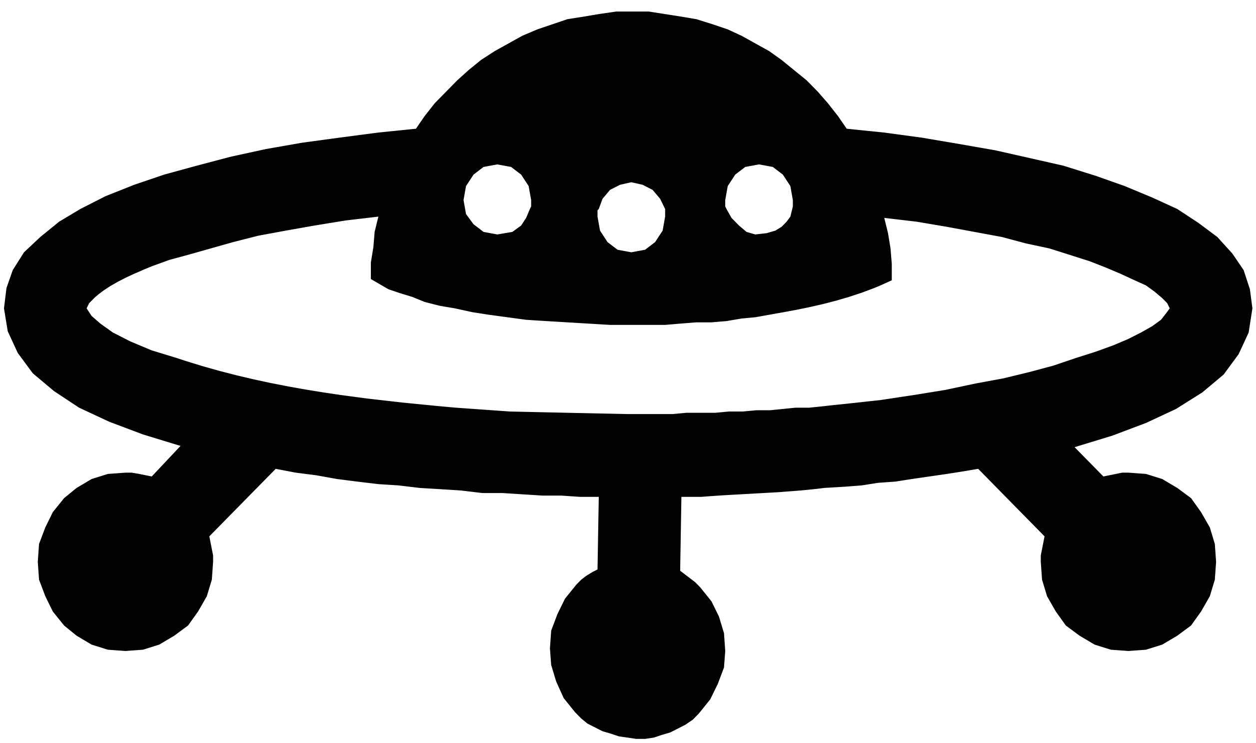

$^{1}/_{5}$

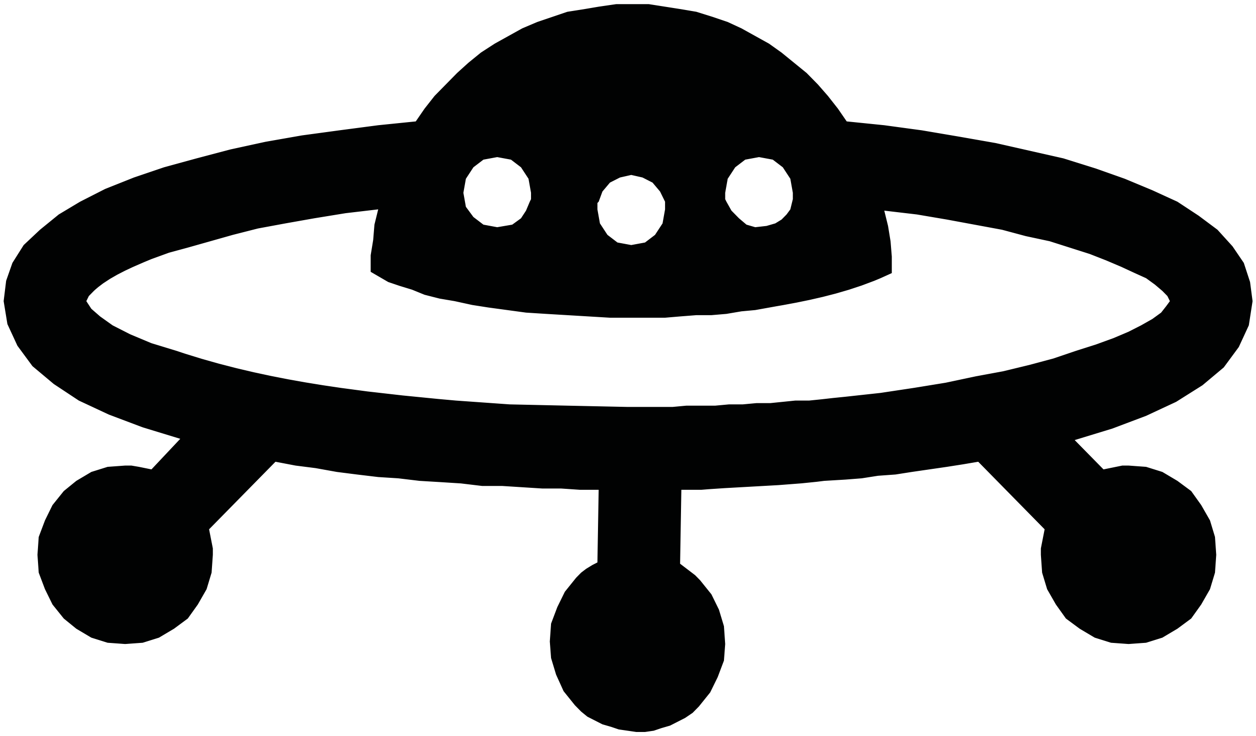

0.06

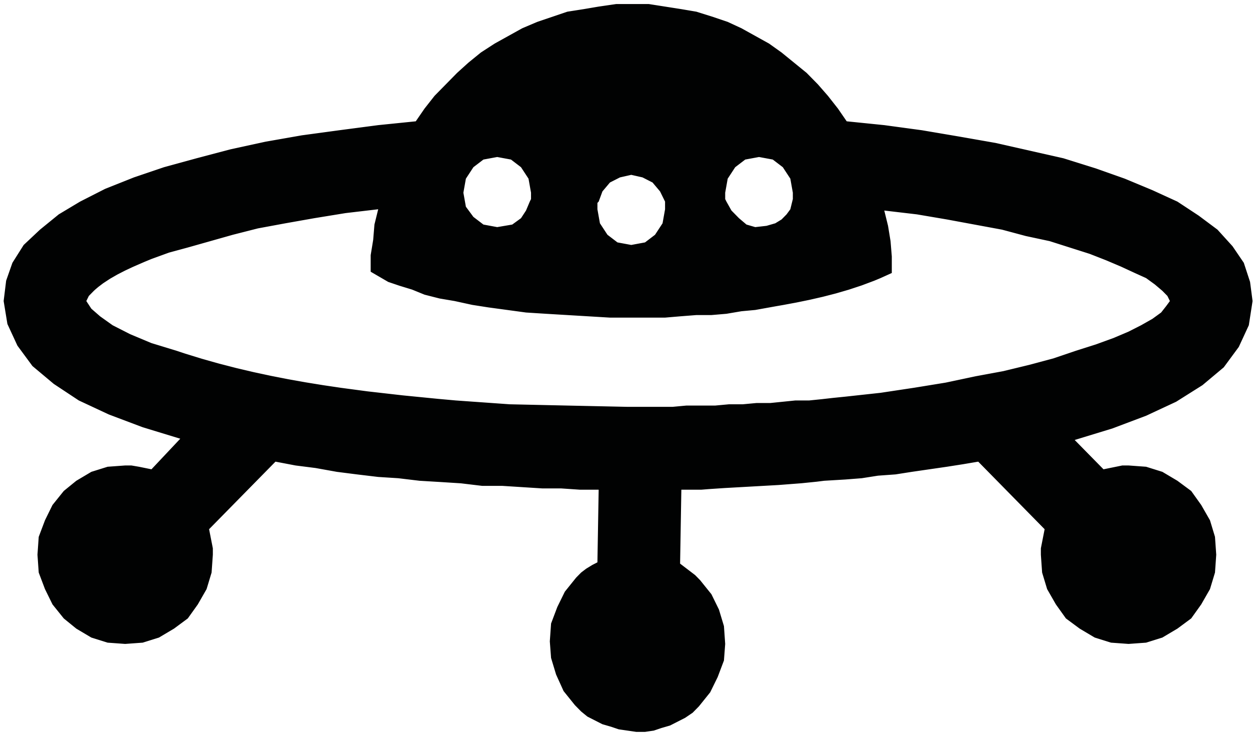

0.18

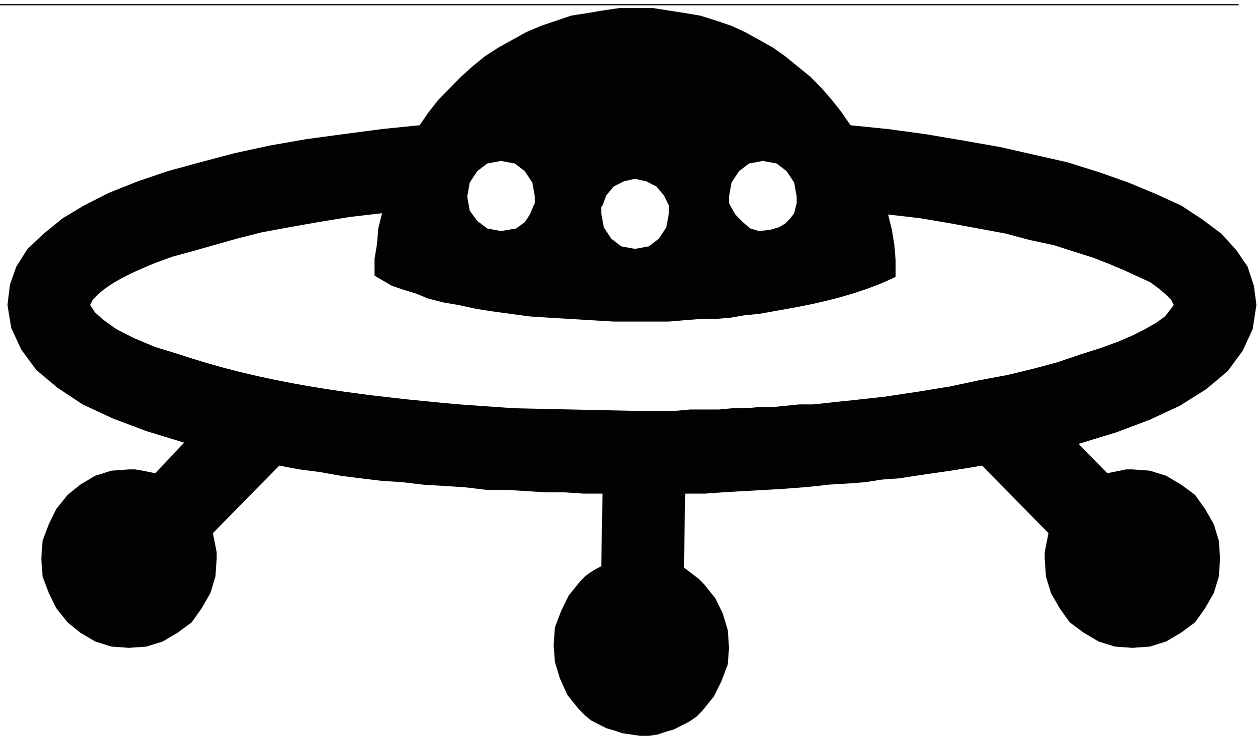

25%

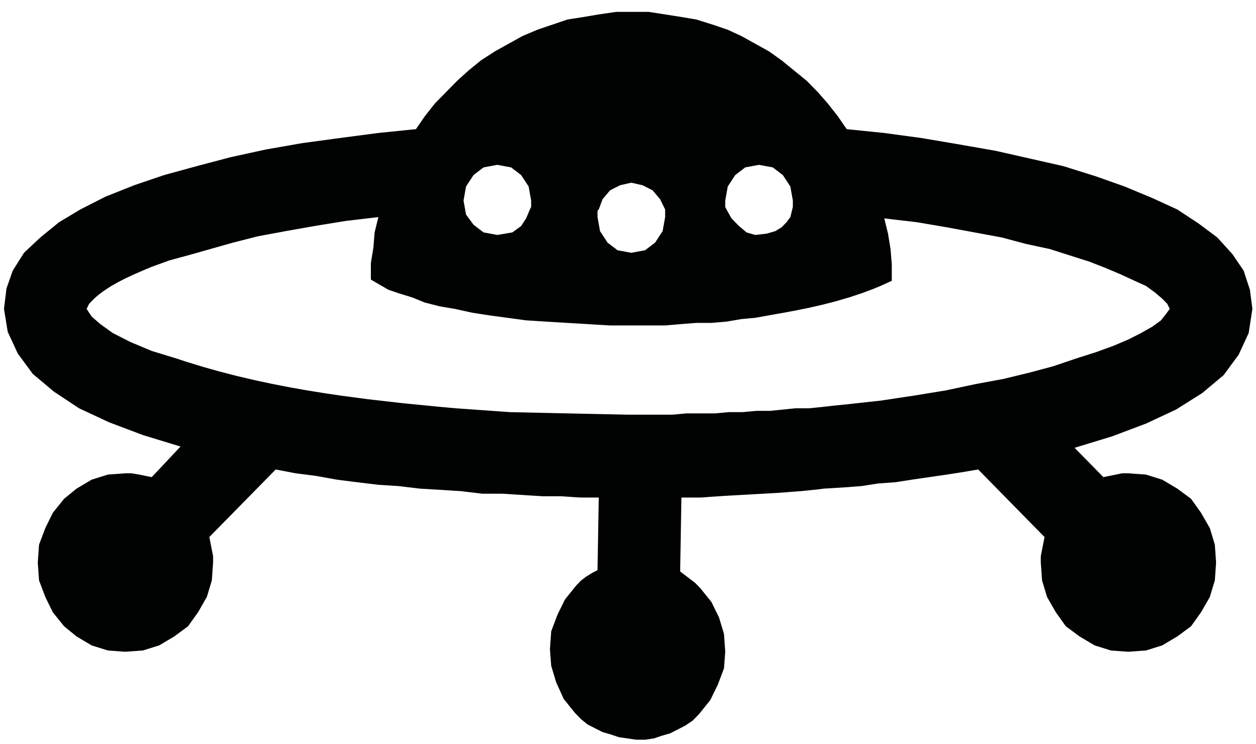


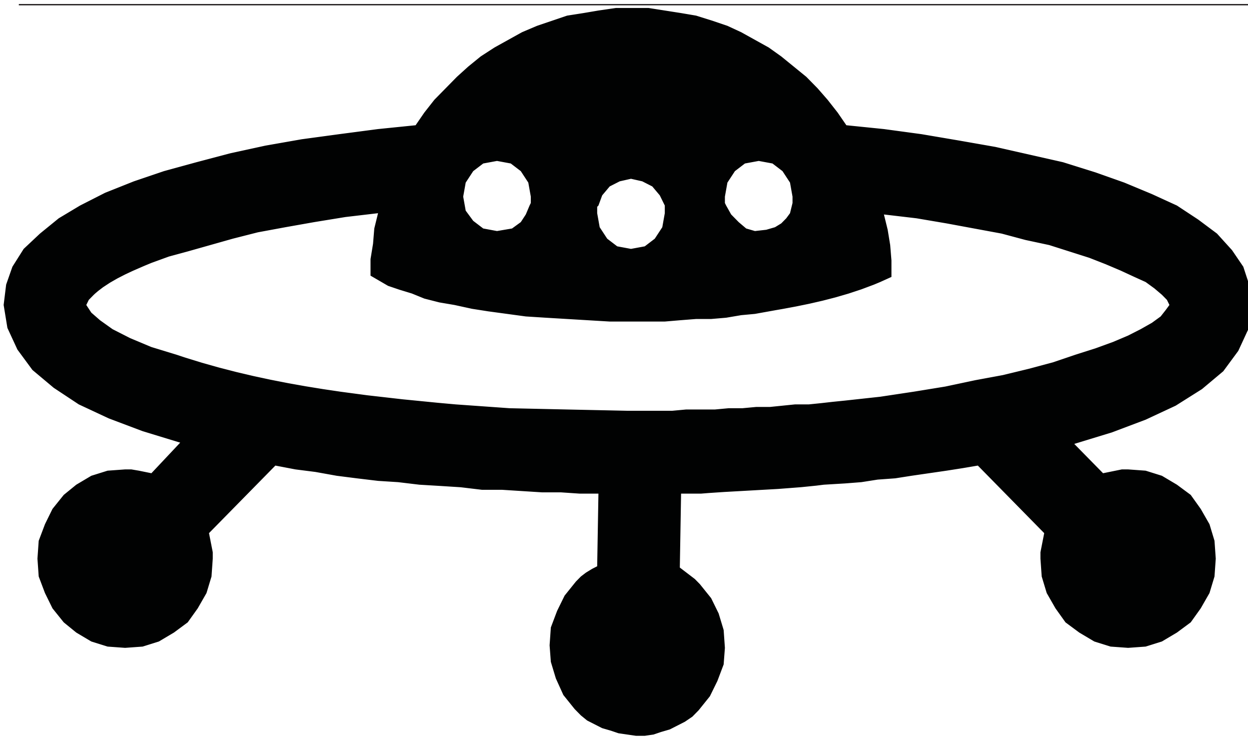


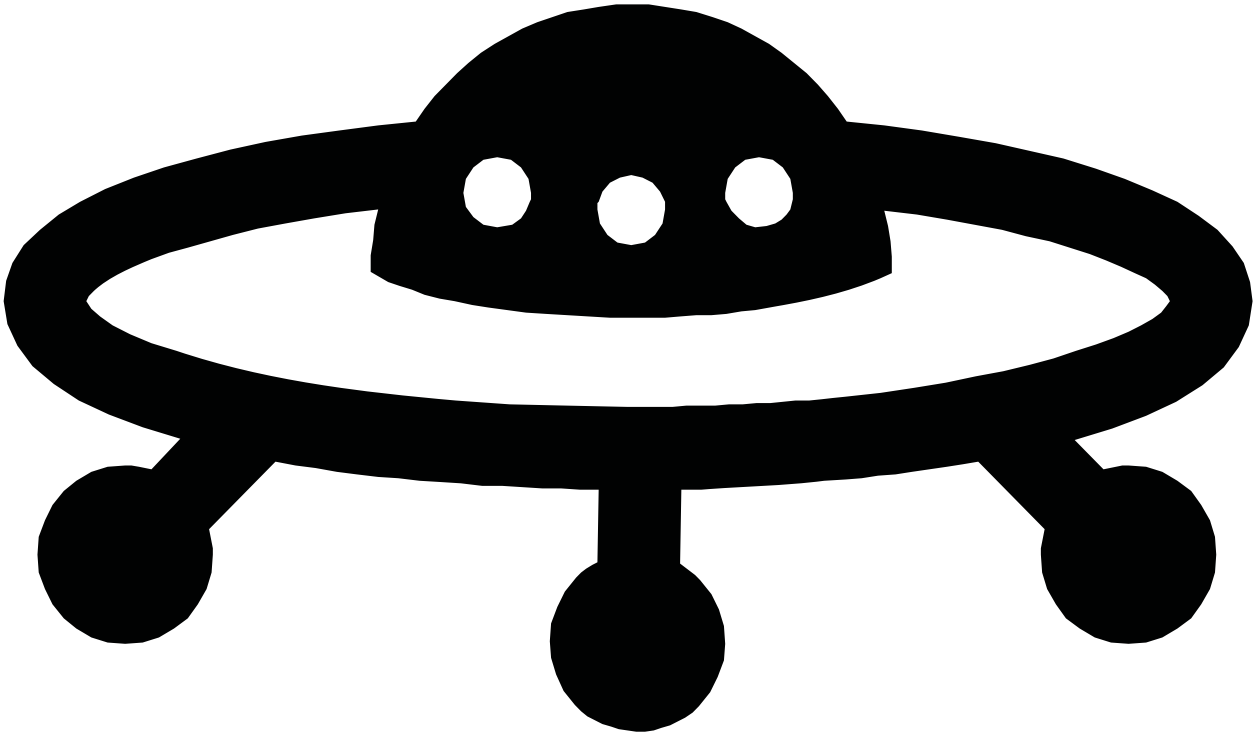


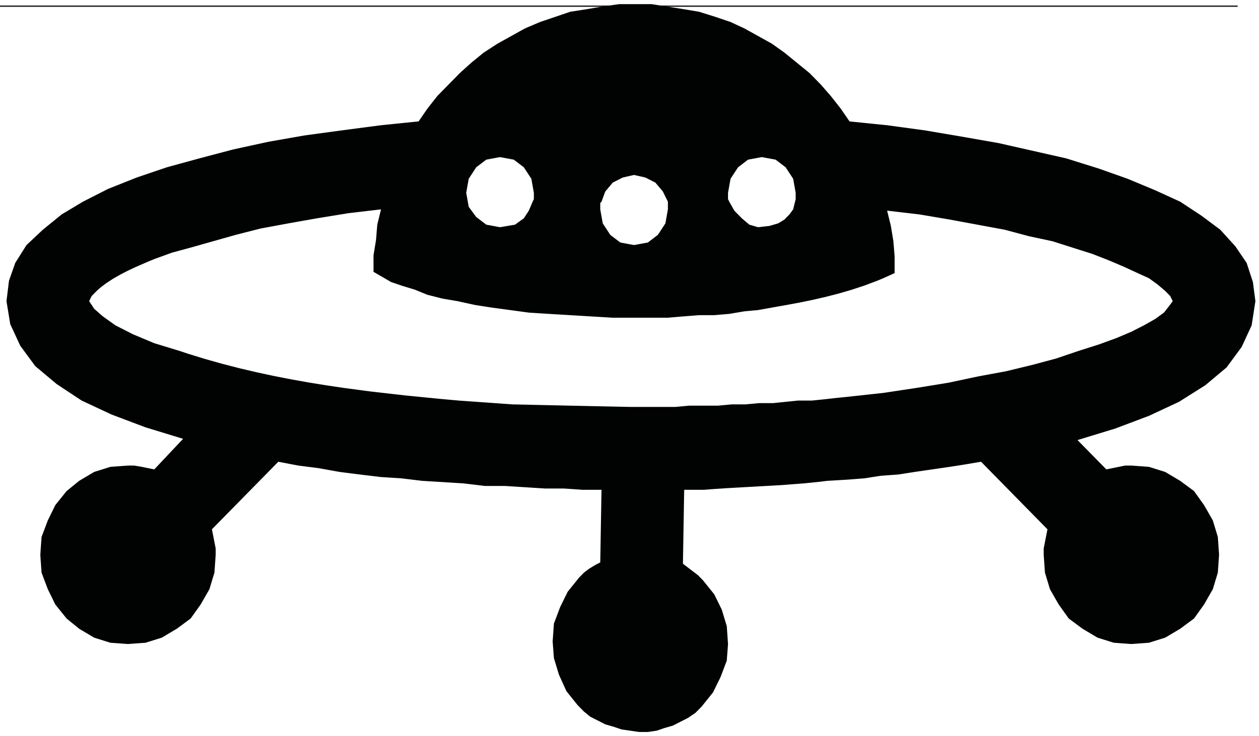


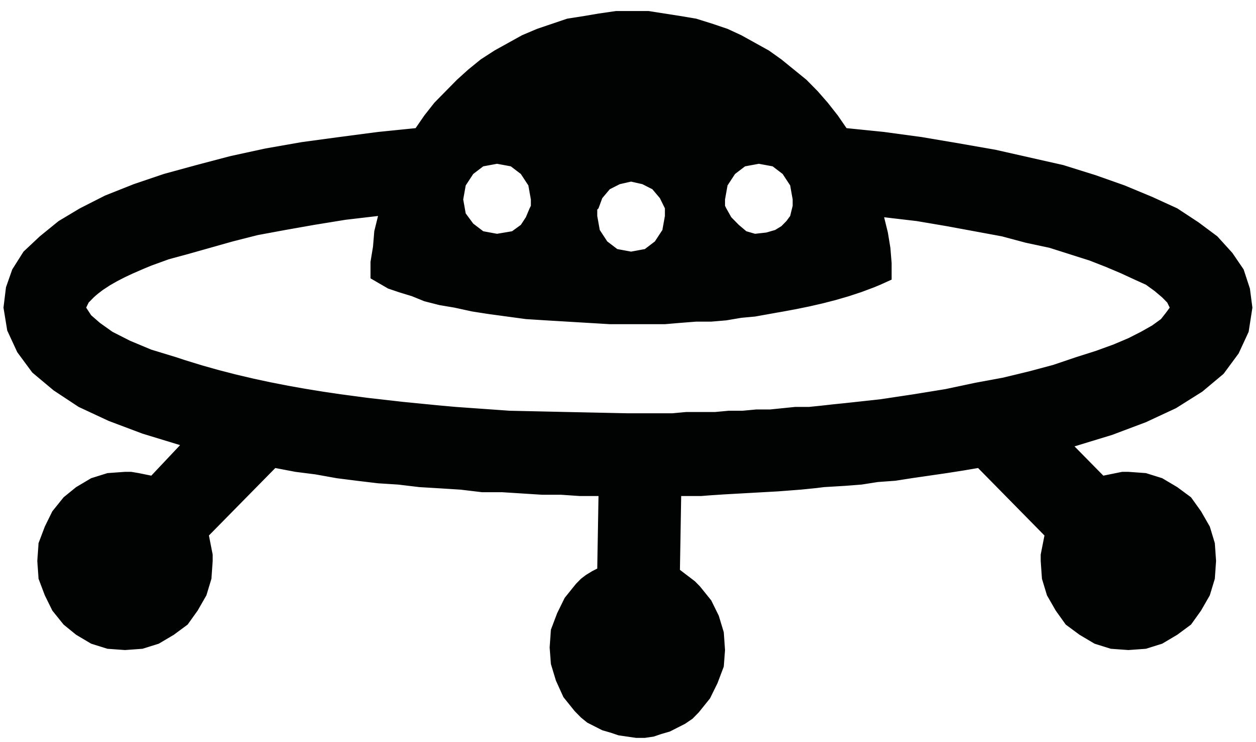












Hands on Banking Activities

http://www.handsonbanking.org/en/instructional-resources.htm

Section 2: Checking Accounts

Checking accounts are another tool banks provide to help individuals manage their finances. In this section, students will investigate the basics of checking accounts and practice writing checks.

Opening Questions

Use these or similar questions to start students thinking about this concept and how it relates to them:

- Describe other ways people can pay for things besides paying in cash.
- When people write checks, why do stores accept them? Isn't a check just a piece of paper?
- What are some reasons that someone might want to pay by check rather than using cash?
- Suppose someone told you that they could write a check to pay for something even if they knew they didn't have enough money in their checking account to cover the amount of the check. What would you tell this person?

Key Points

- Checking accounts, like savings accounts, are part of an individual's personal money management system. Checking accounts are very similar to savings accounts. Both types of accounts keep your money safe, and both are very easy to access if you need cash.
- Checking accounts are designed to be day-in and day-out money-management tools, while savings accounts are designed for long-term money-management. Unlike savings accounts, banks expect people to make frequent withdrawals and deposits to checking accounts.
- An important difference between checking and savings accounts is that checking accounts come with checks!
- A second important difference is that most savings accounts earn interest, while many checking accounts do not.
- Checks can be used to make purchases, just like cash, and they help people pay bills or make simple purchases without carrying around cash or sending it through the mail. People use checks for rent, groceries, clothes.
- Checks are legal documents that function like cash. Knowing how to write a check correctly is fundamental to good money management.
- It's important for students to understand that in order to write a check, there must be sufficient funds in the checking account to cover the amount.

• When you write a check or make a deposit to your checking account, it's very important that you immediately record that transaction in your check register.

• Students can practice using an ATM on the Hands on Banking Web site or CD-ROM.

• Just as with a savings account, you can use an Automated Teller Machine (ATM) to access your checking account. Many ATMs are open 24 hours a day. When the bank issues you an ATM card, they also give you a Personal Identification Number (PIN). The PIN is like a secret password. You should never write your PIN on your ATM card, and you shouldn't tell anyone your PIN.

• Some of the things you can use an ATM to do include:

o Deposit money into a savings or checking account

o Withdraw money from a savings or checking account

o Check the balance in a savings or checking account

o Transfer money between your accounts

• To open a checking account, students and the parent or guardian who accompanies them will probably need the same personal identification required to open a savings account. Students should check with their bank to see what they require.

• Here are some guidelines for how to open a checking account and what you may need to bring with you to the bank:

Opening a Checking Account: What to bring

A parent or guardian must accompany a person under 18. They must bring 2 forms of current identification, with photo, including:

• Driver's license or State ID

• Passport

• U.S. Military ID

• Alien Registration card

• Matricula Consular card

OR they may bring 1 item from the above list and a major credit card or gas card.

If over 18 the student may be asked to provide 1 of the following current IDs with photo, such as:

- Driver's license or State ID
- Student ID
- Passport
- Alien Registration card
- Matricula Consular card

Plus

- Social Security number or individual tax ID number (ITIN)
- Money to deposit – ask if there's a minimum.

Bank requirements may vary, so ask your local bank what they require.

Activity

Students use the following worksheet to practice writing checks. The teacher's copy of this activity follows the students' worksheet.

(Download the worksheets and Teacher's tips provided at http://www.handsonbanking.org/en/instructional-resources.htm)

The following Project is also available in the Appendix Section of the Student Edition as well as the Student Workbook

Financial Literacy Project: Managing My Money

I. Introduction:

As you become older and more responsible, you will be making purchases that are both large and small. Most small purchases can be made using the cash that you carry in your pocket or wallet. For larger purchases you may have noticed that adults often use a plastic card. The store clerk often asks the customer if the card is a charge card or a debit card. Another customer may write a check for the amount. These are alternatives to paying with cash.

This project is to be conducted both by using resources outside class, including accessing the computer and asking adult family members, and in-class discussions. Keep in mind how mathematics plays a role when discussing financial matters.

You will investigate:

- What is the difference between a credit and debit card

- How to balance a check register of withdrawals, deposits, and transfers?
- What is a credit report? What kinds of information are contained in one, where are they kept and for how long?
- Research the features and costs of checking accounts and debit cards offered by different local financial institutions.

II. Investigation on Debit and Credit Cards

A. Investigate what are the differences and similarities between a credit card and a debit card. One internet site with helpful information is at:

http://www.nytimes.com/2009/01/06/your-money/credit-and-debit-cards/primercards.html

B. Use your own words to describe what the two types of cards are and their possible differences and similarities.

Credit Card:

Debit card:

C. Suppose you will use a plastic card to make a purchase totaling $300. Compare the difference in the process of making the purchase if you:

i. Use a debit card that is tied to an account in which you have $400. Describe what will happen when you make the purchase.

ii. Use a debit card that is tied to an account in which you have $200.
Describe what will happen when you make the purchase.

iii. Use a credit card for the purchase. Describe what will happen when you make the purchase and beyond.

iv. Use a credit card for the purchase and you are unable to pay the full amount when the payment is due the following month. Suppose the card issuer charges 1.5% interest per month on the amount you are unable to pay and you can only pay $200. How much will you owe the following month?

III. Investigations on Checking Accounts

A. Read about what a check register is and how checking accounts are balanced. Look up what withdrawals, deposits, and transfers are when using a checking account and how they affect your balance. An internet site with helpful information is: http://banking.about.com/od/checkingaccounts/ss/balancechecking.htm

B. Let's pretend we have a checking account with a $400 balance. Suppose you have the following income and expenses. Create a check register to record the transactions and keep a running balance of what you have at the end of each day.

January 22	Baby sitting Job	$25
February 13	Valentine cards check #23	$15.25
March 9	Birthday from Grandmother	$50
April 1	Baby sitting Job	$20
May 26	Teacher's present check #24	$24.75

Check #	Date	Transaction Description	$ Withdrawal	$ Deposit	$ Balance

C. Pick a bank in your city and investigate what services they offer. Is there a charge for a checking account? How much is the charge? Do they offer any interest on the balance? Is there a minimum balance that must be maintained? Is there a charge for a credit card? How much charge? Is there a charge for a debit card? How much charge? What might be other questions you may have about this bank in comparison with another bank and their fees?

IV. Investigation of credit history and report

A. Investigate what it means to talk about a person's credit history. Possible sites to visit are:

http://www.ftc.gov/bcp/edu/microsites/freereports/index.shtml

http://www.ftc.gov/bcp/edu/pubs/consumer/credit/cre03.shtm

B. Describe some of the information found on credit reports. How might they be of value to borrowers and lenders?

EXPLORATION

Fraction	Decimal	Percent
$\frac{3}{4} = \frac{75}{100}$	0.75	(0.75)(100) = 75%
$\frac{12}{25} = \frac{48}{100}$	0.48	(0.48)(100) = 48%
$\frac{4}{5}$	**0.8**	**80%**
$\frac{9}{15}$	**0.6**	**60%**
$\frac{4}{32}$	**0.125**	**12.5%**
$\frac{7}{10}$	**0.7**	**70%**
$\frac{1}{20}$	**0.05**	**5%**
$\frac{5}{8}$	**0.625**	**62.5%**

SECTION 5.4 FRACTIONS, DECIMALS, AND PERCENTS

EXPLORATION

Select "What is your favorite __________?"

Take a survey and gather data from your class.

Represent your data using fractions, decimals, and percents.

You have learned that fractions such as $\frac{3}{4}$ can be written as the equivalent fraction $\frac{75}{100}$. This equivalent fraction can also be represented by the decimal 0.75. In some instances, this number can then be converted to 75 percent, 75%. The word **percent** means "out of a hundred" in Latin.

Decimals can be converted to fractions by reading the decimal form. For example 0.75 is read "seventy-five one hundredths" which in fractional form is $\frac{75}{100}$. This in turn says 75 out of 100 or 75%. Notice how three different forms, the decimal, fractional, and percent are all referring to the same quantity.

Complete the table by finding the decimal and percent.

Fraction	Decimal	Percent
$\frac{3}{4} = \frac{75}{100}$	0.75	$(0.75)(100) = 75\%$
$\frac{12}{25} = \frac{48}{100}$	0.48	$(0.48)(100) = 48\%$
$\frac{4}{5}$		
$\frac{9}{15}$		
$\frac{4}{32}$		
$\frac{7}{10}$		
$\frac{1}{20}$		
$\frac{5}{8}$		

Activity 2:

Use the activity below to reinforce students' understanding of the connections between the three number forms of fraction, decimal, and percent, and as a way to practice comparing and ordering within the three representations.

In small groups, write 4 or 5 numbers in any form (fraction, decimal, or percent). Practice placing the numbers in order from least to greatest. Switch cards with other groups for additional practice.

What ways did you use to rewrite the fraction to a decimal? One way is to rewrite the given fraction equivalently so that the denominator is a power of 10 such as 10, 100, and 1000. Then rewrite this fraction as a decimal and then a percent.

EXAMPLE 1

How do you convert a fraction like $\frac{3}{5}$ into a decimal and a percent?

SOLUTION

$\frac{3}{5}=\frac{3\cdot 20}{5\cdot 20}=\frac{60}{100}=0.60=60\%$. Another way to convert a fraction to a decimal is to perform the division that is $\frac{3}{5}=3\div 5=0.60=60\%$.

Similarly, you can reverse the pattern of converting percents to decimals by dividing the percent by 100. For example 75% is equivalent to $75\div 100=\frac{75}{100}=0.75$. Even if the percent includes a decimal part, simply divide by 100 to get its decimal equivalent. For example, 6.48% is equivalent to $6.48\div 100=0.0648$.

The following table includes percents that are called benchmarks because they correspond to fractions that are very familiar to you. We look at 10% as an example to show how the number can be written in fractional and decimal forms as well as relate to visual models of various kinds.

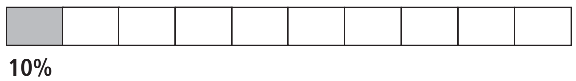
Complete the table and create corresponding visual models for each.

Fraction	Decimal	Percent
		1%
$\frac{1}{10}$	0.1	10%
		25%
		$33\frac{1}{3}\%$
		40%
		50%
		75%
		90%
		100%
		$66\frac{2}{3}\%$

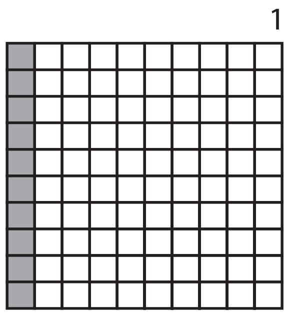
Number line model: 10% on the number line

Fraction strip model:

If the long rectangle is 100% then the 10% is shaded.

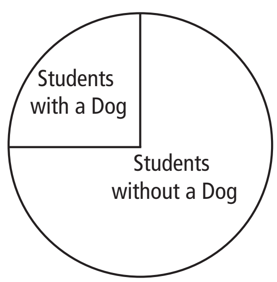


10 x 10 grid model:



10%

How do percents arise in real world problems? Suppose that in a survey of a class of 24 students concerning family pets, 6 had a dog. The fraction of the class with a dog was $\frac{6}{24}$ or $\frac{1}{4}$ of the class. Convert either of these fractions to a decimal. A pie graph is a visual representation of this class.



The portion of the pie graph labeled "Students with a Dog" is 25% of the whole class. What percent of the class does not have a pet dog? What do you notice about the percent of the class with a dog and the percent of the class without a dog? What does that percentage show about the part of the class involved in this situation?

EXAMPLE 2

Ask your students to do this now. Answer is $\frac{12}{25} = \frac{(4)(12)}{(4)(25)} = \frac{48}{100} = 0.48 = 48\%$.

EXAMPLE 3

Even though many times a fraction is not so easily transformed into hundredths, some fractions have a surprisingly nice conversion into hundredths.

PROBLEM

From the 10 x 10 grid model that you created for the benchmarks, explain the pattern you notice about the multiples of 10%? 25%? 33 1/3%

EXAMPLE 2

A class of 25 students has 12 girls. What percent of the class are girls?

SOLUTION

If 12 of the 25 students are girls, the fraction of girls to the total number of students is $\frac{12}{25}$. We can convert this fraction to a decimal and then to a percent. Or we can find an equivalent fraction for $\frac{12}{25}$ that has a denominator of 100. The fraction $\frac{12}{25}$ is equivalent to $\frac{48}{100}$, to the decimal 0.48, and to 48%.

EXAMPLE 3

Amy was shooting hoops in her backyard. She made 9 of 15 baskets. What percent of her shots did she make?

SOLUTION

The fraction of shots made to shots taken is 9 to 15 or $\frac{9}{15}$. To convert this fraction to a decimal, divide 9 by 15 to get $\frac{9}{15} = 0.6 = 0.60 = \frac{60}{100} = 60\%$, the percent of baskets made. It is a little surprising that the fraction $\frac{9}{15}$ is so nice. Many times a fraction is not so easily transformed into hundredths. We could have simplified $\frac{9}{15} = \frac{3}{5}$ and this equivalent fraction may have seemed more friendly.

EXAMPLE 4

A parking lot has 8 cars parked and 3 of them are red. What percent of the cars in the parking lot is red?

PROBLEM:

Using a part to whole idea, 4 out of 32 or $\frac{4}{32}$ of the mixed candy are lemon. $\frac{4}{32} = \frac{1}{8}$. Doing the division gives us 0.125 or 12.5%. Recall $\frac{1}{4} = 0.25$ and because $\frac{1}{8}$ is half of $\frac{1}{4}$, then $\frac{1}{2}$ of 25% is 12.5%.

Encourage your students either to draw a picture of the situation or to make an equation representing the situation as a way to working the exercises.

EXERCISES

1. a. 20% c. 75% e. 23% g. 3% i. 99.5%

   b. 25% d. 43% f. 70% h. 0.5% j. 49.9%

2. a. $\frac{1}{4} = 0.25 = 25\%$ e. $\frac{11}{20} = 0.55 = 55\%$ i. $\frac{1}{250} = 0.004 = 0.4\%$

   b. $\frac{3}{5} = 0.6 = 60\%$ f. $\frac{3}{25} = 0.12 = 12\%$ j. $\frac{1}{500} = 0.002 = 0.2\%$

   c. $\frac{5}{8} = 0.625 = 62.5\%$ g. $\frac{1}{200} = 0.005 = 0.5\%$ k. $\frac{1}{12} = 0.0833... = 8\frac{1}{3}\%$

   d. $\frac{15}{40} = 0.375 = 37.5\%$ h. $\frac{1}{6} = 0.166... = 16\frac{2}{3}\%$ l. $\frac{1}{16} = 0.0625 = 6.25\%$

3. a. $30\% = 0.30 = \frac{3}{10}$ e. $85\% = 0.85 = \frac{17}{20}$ i. $0.6\% = 0.006 = \frac{3}{500}$

   b. $25\% = 0.25 = \frac{1}{4}$ f. $325\% = 3.25 = \frac{13}{4}$ j. $57\% = 0.57 = \frac{57}{100}$

   c. $40\% = 0.40 = \frac{2}{5}$ g. $7\% = 0.07 = \frac{7}{100}$ k. $28\% = 0.28 = \frac{7}{25}$

   d. $2\% = 0.02 = \frac{1}{50}$ h. $24\% = 0.24 = \frac{6}{25}$

4. 90%

5. $\frac{4}{20}$ completed implies $\frac{16}{20}$ incomplete. $\frac{16}{20} = \frac{4}{5} = 0.8 = 80\%$

6. $\frac{5}{8}$ are girls. Decimal notation: 0.625. So, 62.5% of Genius Middle School are girls.

7. a. $\frac{12}{48} = \frac{1}{4} = .25 = 25\%$ being repaired

   b. $\frac{36}{48} = \frac{3}{4} = .75 = 75\%$ with no problems

SOLUTION

The fraction of the red cars to the total number of cars is $\frac{3}{8}$. To convert this fraction to a decimal, divide 3 by 8 to obtain 0.375. In our previous examples we had 0.75 = 75% and 0.48 = 48%. You can see that 0.05 = 5% and 0.5 = 50%. You might notice a pattern that the resulting percent involves moving the decimal place two times to the right. The decimal 0.375 must then be 37.5%. In other words, 37.5% of the cars in the parking lot are red.

PROBLEM 1

In a small bag of 32 pieces of mixed candy, there are 4 pieces of lemon candy. What percent is lemon?

PROBLEM 2

1. Penelope has an 8-ounce measuring cup. Penelope pours the following amounts of water into the cup: a) 4 ounces, b) 2 ounces, c) 6 ounces, and d) 1 ounce. Determine the percent fullness in each case.

2. Victoria has a measuring cup marked from 100 milliliters (ml) to 400 milliliters. Victoria pours the following amounts of water into the cup: a) 200 ml, b) 100 ml, c) 300 ml, and d) 120 ml. Determine the percent fullness in each case.

EXERCISES

1. Convert each decimal below to a percent and a fraction. Use the 10 x 10 grid model to represent the numbers in e, f, and g.

   a. 0.20 c. 0.75 e. 0.23 g. 0.03 i. 0.995

   b. 0.25 d. 0.43 f. 0.7 h. 0.005 j. 0.499

2. Convert each fraction below first to a decimal, and then to a percent. Use the number line to represent the numbers in b, c, and e.

   a. $\frac{1}{4}$ c. $\frac{5}{8}$ e. $\frac{11}{20}$ g. $\frac{1}{200}$ i. $\frac{1}{250}$ k. $\frac{1}{12}$

   b. $\frac{3}{5}$ d. $\frac{15}{40}$ f. $\frac{3}{25}$ h. $\frac{1}{6}$ j. $\frac{1}{500}$ l. $\frac{1}{16}$

8. 75% brought their book to class.

9. Compare the following pairs of numbers. Place the correct relationship, <, >, or = between them.

a. $\frac{1}{4} < 0.27$ b. $\frac{3}{5} > 0.59$

c. $\frac{1}{3} > 0.30$ d. $\frac{3}{4} < 0.79$

10. The numbers from least to greatest will appear in the following order on a number line:

0.25, $\frac{1}{2}$, $\frac{3}{4}$, 125%, 1.5, 190%

11. a. the numbers will appear in the following order on the number line: 0.05, 0.25, $\frac{3}{4}$, $\frac{7}{8}$

b. the numbers will appear in the following order on the number line: $\frac{6}{8}$, $\frac{5}{4}$, 1.6, $1\frac{8}{10}$

c. -0.75, $-\frac{2}{8}$, $\frac{2}{8}$, 1.2

12. 8% wore shirts other than white, blue or black. $\frac{2}{25}$, 0.08

13. 0.20, 0.125, $-\frac{1}{2}$, $-\frac{3}{4}$

14. 0.01, 0.1, $\frac{1}{5}$, $\frac{2}{8}$

15. The numbers should appear on the number lines in these orders:

a. 0.1, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{3}{10}$, 1

b. $\frac{1}{2}$, 0.56, 0.70, $\frac{3}{4}$, $\frac{5}{6}$

c. $\frac{3}{5}$ = 0.60, 0.65, $\frac{2}{3}$

d. -.75, $-\frac{3}{6}$, 0.05, 0.4

3. Convert each percent to a decimal and then to a fraction in simplest form. Use the fraction strip to represent the numbers in a, b, and c.

   a. 30% c. 40% e. 85% g. 7% i. 0.6% k. 28%

   b. 25% d. 2% f. 325% h. 24% j. 57%

4. Dirk Nowitzki made 72 out of 80 free throws in his first 10 games. What percent of his free throws did he make?

5. Lori completed 4 pages of a 20 page report. What percent of the report does she still have left to complete?

6. At Genius Middle School, there are 125 girls out of 200 students in the 6th grade. What fraction of Genius Middle School are girls? Simplify your answer, if possible. Write this as a decimal. What percent of Genius Middle School are girls?

7. Twelve out of 48 cars were being repaired because of transmission problems.

   a. What fraction, decimal, and percent of the cars were being repaired for transmission problems?

   b. What fraction, decimal, and percent of the cars did not have transmission problems?

8. Ms. Garza had 24 students during 7th period. $\frac{1}{4}$ of the students forgot their books at home. What percent of the students brought their books to class?

9. Compare the following pairs of numbers. Place the correct relationship, <, >, or = between them.

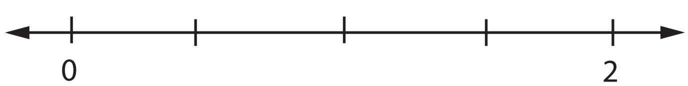
   a. $\frac{1}{4}$ 0.27 b. $\frac{3}{5}$ 0.59

   c. $\frac{1}{3}$ 0.30 d. $\frac{3}{4}$ 0.79

10. Place the following numbers in order on the number line and label them.

   a. 125% b. $\frac{3}{4}$ c. 1.5

   d. 190% e. 0.25 f. $\frac{1}{2}$



11. a. Create a number line between 0 and 1. Place and label each number at the appropriate place on the number line: $\frac{3}{4}$, 0.25, $\frac{7}{8}$, 0.05.

16. 6.100. 6.2, 7.001, 7.321, 7.4

17. 25%

18. 37.5%

19. 10

20. Spiral Review (6.1A)

-3.5, -3, $-1\frac{1}{2}$, $2\frac{1}{4}$, 5, 5.25

b. Create a number line between 0 and 2. Place and label each number at the appropriate place on the number line: 1.6, $\frac{5}{4}$, $1\frac{8}{10}$, $\frac{6}{8}$.

c. Create a number line between -1 and 2. Place each number at the appropriate place on the number line: $-\frac{2}{8}$, 1.2, -0.75, $\frac{2}{8}$

12. A survey at Miller Junior High School showed that on Monday, 20% of the students wore white shirts, 17% wore blue shirts, and 55% wore black shirts. The remaining students wore other colored shirts. What percent of the students wore shirts other than white, blue or black? Represent the value as a fraction, decimal and percent.

13. Order from largest to smallest: 0.20, $-\frac{1}{2}$, 0.125, $-\frac{3}{4}$

14. Order from smallest to largest: $\frac{2}{8}$, $\frac{1}{5}$, 0.01, 0.1

15. For each of the following, create a number line counting by halves. Place the numbers in their appropriate places on the number line. Be as accurate as possible with the spacing.

a. $\frac{1}{5}$, $\frac{3}{10}$, 0.1, 1, $\frac{1}{4}$

b. $\frac{3}{4}$, 0.70, $\frac{1}{2}$, $\frac{5}{6}$, 0.56

c. 0.65, $\frac{3}{5}$, $\frac{2}{3}$, 0.60

d. 0.4, $-\frac{3}{6}$, 0.05, -0.75

16. Lorenz and four runners have finishing times of 7.321, 7.4, 7.001, 6.2, and 6.100. Write the times in order from fastest to slowest.

17. Three-fourths of the planet is covered with water. What percent is not covered with water?

18. Five out of 8 cars are red. What percent are not red?

19. There are 80 students in the classroom. Fifty percent of them ride the bus. What is $\frac{1}{4}$ of the students that do not ride the bus? Use a pictorial model.

Spiral Review:

20. Correctly place the following numbers on a number line.

-3, $2\frac{1}{4}$, -3.5, 5, 5.25, $-1\frac{1}{2}$

21. Spiral Review (6.1E)

    15

22. **Ingenuity**

    Since 23% of the students have neither dogs nor cats, this means that 77% of the students have either dogs or cats (or both). Since 62% of the students have dogs, we know that 77% – 62% = 15% of the students have cats but not dogs. Since 44% of the students have cats, and 15% of the students have cats but not dogs, this means that 44% – 15% = 29% of the students have both cats and dogs. One very helpful way to help students think about this kind of problem is to use a Venn diagram, with one region representing the students who own dogs and one region representing the students who own cats.

Investigation

23. The purpose of this Investigation is to help build students' intuition about proportions. Students will get the greatest benefit out of this exercise if, on each problem, they make a guess, explain why they guessed the way they did, then do the calculation and assess whether their guess was correct or not (and why).

a. 60% of the boys are in the jazz band, but only 40% of the girls are in the jazz band. We can tell that the percentage of boys in the band is greater because there are twice as many girls as boys in the class, but not twice as many girls as boys in the jazz band.

b. 72% of the eighth graders are in honors classes, but only 60% of seventh graders are in honors classes. We can tell the percentage of eighth graders in honors classes is greater because there are 5 times as many eighth graders than seventh grader in the Honor Society, but six times as many eighth graders than seventh graders in honors classes.

c. 71% of the sixth graders live north of Main Street, and 81% of the seventh graders live north of Main Street. We can tell the percentage of the seventh graders living north of Main Street is greater because the sixth graders outnumber the seventh graders by 25% (one-fourth of 400), but the sixth graders living north of Main Street do not outnumber the seventh graders living north of Main Street by 25%.

21. Mrs. Martinez has 60 bottles of blue paint, 45 rolls of tape and 30 protractors. What is the greatest common factor Mrs. Martinez can use to divide the supplies into equal groups?

22. **Ingenuity:**

    At Pascal Middle School, 62% of the students have dogs and 44% of the students have cats. If 23% of the students have neither dogs nor cats, how many of the students have both dogs and cats?

23. **Investigation:**

    Answer each of the following questions. For each question, first make a guess without doing any calculations, and explain your guess. Then check your answer by hand.

    a. In Mr. Byrd's algebra class, there are 10 boys and 20 girls. Six of the boys are in the jazz band, and eight of the girls are in the jazz band. Which is greater: the percentage of the boys who are in the jazz band, or the percentage of girls who are in the jazz band?

    b. In the Honor Society at Fourier Middle School, there are 50 eighth graders and 10 seventh graders. Of these students, 36 eighth graders and 6 seventh graders are taking honors classes. Which is greater: the percentage of the eighth graders who are taking honors classes, or the percentage of the seventh graders who are taking honors classes?

    c. At a certain school, there are 500 sixth graders and 400 seventh graders. Of these students, 355 sixth and 324 seventh graders live north of Main Street. Which is greater: the percentage of the sixth graders who live north of Main Street, or the percentage of the seventh graders who live north of Main Street?

1.

Decimals	Words
0.8	eight tenths
0.16	sixteen hundredths
0.06	six hundredths
8.3	eight and three tenths
0.208	two hundred eight thousandths
3.9	three and nine tenths
100.06	one hundred and six hundredths

2.
a. 2.06
b. 315.2
c. 1
d. 1.10
e. 0.139
f. 20.0

3.
a. $0.076 < 0.7$
b. $0.923 < 0.932$
c. $3.410 = 3.41$
d. $61.75 > 61.570$
e. $0.098 < 1$
f. $0.065 < 0.65$

REVIEW PROBLEMS

1. Complete the following table.

Decimals	Words
0.8	
	sixteen hundredths
0.06	
	eight and three tenths
0.208	
3.9	
	one hundred and six hundredths

2. Round each decimal to the given place values:

a. 2.059 (hundredths) c. 0.648 (ones) e. 0.1386 (thousandths)

b. 315.183 (tenths) d. 1.099 (hundredths) f. 19.99 (tenths)

3. Compare the following pairs of numbers using, $<$, $>$, or $=$.

a. 0.076, 0.7 c. 3.410, 3.41 e. 0.098, 1

b. 0.923, 0.932 d. 61.75, 61.570 f. 0.065, 0.65

4. a. 52.08 b. 1.828 c. 0.07 d. 0.974 e. 2.136 f. 4.655 g. 53.063 h. 25.383

5. a. .0092 b. 281.238 c. 4.743

6. a. 0.1666... or 0.167 b. 0.625 c. 0.3 d. 3

7.

Simplest form	Fractions in tenths, hundredths or thousandths	Decimal form	Percent form
$\frac{3}{5}$	$\frac{6}{10}$	0.6	60%
$\frac{9}{50}$	$\frac{18}{100}$	0.18	18%
$\frac{8}{5}$	$\frac{160}{100}$	1.6	160%
$\frac{9}{20}$	$\frac{45}{100}$	0.45	45%
$2\frac{4}{5}$	$2\frac{8}{10}$	2.8	280%
$\frac{3}{50}$	$\frac{6}{100}$	0.06	6%
$1\frac{3}{4}$	$1\frac{75}{100}$	1.75	175%
$1\frac{1}{10}$	$1\frac{1}{10}$	1.1	110%
$\frac{7}{25}$	$\frac{28}{100}$	.28	28%

4. Compute the sum or difference of each:

a. 35.18 + 16.9	e. 6 - 3.864
b. 0.078 + 1.75	f. 0.15 + 2 + 2.505
c. 2.03 - 1.96	g. 36.98 + 16.083
d. 0.104 + 0.87	h. 98.6 - 73.217

5. Compute the product
   a. 4.6 x 0.002
   b. 74.01 x 3.8
   c. 0.93 x 5.1
6. Compute the quotient:

a. 1 ÷ 6	c. 1.2 ÷ 4
b. 5 ÷ 8	d. 1.2 ÷ .4

7. Compute the following table

Simplest form	Fractions in tenths, hundredths or thousandths	Decimal form	Percent form
$\frac{3}{5}$			
	$\frac{18}{100}$		
		1.6	
			45%
	$2\frac{8}{10}$		
		0.06	
$1\frac{3}{4}$			
			110%

8. a. $\frac{3}{5} = 0.6 = 60\%$ b. $\frac{11}{20} = \frac{55}{100} = 0.55 = 55\%$

c. $\frac{4}{25} = \frac{16}{100} = 0.16 = 16\%$ d. $\frac{1}{4} = 0.25 = 25\%$

e. $\frac{9}{15} = \frac{3}{5} = 0.6 = 60\%$ f. $\frac{6}{24} = \frac{1}{4} = 0.25 = 25\%$

g. $\frac{36}{50} = \frac{72}{100} = 0.72 = 72\%$ h. $\frac{28}{35} = \frac{4}{5} = 0.8 = 80\%$

9. a. $0.26 = \frac{26}{100} = \frac{13}{50}$

b. $0.015 = \frac{15}{1000} = \frac{3}{200}$

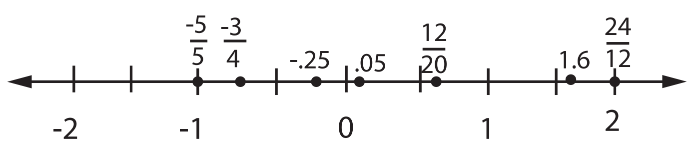
c. $0.98 = \frac{98}{100} = \frac{49}{50}$

d. $3.55 = 3\frac{55}{100} = 3\frac{11}{20}$

e. $12.500 = 12\frac{1}{2}$

f. $0.008 = \frac{8}{1000} = \frac{1}{125}$

10.



$\frac{7}{25}$			

8. Convert each fraction to an equivalent decimal. Use equivalent fractions or division. Then write the answer as a percent.

a. $\frac{3}{5}$ c. $\frac{4}{25}$ e. $\frac{9}{15}$ g. $\frac{36}{50}$

b. $\frac{11}{20}$ d. $\frac{1}{4}$ f. $\frac{6}{24}$ h. $\frac{28}{35}$

9. Convert each decimal to an equivalent fraction. Simplify the fraction if needed.

a. 0.26 d. 3.55

b. 0.015 e. 12.500

c. 0.98 f. 0.008

10. Create a number line from -2 to 2, counting by halves. Place the following numbers at the appropriate location on the number line.

$-\frac{3}{4}$, 1.6, $\frac{12}{20}$, $-\frac{5}{5}$, -0.25, 0.05, $\frac{24}{12}$

For the following word problems, write an expression that can be used to solve the problem. Then solve the problem and write your answer in a complete sentence.

11. $\frac{28}{100} = \frac{7}{25}$; $\frac{7}{25}$ of the money to feed the animals is spent on elephants.

12. (315 – 245.85) = 69.15
Everest made $69.15 more than Marci.

13. $\frac{16}{20} = \frac{4}{5} = .8 = 80\%$.

14. $\frac{1}{8} = .125$
Uncle Lloyd ate .125 of the pie.

15. 25 – (12.5 + 11.75) = .75 feet
The hardware store has .75 feet of cable wire left.

16. Car A traveled the smallest distance- 3.35 feet
Car C traveled 3.5 feet
Car D traveled 3.58 feet
Car B traveled the furthest- 3.85 feet

17. Karla will travel 8.45 miles.

18. 0.7 feet

19. 45.36 minutes

11. At the local zoo, 15% of the money to feed the animals is spent on the monkeys, 25% on the lions, 28% is spent on the elephants, 14% is spent on the giraffes, and the remaining 18% is spent on the other animals. What fraction, in simplest form, of the money is spent on food for the elephants?
12. Marci was paid $245.85 for books she sold online. Everest was paid $315. How much more money did Everest make than Marci?
13. Allison completed 16 of the 20 pages of her science project over the weekend. What percent of her project has she completed?
14. Uncle Harry ate $\frac{1}{6}$ of the cherry pie and Uncle Lloyd ate $\frac{1}{8}$ of the same pie. What decimal amount of the cherry pie did Uncle Lloyd eat?
15. The Hardware Store had a 25 foot spool of cable wire. They sold 12.5 feet in the morning and 11.75 feet in the afternoon. How much cable wire was left on the spool at the end of the day?
16. Four cars were pushed across a table during an experiment. Car A traveled 3.35 ft. Car B traveled 3.85 ft. Car C traveled 3.5 ft. and Car D traveled 3.58 ft. Place the cars in order rom least to greatest.
17. Karla lives 5.6 miles away from the doctor's office. The pharmacy is 2.85 miles away from the doctor's office. If Karla goes from her house to the doctor's office and then to the pharmacy, how far will she have to travel to go back home from the pharmacy following the same route?
18. A ribbon of length 5.6 feet is cut into 8 equal pieces. What is the length of each piece?
19. Jeremy averaged 7.2 minutes a mile in a 6.3 mile race. How many minutes did Jeremy take to complete the race?

[redacted]

MATH EXPLORATIONS

Equations, Inequalities, and Functions

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

The four sections in this chapter are connected to the big idea of how functions and rules relate numbers between two sets. In Section 6.1 students examine numerical and figurate sequences and investigate patterns and how to express the general rule for an nth term. Section 6.2 introduces equations as a mathematical statement that relates quantities including variables and algebraic expressions that represent quantities. The students informally solve equations using a balance scale and the subtraction property of equations. A number line method for solving equations and inequalities is presented in section 6.3 and revisits students exposure to operations on the number line from Chapter 1. Section 6.4 develops the concept of functions and examines relationships between input and output values often given by an equation or a graph that may define a rule for a function.

Section 6.1 - Patterns and Sequences

Big Idea:

Describing and generalizing patterns in sequences of numbers and figures.

Key Objectives:

- Determine terms in sequences using the observed pattern.
- Determine a rule for finding the nth term of a sequence from observed patterns.
- Use a rule for generating the nth term.
- Identify an arithmetic sequence.
- Foreshadow the idea of functions.

Pedagogical/Orchestration

- Teachers may want students to individually see the patterns they observe before having small groups or large group discussions. Once a pattern is stated, it is often "easy".
- Finding the next three terms may be relatively easy for many students. Having students relate the position with the term itself may be more difficult. Have students articulate both the term as it is related to the term before as well as to its position. It is the latter relationship that will help lead the students to thinking about functions..
- Consider thinking of the position and term pair as an input and output idea.
- Organizing the position and term vertically as a t-chart may provide a clearer relationship between the "input" and "output".

Materials:

None needed.

Activity:

Patterns Patterns Patterns

Exercises:

Students are encouraged to see how the position of the term and the term number or figure are related. This foreshadows the idea of functions.

Encourage students to describe in full sentences the pattern that they observe.

Vocabulary:

Pattern, sequence, constant difference, term, position, nth term, arithmetic sequence

TEKS:

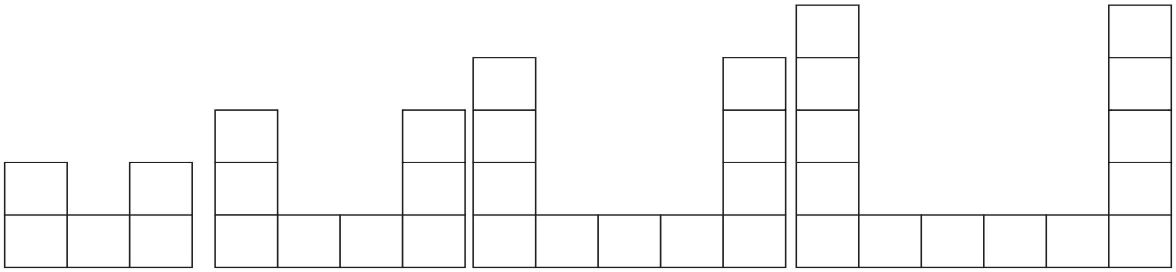
6.4 (A) New: 6.6(B)

Launch for Section 6.1:

Have each student come up with a list of eight terms. Have them write the list down and also describe in words the way they generated each term in the list. Did they have one common rule? Did they do things differently at each term? There are ways to describe patterns that they observe. Are there patterns that they observe in their lives? For example, in some cities, the names of streets go in alphabetical order for names of trees or names of birds. If a riverfront were the beginning of the street names, then Apple Street might come first, then Banana Street, and that the 6th street could be Fir while Elm Street would be 5th. Where would say Magnolia Street be relative to the river? How about Pecan Street?

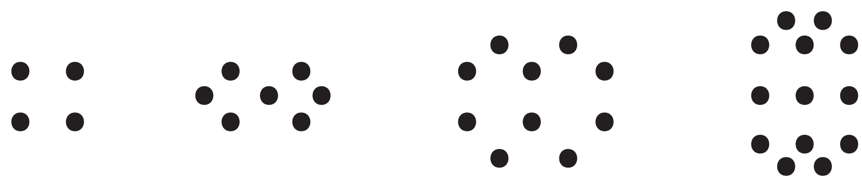
6.1
Patterns, Patterns, Patterns

1. How many square tiles would be in the next figure? In the 10th figure?



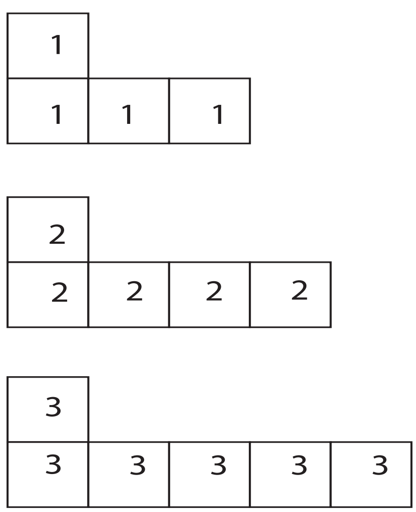
(Solution: Next figure has 17 square tiles. 10th figure has 32 square tiles.)

2. Find the pattern in the sequence started below and draw the fifth and sixth figures. Write an expression to find the nth figure and explain.



(TE Solution: 5th term has 16 dots, 6th term has 19 dots, nth term has 4 + 3(n-1) or 3n + 1 dots.)

3. Karen and Nama are building a sequence of block buildings. The first three are shown in the figure. The number shows how many blocks are stacked up on each spot. How many blocks are needed for the fourth building? The tenth building? The nth building?



(TE Solution: 4th building uses 4(7) = 28 blocks; 10th building uses 10(10+4) = 140 blocks; nth building uses n(n+4) blocks.)

4. Continue the pattern by listing the next six terms in this sequence:

3, 8, 13, 4, 9, 14, 5, 10, 15, ...

(TE Solution: 6, 11, 2, 7 where you add 5 to the previous if it is a one digit number, if it is a two digit number you sum the digits, then continue with adding 5 to single digits until a two digit number results in which case you sum the digits and so on.)

5. Determine the next value in the following pattern of numbers:

124/186, 231/396, 50%, 460/1104, 1/3, 0.25, 1/6

(Rewriting the fractions in equivalent form using a common denominator of 12, you see that the fractions are descending from 8/12, 7/12, 6/12, 5/12, 4/12, 3/12, 2/12 and so the next term must be 1/12.)

6. Find the missing number in the pattern below, and justify your reasoning for the solution.
1, 3, 6, 11, 18, _____, 42

(TE Solution: 29. The difference between each consecutive pair of numbers is a prime number. The differences are 2, 3, 5, 7, so the next difference would be 11 and 18 + 11 = 29.)

17. Describe any patterns that you observe in the sequence

1, 11, 111, 1111, 11111, 111111,

and so on. What relationship can you find between the numbers of digits in the number d, and the square of the number?

(TE Solution:

Number	Number of digits in number	Square of number
1	1	1
11	2	121
111	3	12321
1111	4	1234321
11111	5	123454321
111111	6	12345654321)

Shikaku-Rectangle Puzzles

Objective: The goal of a Shikaku puzzle is to generate rectangles whose area is the same as the number encompassed in each rectangle.

Material:

Copy of puzzle sheet

Activity Instructions:

1. Each of the following Shikaku puzzles needs to be sectioned into rectangles (and squares) along the grid lines, so that the number in each rectangle refers to the area of that rectangle.
2. Only one number can appear in each rectangle.
3. Each square on the gird is used in exactly one rectangle (that is, no rectangles may overlap).

4		2			
	9				
				10	
			5		6

9			12			5			
								6	
8		6	8						
				6		8		12	
4									
		3		9				4	

EXPLORATION:

a. 3, 6, 9, 12, 15, 18,...

3n = rule; 3 (20) = 60, so 60 is the 20th term.

b. 2, 6, 10, 14, 18, 22, ...

4n - 2 = rule

4 (20) - 2 = 78. So, 78 is the 20th term.

EQUATIONS, INEQUALITIES, AND FUNCTIONS

6

SECTION 6.1 PATTERNS AND SEQUENCES

We have many different ways to describe any given number. For example, a number can be even or odd, prime or composite, a power of just one number or a product of many different numbers. Think of other ways to classify numbers and examples for each.

A list of numbers can have interesting characteristics. For example, what **pattern** do you notice in the following list of numbers:

1, 3, 5, 7, 9, 11, 13, 15, ...

One pattern is that the numbers are positive odd number. Another observation is that the next number in the list is two larger than the number before it. You might even notice that the first number in the list is 1, the second number in the list is 3, the third number in the list is 5, and that the number in the list is one less than twice its location in the list.

If you want the number following 15 on the list, you would probably agree that 17 is reasonable. We could also say that the 9th number or **term** in the list is 1 less than twice 9. We could write the numerical expression as $2 \cdot 9 - 1 = 18 - 1 = 17$, which is indeed two more than 15. To summarize, we say that a number in any position, n, in this **sequence** or number list must be $2 \cdot n - 1$ or $2n - 1$. We also say n is the term number.

EXPLORATION

Consider the following three lists of numbers or sequences. Write all the patterns that you observe. Explain your observations. Predict what is the next term. Explain why you predict this term. Use the process that you used to find the 20th term. Do this without writing all the terms between the 7th and 20th.

a. 3, 6, 9, 12, 15, 18, ...

b. 2, 6, 10, 14, 18, 22, ...

c. 2, 6, 18, 54, 162, ...

d. 1, 4, 9, 16, 25, ...

Another way to write a sequence is by using a table. The table below for exploration 1 would look like this:

Position or term number	Term
1	3
2	6
3	9
4	12
5	15
6	18

One observation is that the terms are increasing by 3. We say there is a **constant difference**, 3, between the adjacent terms. Lists of numbers with a constant difference are called **arithmetic sequences**. Are any of the other sequences in the exploration above arithmetic sequence? Identify the constant differences.

Another observation that you may have noticed is "horizontally". Is there a pattern that relates the position of the number with the number in the list? The pattern should relate 1 to 3, 2 to 6, 3 to 9, 4 to 12, and so on.

We can describe the pattern between the position, n, and the corresponding term as 3n. Check to see if this expression for the nth term works. The 20th term should be $3 \cdot 20 = 60$. Does this check with your work in the exploration?

PROBLEM 1

Consider the sequences below.

a. 10, 100, 1000, 10000, ...

b. 7, 12, 17, 22, 27, 32, ...

c. 1.2, 2.2, 3.2, 4.2, 5.2, ...

Use tables to describe the number sequences above. Include the next three terms in the sequence. Write an expression for what you think will be the nth term. Some patterns may involve shapes that change in a predictable manner. The example below involves shapes created by dots. A pattern is observable both as a shape and as a number sequence.

EXERCISES

1. a. 17, 21 (rule: $4n - 3$) b. 24, 29 (rule: $5n - 1$) c. 512, 2048 (rule: 2^{2n-1})

2. Next picture has 17 squares, then 21 squares.
   Sequence: 5, 9, 13, 17, 21,...

   Rule = $4n + 1$

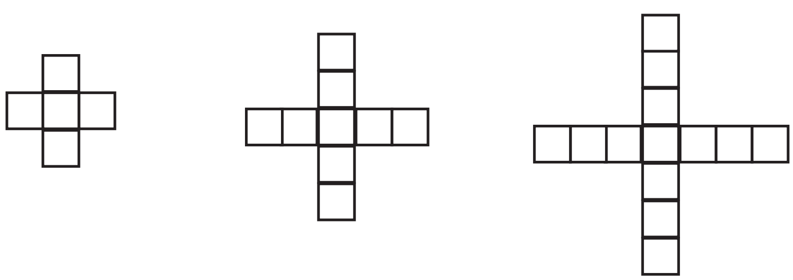
Position	Figure	Term
1	· ·	2
2	· · · · · ·	6
3	· · · · · · · · · · · ·	12
4	· · · · · · · · · · · · · · · · · · · ·	20

If the pattern continues, determine how many dots would be in the 5th figure; in the 6th figure. What pattern do you observe? Write an expression for the number of dots in the n^{th} term.

EXERCISES

1. For each of the following sequences, write the next two terms.
   a. 1, 5, 9, 13, ...
   b. 4, 9, 14, 19, ...
   c. 2, 8, 32, 128, ...
   d. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$

2. Observe the pattern below. Draw the next two figures of the pattern. Then create a list of numbers that describes the sequence of the figures.



Numerical Sequence: ______________________________

Describe the pattern that you created in the numerical sequence.

3. Sequence: 3, 6, 9, 12, 15. Rule/Pattern: 3n

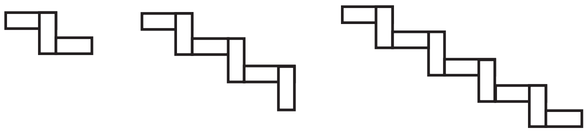
4.

Rick's Age	Numerical Process	Bill's Age
1	3 • 1	3
2	3 • 2	6
3	3 • 3	9
4	3 • 4	12
5	3 • 5	15
x	3 • x	y

5.

Pam's Age	Numerical Process	Rudy's Age
6	6 - 5	1
7	7 - 5	2
8	8 - 5	3
9	9 - 5	4
10	10 - 5	5
x	x - 5	y

3. Observe the pattern below. Draw the next two figures of the pattern. Then create a list of numbers that describes the sequence of figures..



Numerical Sequence: ____
Describe the pattern that you created in the numerical sequence.

4. Complete the table from the data below.
Bill is three times Rick's age.

Rick's Age	Numerical Process	Bill's Age
1		
2		
3		
4		
5		
x		y

5. Complete the table from the data below.
Decreasing Pam's age by 5 years will give her brother, Rudy's age.

Pam's Age	Numerical Process	Rudy's Age
6		
7		
8		
9		
10		
x		y

6. Answer: y = 12n.

7. Answer: 96 times

8. 92 seconds. Yes.

9. **Ingenuity:**
There are several different ways to approach this problem. One approach is to group the terms in a convenient way:

(300 - 299) + (298 - 297) + ... + (4 - 3) + (2 - 1) = 1 + 1 + 1 + ... + 1 + 1

There were 300 terms in the original series, and therefore 150 pairs of terms. So there are 150 1's on the right side of the equation, and the sum is 150.

Another possible approach is to try finding the first few partial sums of the series:

300 = 300
300 - 299 = 1
300 - 299 + 298 = 299
300 - 299 + 298 - 297 = 2
300 - 299 + 298 - 297 + 296 = 298
300 - 299 + 298 - 297 + 296 - 295 = 3
....

Notice that the even-numbered (second, fourth, sixth, etc.) partial sums are 1, 2, 3, and so on. So the 300th partial sum will be one half of 300, which is 150.

6. Molly is making bracelets to sell at her booth at the school carnival. The table below shows the relationship between the number of bracelets she made and the number of crystals she used to make each bracelet. Write an equation to find y, the number of crystals needed, in terms of n, the number of bracelets she made.

Number of Bracelets (n)	Number of Crystals (y)
1	12
2	24
3	36
4	48
n	y

Spiral Review:

7. Stephanie's heart beats 8 times per 10 seconds while she is resting. How many times would Stephanie's heart beat during 2 minutes of rest?

8. Emmie can run 100 meters in 23 seconds. If she competes in the 400-meter race, how many seconds will it take Emmie to run the race? Did she run the race in under 2 minutes?

9. **Ingenuity:**

   Find the value of the following expression:

   300 - 299 + 298 - 297 + 296 - ... + 4 - 3 + 2 - 1

10. **Investigation:**

    The *Fibonacci sequence* is a well-known sequence of numbers that begins with the following terms:

    1, 1, 2, 3, 5, 8, 13, 21, 34, ...

    a. What is the pattern here? What are the next five terms of the sequence?

    b. Suppose we pick a positive integer N, and add up the first N terms of the Fibonacci sequence. For example, if we pick N = 4, then the sum of the first N terms is 1 + 1 + 2 + 3 = 7. Try doing this for other values of N, and make a table of the sums you get. Do you see a pattern?

10. **Investigation:**

a. The pattern is that each term, beginning with the third term, is the sum of the two terms before it. The next five terms are 55, 89, 144, 233, and 377.

b. We can make a table of the partial sums of the Fibonacci sequence:

N	Nth partial sum
1	1
2	2
3	4
4	7
5	12
6	20
7	33

Notice that the Nth partial sum of the Fibonacci sequence is one less than the (N+2)th term of the sequence.

c. Again, let's make a table of our observations:

n	Square of nth term	Product of (n-1)th and (n+1)th terms
2	1	2
3	4	3
4	9	10
5	25	24
6	64	65
7	169	168
8	441	442

When n is even, the square of the nth term is one less than the product of the two neighboring terms. On the other hand, when n is odd, the square of the nth term is one more than the product of the two neighboring terms.

c. Suppose we pick one of the terms of the sequence and square it. Then we take the product of the terms on the left and right of that term. For example, if we choose 8 and square it, we get 8^2 = 64. If we take the product of the two neighboring terms, we get 5 x 13 = 65. Try this for several terms of the sequence. What do you notice?

Section 6.2 – Equations

Big Idea:

Translating a word problem by writing an equation.

Key Objectives:

- Setting up equations and informally solving them using a number line, picture, or balance scale.
- Solve one step equations by using the subtraction property.

Pedagogical/Orchestration:

- Teacher may remind students that an equation is just like setting up problems that they have seen before, for example: $3 + \_\_\_\_ = 10$ and instead of "box" we call it *N* or variable.
- The teacher needs to be aware that a numerical expression is not an equation. The definition of equation needs to be made clear and reinforced at this point.
- Teacher may lead students to avoid setting up an equation incorrectly; for example: x less than 10 equals 8 should be written as: $10 - x = 8$. 10 less than a number equals 6 should be written as: $x - 10 = 6$.
- Introduce Polya's 4 step problem solving process.
- The subtraction property on the balance scale gives the students a concrete way of seeing an equation as a balance scale.

Materials:

Legal size paper, markers.

Activity:

"Match My Algebra", "4 Step Process Poster Activity"

Exercises:

Exercises 8 maybe the best on balance scale.

Exercises 4 & 5, 6 & 7 may be set up by role playing them in front of class.

Vocabulary:

equation, translate(change). equivalent, equality, solve.

TEKS:

6.1(C); 6.5; 6.11(B); 6.12(A) New: 6.9(A); 6.6(C); 6.7(B)

Launch for Section 6.2

Write the following words on the blackboard leaving room to write in student responses:

Words Phrases Sentences

Ask students for words and phrases found in algebra and write down their responses. The "Words" column should include things like numbers, variables, and operators. The "Phrases" column should include the word "expressions" with examples such as $x + 5$ and others. Tell students, "Today we will expand our algebra language to include complete sentences. You have heard quite a few algebra sentences already. Who can give me an example of one?" Accept responses such as $x + 2 = 5$ or $x > 5$. Tell students any equations or inequalities are considered a sentence in algebra. Ask students, "What is the difference between a phrase and a sentence?" Listen to responses and make sure it comes out that a sentence has a verb and is a complete thought whereas a phrase is a sentence fragment or part of a sentence. Sentences are made up of phrases just as equations and inequalities are made up of two expressions that are being compared. An equation always has an equal sign which is the verb of the sentence. Tell your students, "Today we will be translating phrases and sentences into the language of algebra." Throughout the lesson continue to add examples of algebra words, phrases and sentences to the board.

Match My Algebra

Objective: This game will help students review translating word expression to algebraic expression for Section 6.1

Materials:

Teacher will prepare 10 to 20 algebraic expression cards written as in the following example:

7 decreased by 31 51 is greater than x, 3 more than 20.

Activity Instruction:

The class will be divided into two groups/teams. Each group/team chooses a player. The player goes up to the board and uses the number line to "act out" the algebraic expression. The player's team tries to translate the phrase, while a timer goes on for 2 minutes (adjust time as needed.) If the team correctly answers, the team gets 1 point. If the teach is unable to answer correctly, the opposing team gives their translation. If the opposing team provides the correct answer, the team gets the point from their opponent. Game continues with the player from the opposing team. The team /group with the most points wins.

Rules - "Act Out" will allow students to use:

Body language

Gestures

Forward/backward motions on the number lline

No spoken clues

4 Step Process Poster Activity

Materials:
legal sized paper (1 per pair of students), markers

Instructions:

1. Pair up students.
2. Assign each pair one of the problems from the equation statements provided (you may create more or different statements if needed).
3. Direct students to create a poster showing the problem they are solving and how each step is used to solve it.
4. Make sure students are following the 4 step process correctly. Monitor the translation step closely as this is the step students struggle with.
5. Encourage students to display their information in a colorful creative way.
6. After students present posters, you may wish to display them.

Lead your students through Examples 1 through 3 so they can consider and discuss them. Also offer the example from the Exploration, but don't tell your students to open their books to that page yet. Instead, let them consider the steps they might use to translate the Exploration scenario and solve for x. Then have them compare their steps to those in the book.

The focus of these problems is to practice using the four-step process as a way of setting up algebraic problems. The numbers here are fairly simple, but the students should be encouraged to learn to set up problem and to organize their information. Steps 1 and 2 are the important steps here. Step 3 may seem very simple. Step 4 is important. Students should check their own work and determine if their answers are reasonable and determine if they have answered the question posed.

SECTION 6.2 EQUATIONS

Let's begin with the sentence "A number is 3 more than 7." You could figure out what this number is with relative ease, but how can you write this mathematically? You may recall that you can use numbers, variables, and operations to form expressions. We can now combine these expressions to form a mathematical sentence called an equation. An **equation** is a math sentence with an equality sign, **=**, that relates two expressions.

EXAMPLE 1

Translate the sentence "A number is 24 more than 17" into an equation. What is the difference between the expressions in your answer and the equation? Explain verbally, numerically, and algebraically.

SOLUTION

Step 1: We give the unknown number a name, N, and write "$N =$ the number." N is a variable. It represents the number we are trying to find.

Step 2: We translate the sentence into an equation.

A number	is	24 more than 17
N	$=$	$17 + 24$

So the equation form of the sentence is $N = 17 + 24$.

Since $17 + 24 = 41$, we can conclude that $N = 41$. N now represents a known quantity, 41, instead of an unknown quantity. Therefore, we say that we have **solved** the equation for N.

Step 3: The expressions that you used may be numerical such as 17+24, or algebraic such as N. Numerical expressions only contain numbers, whereas algebraic expressions may include both variables and numbers.

The equation is the statement that these two expressions are equal, N = 17+24. Every equation has an equality sign, =.

PROBLEM 1

a. $10 - x = 8$

b. $x - 10 = 8$

PROBLEM 2

Let C = number of milk cartons Terry purchased

$C = (2)(12)$

$C = 24$

PROBLEM 1

Translate the sentences below where x is a number.

a. x less than 10 equals 8.

b. 10 less than x equals 8.

EXAMPLE 2

What number is twice as large as six?

Use the number line to also illustrate the solution.

SOLUTION

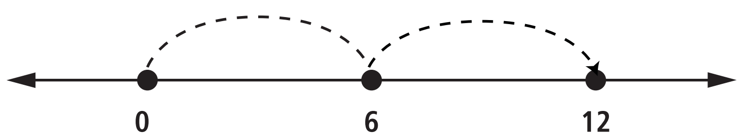
Step 1: We define a variable to represent our number. Let T be a number that is twice as large as six.

Step 2: The statement "A number is twice as large as six" translates as $T = 2 \times 6 = 2 \cdot 6 = (2)(6)$.

When we put the symbols 2 and 6 next to each other with parentheses around each, it is understood that we mean to multiply them. The small dot is also a symbol for multiplication. So $(2)(6) = 2 \cdot 6 = 2 \times 6$. We usually do not use the symbol $\times$, however, since it could be confused with a variable x.

Step 3: We know $2 \cdot 6 = 12$, so $T = 12$.

Step 4: Check. Is 12 a number that is twice as large as 6? Yes.



PROBLEM 2

Pat purchased a dozen doughnuts. The number of milk cartons Terry purchased is twice the number of doughnuts Pat purchased. How many milk cartons did Terry purchase?

Using variables to model problems is the beginning of learning algebra. Algebra enables us to translate problems into mathematical expressions and equations. We then use mathematics to solve for the unknowns, which provides solutions to our problems.

Give the Jeopardy cards back and ask each student to write a new sentence for their secret number, one that can be written as an equation. By now the class should have some examples to refer to, if they need help. The sentences could be thought of as clues for someone to discover their number. For example, "F is 2 more than 5 times 7 becomes: $F = 5(7) + 2$ so $F = 37$. Some of these clues are more help than others. Generally, the clues that can be translated into equations are the most help.

Have students share their sentences for the class to translate and, possibly, solve. Do as many as you have time for. Collect the rest to use in tomorrow's summary.

Teacher Tip: Lead your students to create a poster or post the steps of an equation-solving procedure for students to use as a guide until they have demonstrated a solid understanding of the process.

EXAMPLE 3

Translate the sentence "A number is 2 less than four times 10" into an equation and solve for the unknown variable. Does your answer make sense?

SOLUTION

We use the 4 - Step process.

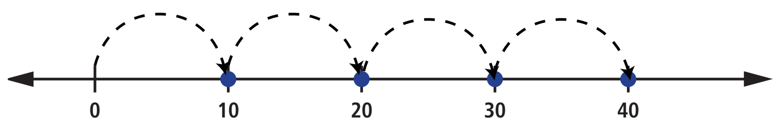
Step 1: Let's call our unknown number N.

Step 2: $N = (4)(10) - 2$ is our equation for the statement above.

Step 3: The right side is equal to 38. So $N = 38$. We have solved the problem! Now let's check our answer.

Step 4: Is 38 equal to 2 less than 4 times 10? Four times 10 equals 40, and 2 less than 40 is 38, so our answer is correct.

Notice on the number line below that 4 times 10 is 40.



Then 2 less would mean backing up from 40 to 38, which is our solution as shown below.

EXPLORATION: CHARTING THE PROCESS

We have seen how we can use numbers and variables to translate problems into equations. Consider the problem, "Jeremy is 9 years old. In how many years will Jeremy be 15 years old?"

How might you begin this problem? Did you define a variable? If so, how did you use this variable?

Here is a step-by-step approach. Do your steps resemble the following?

Step 1: Define your variable

PROBLEM 3

a. N – 6 = 46 N = 52 b. N = 25 + 12 N = 37

c. N = (2)(15) - 9 N = 21 d. N = (2)(55) N = 110

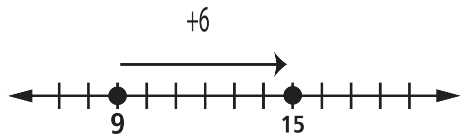
Y = the number of years it takes for Jeremy to reach 15.

Step 2: Translate the problem into an equation

We know that 15 is Y more than 9, so we write $15 = 9 + Y$, an equation with one variable, Y.

Step 3: Solve for the unknown variable

If you look on the number line, you'll notice you have to move right 6 units to go from 9 to 15. So $Y = 6$.



Step 4: Check your answer

Substitute $Y = 6$ into the original equation to see that $15 = 9 + 6$.

Using the 4 step process discussed above, create a poster showing how the steps should be used to solve a problem. Begin by writing the problem and continue by showing how each step helps you solve the problem.

PROBLEM 3

Write the following statements in equation form. Let N = the number.

a. 6 less than a number is 46.

b. A number is 12 more than 25.

c. A number is 9 less than twice the number 15.

d. A number is twice as large as 55.

This is a **balance scale**:

When we put a weight on one side of the scale, we must place the same weight on the other side in order for the scale to be balanced. If a scale is balanced and equal weights are added or subtracted from both sides of the scale, the scale will remain balanced.

PROBLEM 4

John = 5 + Wesley

11 + 5 = Wesley

6 = Wesley. So, Wesley has 6 marbles.

In much the same way, an equation is a statement that two expressions are equal. We can think of the expressions on each side of the equality sign as representing the weight placed on each side of a balanced scale. When we add or subtract the same amount from each side of the equation, the equation will remain **balanced**.

PROBLEM 4

If Wesley finds 5 more marbles, he will have the same number of marbles as John. John has 11 marbles. How many marbles does Wesley have?

The rule we are using to solve these problems is called the **subtraction property of equality,** because in each, we are subtracting the same number (removing the same number of blocks) from both sides of an equation. The new equation we obtain is said to be **equivalent** to the original equation because the two equations have the same solution: any value for a variable that makes one of the equations balance will make the other balance as well.

DEFINITION 6.2: SUBTRACTION PROPERTY OF EQUALITY
If $A = B$ then $A - C = B - C$.

Remember the sequence 1.2, 2.2, 3.2, 4.2, ... from the previous section. We saw that the term in a particular position, n, is n + 0.2. We can write this as the term = position + 0.2 or t = n + 0.2, where we let the variables t = term and n = position. According to this pattern or "rule", the 20th term would be 20.2 and the 200th term would be 200.2.

Suppose you were given the term 187.2. Can you determine which position this number is in the list? Remember that t = n + 0.2 and 187.2 is the term or symbolically, t = 187.2. If we rewrite the equation as 187.2 = n + 0.2, how can we determine n? One way is to use the subtraction property. We have 187.2 = n+0.2, but we want n so consider subtracting 0.2 from both sides of the equation. Then 187.2 - 0.2 = n + 0.2 - 0.2 and 187 = n.

Finding the value that makes an equation true is referred to as **solving an equation**. Solving equations is a very important part of doing algebra, and the subtraction property is an important tool for solving equations.

EXPLORATION

The total cost of a meal for three people is $51. If the three people agree to split the cost equally, what would each person's cost be? Write two equations that could be used to model the problem. You do not have to solve the problem. Use x to represent the cost each person pays.

You may have found that one equation is $3 \cdot x = 51$. Another equation could be written as $x = \frac{51}{3}$. You may recall this important connection between multiplication and division. We will talk more about this when we multiply by fractions in the next chapter.

We can relate the idea above with the sequence 3, 6, 9, 12, 15, ... Suppose we ask the question, "what position is 51 in this list?" We write this question mathematically as $3 \cdot x = 51$, where we let x represent the position the term 51 occupies in the list. From above, $3 \cdot x = 51$ is equivalent to $x = \frac{51}{3} = 17$. We conclude that 51 occupies the 17th position in the list above.

EXAMPLE 4

If Jeremy were three years older, he would be the same age as his twelve-year-old sister. What is Jeremy's age?

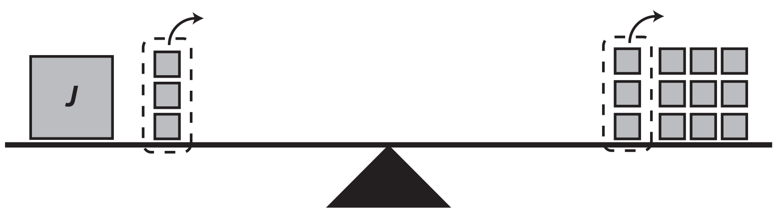
SOLUTION

We let J be Jeremy's age, and translate the sentence into an equation as follows:

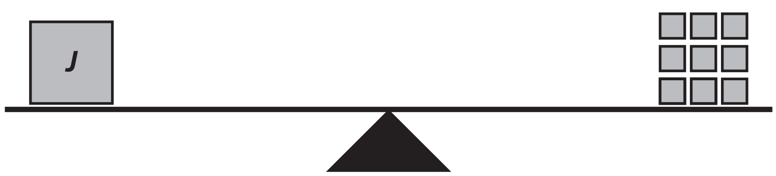
Jeremy's age	three years older	same age as	twelve-year-old sister
J	$+ 3$	$=$	12

Now we have the equation $J + 3 = 12$. The unknown is J, Jeremy's age. Visually, this sentence says that $J + 3$ is the same as 12, which we can show on a balance scale:

In order to solve this equation for J, we must find what balances J. To do this, we remove three "blocks" from each side of the balance scale:



This is what we have left:



We can express this algebraically as follows:

$$
\begin{aligned}
J + 3 &= 12 \\
(J + 3) - 3 &= 12 - 3 \\
J + (3 - 3) &= 9 \\
J + 0 &= 9 \\
J &= 9
\end{aligned}
$$

EXERCISES

1.

a.	N = 38 + 3;	b.	N = 16 - 5	c.	N = 2 (65)	d.	N - 6 =16
e.	N = 2 • 15 + 4	f.	N - 7 = 40	g.	N = 2 • 12 - 7	h.	N - 6 =22
i.	N = (1/2) (24) - 5	j.	N (19)	k.	N = 2 (18) + 5	l.	N + 8 = 13

Because we have now solved for J, we can go back and check the solution. Substituting $J = 9$ into the original equation $J + 3 = 12$ gives us $9 + 3 = 12$, which is correct. Jeremy's age is 9.

EXERCISES

Try the following exercises using the four-step process. When you "solve" your problem, you should not only find the answer, but also show the way you got your answer, which is just as important.

1. Write an expression that represents each number below. Which of these expressions are numerical, and which are algebraic?
   a. 3 more than 17
   b. 5 less than 16
   c. Twice as much as 65
   d. 6 less than a number
2. Write the following statements in equation form. In each case, let N = the number. Explain verbally what are the numerical expressions, the algebraic expressions, and what are the equations for each problem.
   a. A number is 3 more than 38.
   b. A number is 5 less than 16.
   c. A number is twice as large as 65.
   d. 6 less than a number is 16.
   e. A number is 4 more than twice the number 15.
   f. 7 less than a number is 40.
   g. A number is 7 less than twice the number 12.
   h. 6 less than a number is 22.
   i. A number is 5 less than half of 24.
   j. A number is double 19.
   k. A number is 5 more than twice the number 18.
   l. 8 more than a number is 13.

3. $32

4. 11 years old

5. 9 years

6. $x = 72 - 29$

Answers will vary for 7 and 8.

9.

a.	$p = 209$	b.	$q = 983$	c.	$r = 4\frac{1}{2}$	d.	$t = \frac{3}{10}$
e.	$n = .14$	f.	$y = 1.2$	g.	$m = 1.318$	h.	$x = .48$

10. Mike = m, Jill = m + 3, Ramon = 2 (m + 3)

a.	14 cards	b.	26 cards	c.	2 (x + 3) cards

11.

Ted = Juan - 5; Sophia = 3 (Ted) = 3 (Juan - 5)

So. when Juan = 12, Sophia = 21.

Do steps 1 and 2 of our four-step process for Exercises 3–6. See TE.

3. Jake has \$65. How much more does he need if he wants to have \$97?
4. If Lori will be 19 in 8 years, how old is Lori now?
5. Mark is 12 years old. In how many years will Mark be 21?
6. Sean has \$72 and lends George \$29. How much does he have left?

Write a story problem for the equations in Exercises 7 and 8:

7. $x + 9 = 27$ Answers will vary.
8. $81 - x = 17$
9. Solve the following equations.

a. $p + 4 = 213$ $x = 0.11$ c. $\frac{1}{2} + r = 5$

b. $17 + q = 1000$ $y = 0.5$ d. $t + \frac{2}{5} = \frac{7}{10}$

e. $n + 0.18 = 0.32$ $x = 0.11$ g. $m + 5.382 = 6.7$

f. $y + 0.6 = 1.8$ $y = 0.5$ h. $x + 0.03 = 0.51$

Use the 4 step process you learned to solve the following problems. Remember to use the correct labels (ºC, ºF, years, cards, etc.) when defining your variable and in your solution.

10. Mike has a certain number of baseball cards, let **m** represent the number of cards that Mike has. Jill has 3 more cards than Mike, and Ramon has twice as many cards as Jill. How many cards does Ramon have if Mike has: See TE.

a. 4 cards? b. 10 cards? c. x cards?

11. Sophia has two nephews, Juan and Ted. Ted is five years younger than Juan. Sophia is three times older than Ted. If Juan is 12 years old, what is Sophia's age?

12.

Sara = Jane - 3; Gloria = 2 (Sara) = 2 (Jane - 3);

So, if Jane is 9 then Gloria = 2 (9 -3) = 12

13. 20602 feet

Spiral Review:

14. 29 ˚F

15. 13 ˚C

12. Gloria has two nieces, Sara and Jane. Sara is three years younger than Jane. Gloria is twice as old as Sara. If Jane is 9 years old, how old is Gloria?

13. The lowest and highest points in North America are Death Valley in California and Mount McKinley in Alaska, respectively. Death Valley is below sea level, in fact, 282 feet below sea level! On a number line, we represent this elevation by $^-282$. Mount McKinley is 20,320 feet above sea level. There is a big difference between the two elevations. Use D to represent the height we must climb to go from the elevation of Death Valley to the elevation of Mount McKinley. The equation that models this situation is $^-282 + D = 20{,}320$. Solve for D.

Spiral Review

In exercises 14 and 15, define a variable, set up an equation, solve, and check your work. Remember to include your units of measurement, such as °F, °C, feet, inches and years.

14. In the morning it was a cool 58 °F. By mid-afternoon the temperature had reached 87 °F. What was the increase in temperature from morning to mid-afternoon? 22 °F

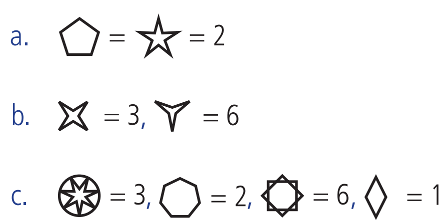
15. On a cold day in Canada, the temperature was $^-5$ °C at 6:00 AM How many degrees must it warm up to reach 8 °C?

16. **Ingenuity:**

Since 41 students went to the park, and 2 students went in the car, the other 39 students went in the vans. There were three vans, and each van carried the same number of students, so there were 39 [divided by] 3 = 13 students in each van.

Another approach is to write an equation. Letting S be the number of students in each van, we get 3S + 2 = 41. We then have 3S = 39, and thus S = 13.

17. **Investigation:**



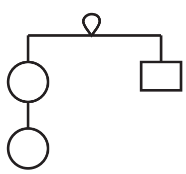
This is a good place to mention that different symbols (whether weights or variables) can have the same values. For a practical application, suppose x is the number of nickels, y is the number of dimes, and z is the number of quarters. Then surely you can have, for example, 5 nickels and 5 quarters, in which case $x = z$.

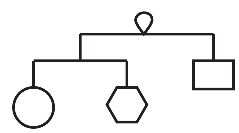
16. **Ingenuity:**

The Banneker Middle School band took a field trip to an amusement park to celebrate their success at a recent contest. They rented three vans to take students to the park. It turned out that the vans were not enough, so two students had to ride in a teacher's car. If each van had the same number of students, and a total of 41 students went to the park, how many students rode in each van?

17. **Investigation:**

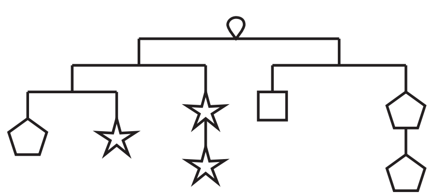
A mobile is a type of hanging sculpture in which several objects are suspended in balanced equilibrium. In the mathematical mobiles below, each shape has an associated weight, and for each horizontal beam, the total weight hanging on one side is equal to the total weight hanging on the other (we assume that the wire has no weight). For example, in the mobile on the left, the two circles together weigh as much as the rectangle: the rectangle weighs twice as much as a circle. In the mobile on the right, the circle has the same weight as the hexagon, and the rectangle has twice the weight of the hexagon.





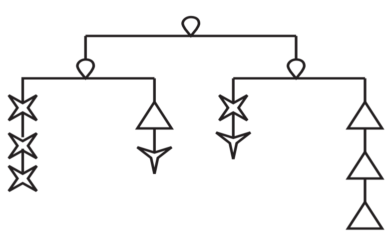
If the rectangle has weight 4, the hexagon and circle each have weight 2. We can write this symbolically as □ = 4, ⬡ = ○ = 2. Based on the weight given and the mobile balance property, deduce each shape's weight.

a.



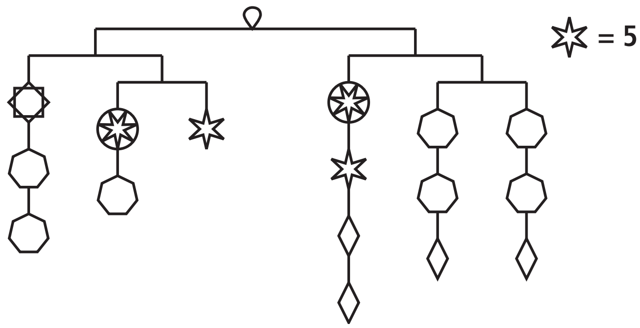
□ = 4

b.



△ = 3

c.



Section 6.3 - Equations and Inequalities on Number Line

Big Idea:

Develop the idea of solving equations using a number line.

Key Objectives:

- Plot points on the number line to represent various expressions
- Reflecting on the effects of adding/subtracting a positive integer to/from a variable
- Visually connecting that $x+3 = 7$ is equivalent to $x+2 = 6$ and $x+1 = 5$ and $x = 4$.

Vocabulary:

equation, inequality

TEKS:

6.9(A, B, C)

EXPLORATION 1

TE: If $x<0$, then $-x>0$ and $2x<x$. Students might object to $-x$ being a positive number. An extension question could be to plot points that represent $a/2$ and $3a/2$ in part 1.

EXPLORATION 2

TE: Students can make a little strip of paper with marks showing the distance of 1 and use it to measure 1 unit to the right or left of x. Students may try to estimate the value of $x+1$ or $x-2$. This is OK but it is not the purpose of this activity.

TE: Reflect with students about any patterns that they notice. In particular, what is the effect of adding a positive integer to a variable? What is the effect of subtracting a positive integer from a variable?

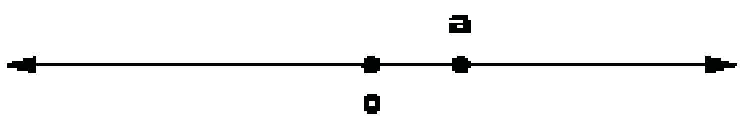
SECTION 6.3 EQUATIONS AND INEQUALITIES ON NUMBER LINE

In Section 6.2, we used the balance scale to solve equations such as "$x + 3 = 5$". We can also explore equations using a number line. We begin by investigating how to visualize expressions on a number line.

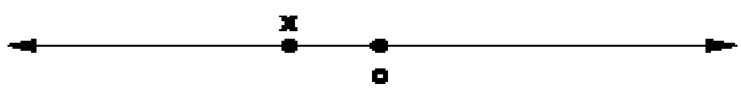
EXPLORATION 1

Suppose a and x are numbers located on the number line as seen below. Locate and label the points that represent the indicated numbers. Use string to act out how you determine your answer.

1. Plot points that represents each of the following: 2a, 3a, -a, -2a, -3a



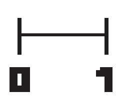
2. Plot points that represents each of the following: 2x, 3x, -x, -2x, -3x



3. Compare the results from parts 1 and 2. What do you notice?

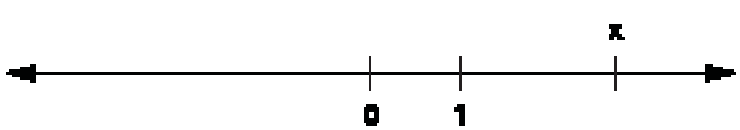
EXPLORATION 2

Part A: Suppose x is a number that is located on the number line as seen below. Locate and label the points that represent the indicated expressions. The numbers 0 and 1 are also labeled. The length of the line segment below is 1:



Plot a point that represents each of the following expressions:

$x + 1$, $x + 2$, $x + 3$, $x - 1$, $x - 2$



TE: Expect students to see that

$x+3=7$ is equivalent to $x+2=6$ and $x+1=5$ and $x=4$. That this process is subtracting 2 from "both sides".

$y-2=1$ is equivalent to $y=3$

$z+2=-3$ is equivalent to $z+1=-4$ and $z=-5$

Reflect with class that we are showing that if $x+3=7$, then $x=4$. This is a visual way of "solving" the equation $x+3=7$. So, the equation $y-2=1$ has the solution $y=3$ and the equation $z+1=-3$ has the solution $z=-4$.

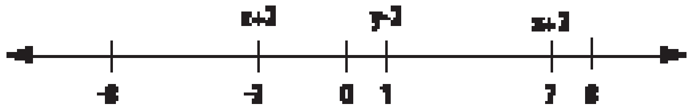
PROBLEM 1

TE: $x = 2$, $y = -3$, $z = 6$

EXAMPLE 1

1. TE: You can also say that $A > 2B > 0$.
2. TE: You can say $0 < J < 40$.

Part B: Suppose we know the location of each of the expressions as indicated on the number line below. Find the locations for x, y, and z. Explain how you locate each of these points on the number line.



In Part A in this exploration we used the location of a variable on the number line to locate expressions on the same number line. In Part B, we were given the location of an expression, such as x+3 = 7, and used it to find the location of the variable x on the number line. We see that x = 4. In other words, we solved the equation using the number line.

PROBLEM 1

Use the number line to solve each of the following equations:

a. $x + 3 = 5$

b. $y + 5 = 2$

c. $z - 4 = 2$

d. Discuss how solving these equations on the number line compares with the balance scale method.

Recall that an equation is a statement that two expressions are equivalent. A statement that one expression is always less than (or greater than) another is called an **inequality**.

EXAMPLE 1

1. The number of apples, A, consumed is more than twice the number of bananas, B.

2. Jack's age, J, is less than 40 years.

SOLUTION

1. $A > 2B$.

2. $J < 40$. Sometimes an inequality is a statement of comparison between two quantities, such as, $4 < 7$. But we can also use an inequality to describe a condition that a variable satisfies, as in Jack's age, J, is less than 40. So we say $J < 40$.

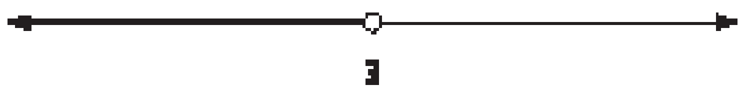
EXAMPLE 2

TE: We say -2 is in S because the statement that -2 is less than or equal to -2 is true because -2=-2.

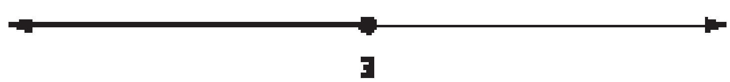
We can use a number line to represent all the possible numbers that satisfy an inequality. For example, suppose S is the set of all numbers less than 3. Another way to describe this set is:

"S is the set of all numbers x so that $x < 3$."

We use the inequality $x < 3$ to defined the set S. We can represent this set S on the number line below.



Notice that the part of the number line to the left of 3 represents the set S. This means that each number to the left of 3 is in S and every number of S is located on the line to the left of 3. Note that the point representing 3 is not filled in to indicate that 3 does not satisfy the condition that $x < 3$. We write $x \le 3$, read numbers x less than or equal to 3, to mean $x = 3$ is included. The number line representation looks like this:



EXAMPLE 2

Draw a number line and represent the set T of all numbers x such that $-2 \le x$.

SOLUTION

Notice that the point at -2 is filled in to indicate that -2 does satisfy the condition that $-2 \le x$.

In solving an equation, such as $x + 3 = 5$, we want to find all numbers x that satisfy this statement. Since the only solution is $x = 2$, we say that the solution set is {2}.

If we start with an inequality, such as $x + 3 < 5$, we can ask:

TE: Discuss how the inequality $x<2$ and $2>x$ are equivalent.

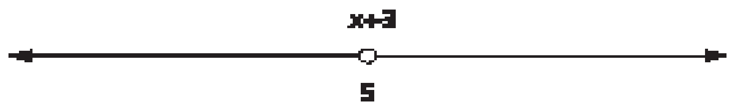
EXERCISES

For 1. and 2. Students can make a little strip of paper with marks showing the distance of 1 and use it to measure 1 unit to the right or left of x

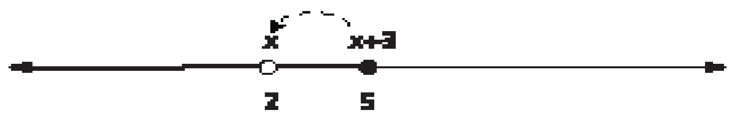
3. x = -3. Use the number line to represent the solution.

What numbers x satisfy this inequality?

We can represent the inequality on the number line as shown below.

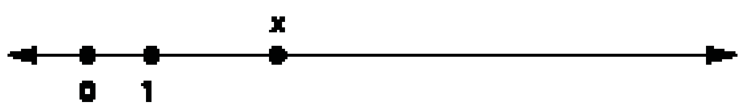


The shaded part of the number shows where the expression x+3 could be located on the number line. We draw a new number line to represent where the variable x can be.



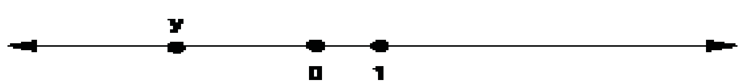
EXERCISES

1. Plot a point that represents each expression: 2x, 2x + 1, 2x – 1, 3x – 1

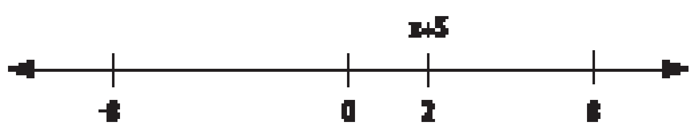


a.

2. Plot a point that represents each expression: y+1, y – 1, 2y, 2y + 1, 2y – 1, 2y + 8

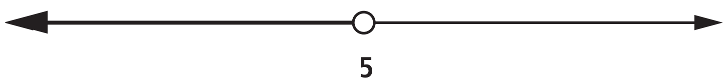


3. Solve the equation x + 5 = 2 using the number line.

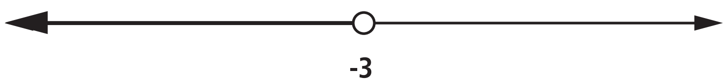


4. a. $x + 4 = 6$ TE: $x = 2$
   b. $x + 2 = 7$ TE: $x = 5$
   c. $x - 4 = 2$ TE: $x = 6$
   d. $x + 6 = -2$ TE: $x = -8$
   e. $x - 5 = 2$ TE: $x = 7$
   f. $x - 4 = -6$ TE: $x = -2$
   g. $x + 4 = 4$ TE: $x = 0$

5.



6.



8.

9. a. $x < -1$ b. $x < 5$ c. $x < 1$ d. $x < -3$
   e. $x > 3$ f. $x > 6$

Make sure that students graph their solutions on the number line.

4. Use a number line to solve each of the following equations:

   a. $x + 4 = 6$
   b. $x + 2 = 7$
   c. $x - 4 = 2$
   d. $x + 6 = -2$
   e. $x - 5 = 2$
   f. $x - 4 = -6$
   g. $x + 4 = 4$

5. Draw a number line and represent the set of all numbers x such that $x < 5$.

6. Draw a number line and represent the set of all numbers x such that $x < -3$.

7. Draw a number line and represent the set of all numbers x such that $x > 1$.

8. Draw a number line and represent the set of all numbers x such that $-2 < x$.

9. Solve each of the following inequalities and graph their solution sets on the number line:

   a. $x + 3 < 2$
   b. $x - 3 < 2$
   c. $x + 5 < 6$
   d. $x + 5 < 2$
   e. $x + 3 > 6$
   f. $x - 4 > 2$

10. Graph the solution sets for each of the following inequalities. Show your work on the number line

   a. $2 < x + 3 < 5$
   b. $0 < x - 3 < 2$
   c. $2 < x + 5 < 8$

Investigation

12. a. $x = 2$
    b. $x = 5$
    c. $x = 4$

d. $0 < x + 4 < 3$

11. Write word problems that can be modeled using the following inequalities. Be sure to state what the variable stands for in each of your word problem, then solve the inequality. Check to see that it makes sense.

a. $x + 5 < 13$ b. $x - 7 > 2$ c. $x - 3 < -1$

Investigation

12. Use a number line to solve each of the following equations:

a. $2x + 1 = 5$

b. $2x + 3 = 9$

c. $2x - 3 = 5$

Section 6.4 - Functions

Big Idea:

Develop the idea of function: A relationship between two sets in which one input value from a first set yields only one output value from a second set.

Key Objectives:

- Discover that inputs are first coordinates (from domain) represented by the independent variable and outputs are second coordinates (from range) represented by the dependent variable.
- Use function rule to generate input/output pairs.
- Discover the use of tables to see patterns.

Materials:

Sticky Notes, Handouts of coordinate grids with axes (several per student for notes and exercises), Coordinate grid for whole-class demonstration, markers or map colors

Pedagogical/Orchestration:

- Reinforce also that given an input of a function, it generates only one specific output and no more.
- Reinforce the importance of showing the process by which the output results from the rule of the function.
- Organizing the input helps to see the function rule more easily than to randomly list the inputs.

Activities:

"Functions, Functions, Functions", "Guess my Function"

Vocabulary:

function, domain, range, notation, rule, table, graph, input, output

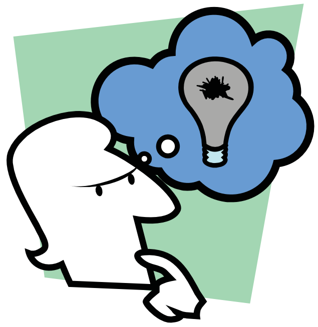
TEKS:

6.2(C,E); 6.4(A,B); 6.5; 6.7 New: 6.6(A, B, C); 6.11

Launch for Section 6.4:

Tell students today they will be creating a machine that performs a function. Ask for two volunteers and have the students stand in front of the class with arms raised and hands touching, "London Bridge" style. These students are the function machine. There will be subjects called "inputs" that will go into the machine. The machine will perform a function on the inputs and change them in some way. When they leave the machine the inputs have been transformed and are now an "output." The machine always performs the same function on each input. Ask for volunteers brave enough to go through the function machine. As students line up to go through the function machine, ask audience students to guess what the function of the machine is. Have a large coordinate grid set up to graph the input and output values as coordinate pairs. Give the first input student a sticky note with a 4 on it. As he/she comes out of the machine switch the 4 with a 12. Have a student plot the point (4,12) on the grid. Have the input student place the sticky notes with the 4 and 12 on the board under the titles of input and output. Do this with several students plotting the input and output values of (2, 6), (1,3) and others on the graph and placing the sticky notes on the table. Guide students to the idea that rearranging the sticky notes so the input values are in numerical order helps in noticing patterns. Students will hopefully guess that the function is to multiply the input by 3. Have a discussion of the arrangement of the points on the grid. Students should notice the linear pattern. Ask students if the pattern they see will continue for all inputs going through the function machine. Tell students, "Pay close attention to the lesson today and apply what you learned about function machines to the mathematical functions we will discuss."

Functions, Functions, Functions



Objective: The students will use their understanding of functions to find the value of a function given one input and its corresponding output.

Materials:
Functions, Functions, Functions Worksheet (one copy per student)

Activity Instructions:

1. Divide your class into equal groups. Four students per group would work best, if possible.
2. Make one copy of the function worksheet for each student.
3. Explain to the students that they are to find a function that will work for each table given one input and output per table. The challenge is that, each member of the group must have a different function.
4. After discovering a function that works, each student will complete their table with at least 3 more values for each input and output.
5. It is expected that the group will work together, as a whole, even though they each have their own worksheet. If one student in the group opts to take the easy way out and find a simple function that works, they are still responsible for helping their group complete their worksheets.

Name ______________________________ Date ____________________

Functions, Functions, Functions

Directions: You will need to consider the input and output that is given for each table, and use this information to discover a function that will work for the data given. After finding and recording a function that works, find at least 3 more input and output values for each table using your function.

Table 1		
Input (x)	**Function**	**Output (y)**
1		7

Table 2		
Input (x)	**Function**	**Output (y)**
2		11

Table 3		
Input (x)	**Function**	**Output (y)**
3		15

Table 4		
Input (x)	**Function**	**Output (y)**
4		19

Table 5		
Input (x)	**Function**	**Output (y)**
5		23

Guess My Function

Objective: Students will work in small groups to find patterns in numbers to discover functions. Students will state all functions using function notation.

Materials:
Set of Guess My Function cards
Scratch paper, or small dry erase boards
Dice (one die per group)

Activity Instructions:
The class will need to be divided into small groups of 4 to 6 students. Each group will be given a set of the Guess My Function cards and one die. The students in each group will roll the die to see who will go first.

Once it has been established who will go first in each group, the student who goes first will take the top card from the deck. The student will then use the information on this card to give clues to the person to their left. For example, if the card reads "add five", the student holding the card will tell the player to his left, $f(2) = 7$ and $f(0) = 5$. (This reads, "f of two equals seven" and "f of zero equals five".) The player on the left will use these clues to try to guess the function. Once the player on the left thinks they know the function, they will give a third value for x. For example, they could state that $f(3) = 8$ (read as f of three equals 8.) The student holding the card will tell this player whether or not he is correct, but the round isn't over until this player can state the correct function. If the player can state that the function is $f(x) = x + 5$ or $f(x) = 5 + x$, then that player gets a point.

It might be a good idea to set a time limit on each round to make sure that each player in the group has time to play. A good time limit might be 2 minutes per person. You can either use a sand timer, or have a group member watch the clock. Within this two minute time period, the player that is guessing the function can have unlimited guesses to try to find the correct function.

The game will continue in a clockwise direction until either all Guess My Function cards are used or until the designated time for the game is up. The player with the most points at the end of the game wins.

Add four $f(x) = x + 4$	Subtract six $f(x) = x - 6$	Multiply by two $f(x) = 2x$
Divide by seven $f(x) = \frac{x}{7}$	Multiply by three, then add five $f(x) = 3x + 5$	The absolute value of $f(x) = \lvert x \rvert$
Add three, then divide by two $f(x)=\frac{x+3}{2}$	Add ten $f(x) = x + 10$	Subtract five $f(x) = x - 5$

Multiply by four $f(x) = 4x$	Divide by three $f(x) = \frac{x}{3}$	Multiply by five, then subtract two $f(x) = 5x - 2$
Multiply by six $f(x) = 6x$	Add twelve $f(x) = x + 12$	Subtract eight $f(x) = x - 8$
Add six, then divide by three $f(x) = \frac{x+6}{3}$	Add twenty $f(x) = x + 20$	Subtract nine $f(x) = x - 9$

Teacher's Note: Check for Understanding

Since the input represents number of days, make sure students understand that only positive whole and decimal numbers are valid answers. You cannot work a negative number of days.

Have the students write these pairs of numbers, i.e. (3 days, 6 planes) on pieces of paper or cards. Have individual students put these on the board with tape. Don't organize the data for them. After putting up 7-9 pairs, reflect on what they notice. Ask if there is any way to "organize" the information in such a way that could help see any patterns.

You want the students to articulate the pattern that the number of airplanes is always twice the number of days.

Things to listen for: The table starts with 0 in the first column. Be sure to make sense of $x = 0$ for your students. The numbers in the first column increase by 1 for the first few slots and then they make some jumps. The numbers in the second column are increasing by 2 for the first few lines.

Teachers, if students are having trouble organizing information then you might want to re-copy the table on the board so that you can write the corresponding pair notation next to each row in the table. Realize that the end goal is to get students to develop tables themselves. But this will help your students to visually link the table and the pairs. Note x is a variable not multiplication in 2x. Also, recall input-output.

For example:

Days	Total Number of Planes	Coordinates
0	0	(0, 0)
1	2	(1, 2)
2	4	(2, 4)
3	6	(3, 6)
5	10	(5, 10)
10	20	(10, 20)
x	$2x$	$(x, 2x)$

The domain consists of numbers that can be used as inputs so that the output make sense. 7 as an input makes sense, because in 7 days Sarah will have made $2 \cdot 7 = 14$ airplanes. However, -7 as an input would refer to -7 days or before she started making airplanes. -14 as an output does not make sense. Using a decimal such as 0.5 could represent half a day, usually we will refer to whole or whole number days for input.

SECTION 6.4 FUNCTIONS

In our daily lives, we often encounter situations where we receive a set of instructions and then perform certain tasks based on those instructions. In mathematics, this is the role of a function. A **function** is a rule that assigns a unique **output** value to each number in a set of **input** values. Let's explore this.

EXPLORATION 1

Sarah builds model airplanes. She makes two airplanes each day. How many airplanes will she make in 4 days? 10 days? Organize the information to reveal a pattern in the number of airplanes she makes in a given number of days.

How did you organize the information? Do you see a pattern in the number of airplanes Sarah can make in a given number of days?

One way to organize such information is to build a table like the one to the right. Notice that the first column is the number of days and the second column is the total number of airplanes that she can make in the corresponding number of days.

Days	Total Number of Planes	Ordered Pairs
0		
1		
2		
3		
5		
10		
x		

What do you notice about this table? Why is this a good way to organize the information?

This is an example of a function. There is a rule, or function, to determine how many planes Sarah has produced based on the number of days she has worked. You can think of a function as a machine with inputs and outputs. The input is the number of days Sarah has worked. The output is the number of planes produced.

Articulate multiple use of the word range. Include domain and range in the definition but view as inputs and outputs.

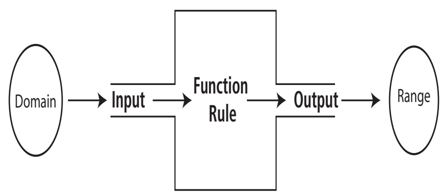
Many students do not understand that different inputs can have the same output, especially when they first see the concept. This confusion is only compounded when they are told that any input can only have one corresponding output. You might not want to go into detail with your students about this now, but you might try to develop extra examples for the class being careful not to produce examples of relations that are not functions.

Address ellipsis (dots) as continuing a similar pattern.

Think of a number to input into the machine. Are there some restrictions on the number you can use? Does it make sense to input 7? -7? 0.5? Explain your reasoning. Now use the number you chose for the input and determine the output corresponding to your number using this function.

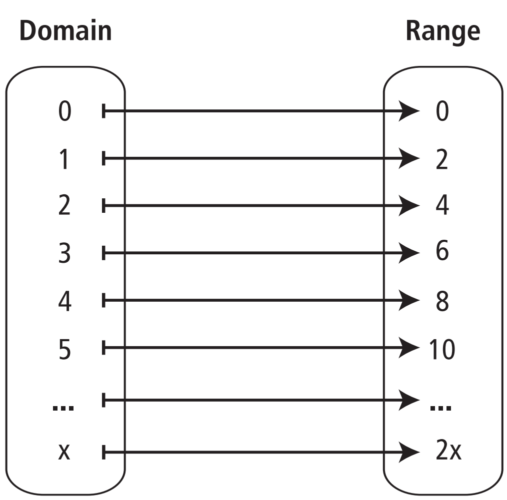
Now that we have an idea of what a function does, let's go ahead and make a more formal definition.

DEFINITION 7.1: FUNCTION
A **function** is a rule which assigns to each member of a set of inputs, called the **domain**, a member of a set of outputs, called the **range**.



For example, consider Sarah's function from Exploration 1 again. The domain is the set of nonnegative integers 0, 1, 2, 3, … and the range is the set of even nonnegative integers 0, 2, 4, 6…

In general, the domain = set of inputs and the range = set of outputs.

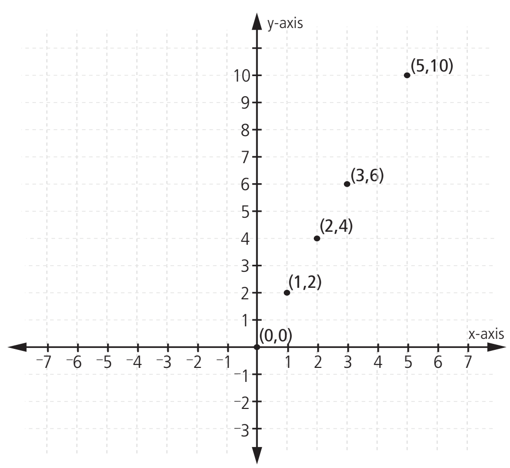


Finally, take time to discuss the relationship between the inputs and outputs. This is especially powerful in the coordinate notation. If you take the x coordinate and double it you will get the y coordinate. As students progress through this chapter, have them look for patterns between the x and y coordinates. Encourage the habit of observing patterns and the students will become better at finding others.

Notice that a function produces input and output pairs of numbers.

Input	Function Rule	Output
x	Multiply input by 2 to get the output y = 2(x) = 2x	y
0	y = 2(0)	0
1	y = 2(1)	2
2	y = 2(2)	4
3	y = 2(3)	6
x	y = 2(x)	2x

Let's call Sarah's function *F*. From the table and the picture above, we can see that (0, 0), (1, 2), (2, 4), (3, 6), and (5, 10) are some of the pairs that belong to the function *F*, where the *x* or first-coordinate is the input and the *y* or second-coordinate is the output. In other words, the ordered pairs are of the form (input, output) or (*x*, *y*). Each pair of numbers can be thought of as a point on the coordinate system, so we can also talk about the **graph** of a function. The graph of the function is the visual representation of the function.



In mathematics, a **notation** is a technical system of symbols used to represent unique objects. We can write "The function *F* pairs the number 1 with 2" using the function notation as "*F*(1) = 2." We read this as "*F* of 1 equals 2." This

means F sends the input 1 to the output 2. Similarly, because the function F pairs the number 2 in our domain with the number 4 in the range to give us the pair (2, 4), we write "$F(2) = 4$."

So $F(x)$ = the number of planes that can be produced in x days. We can express this rule in general as $F(x) = 2x$.

We sometimes express this rule as the equation $y = 2x$. Note that if the rule for the function F is $y = 2x$, then we say the point (x,y) belongs to the graph of F. Since the value of an output y depends on the value of its corresponding input x, we call y the **dependent variable** and we call x the **independent variable**.

EXPLORATION 2

Consider the following tables of numbers that describe a function.

Function f

Input	Function Rule	Output
x		y
1		13
2		14
5		17
7		19
8		20

Function g

Input	Function Rule	Output
x		y
0		10
1		10
2		10
3		10
4		10

1. Describe in words all the patterns that you observe between the input and the output of the two functions above.

EXPLORATION 3

1. TE: They have different steepness. They each go through the origin (0,0).

2. TE: They are parallel to each other. They have the same slant.

3. TE: The lines with y = mx vary in steepness which depends on m while the lines with equation y = x + b have the same "slant" and are parallel to each other.

EXPLORATION 4

1. Function R

Input	Process	Output
0	4 (0) + 7	7
1	4 (1) + 7	11
2	4 (2) + 7	15
3	4 (3) + 7	19
4	4 (4) + 7	23
10	4 (10) + 7	47
15	4 (15) + 7	67

2. Function S

Input	Process	Output
0	none	4
1	none	4
2	none	4
3	none	4
4	none	4
10	none	4
15	none	4

3. 55 = 4x + 7 => 48 = 4x => x = 12

2. Use your pattern rule to determine the output if the input is 10 in each of the functions above.
3. Write an expression for the output (dependent variable) y in terms of the input (independent variable) x.
4. Extend the tables for functions f and g by selecting inputs that are rational numbers that are not whole numbers. For example, select numbers like x = ½, -12.5, -11½ for the inputs, then determine the corresponding outputs. Do the same for the function g.
5. Plot the points for each of these functions on a coordinate system.

In Exploration 2, the table of inputs and outputs for functions f and g determine a pattern that can be expressed using independent and dependent variables. In function f, y = x + 12. In function g, y = 10.

EXPLORATION 3

1. Make a table for each of the following functions. Plot these points on a coordinate system. The points from each table lie on a line. Draw each line in a different color. What do they have in common? How are they different?
   a. $f(x) = x$
   b. $g(x) = 2x$
   c. $h(x) = 3x$
   d. $j(x) = 4x$

2. Make a table for each of the following functions. Plot these points on another coordinate system. The points from each table lie on a line. Draw each line in a different color. What do they have in common? How are they different?
   a. $f(x) = x + 1$
   b. $g(x) = x + 2$
   c. $h(x) = x + 3$
   d. $j(x) = x + 4$
3. Look at these two groups of graphs. What do they have in common? How are they different?

1. a) $x + 5 = y$
   b) $x (0) +3 = y$
   c) $5x + 1 = y$

EXPLORATION 4

The functions below have rules for the input, x, and the output, y. Make a table to indicate the outputs for the inputs, 0, 1, 2, 3, 4, 10, 15. Make sure your table includes a column that shows how you got the output.

1. The function rule for R is given by the equation $y = 4x + 7$.
2. The function rule for S is given by the equation $y = 4$.
3. Use the R function to determine the input if the output is 55.

EXERCISES

1. Consider the following pattern of inputs and outputs. Write a rule for each that gives the output in terms of the input x. Plot the points for each of these functions on a coordinate plane.

a.

Input	Output
2	7
3	8
4	9
5	10
6	11

b.

Input	Output
1	3
2	3
3	3
4	3
5	3

c.

Input	Output
1	6
2	11
3	16
4	21
7	36
10	51

2

a. 13 b. 15 c. 17

3.

a. 11 b. 15 c. 19

4.

Input	Rule: $x^3 + 1$	Output
0	$0^3 + 1$	1
1	$1^3 + 1$	2
2	$2^3 + 1$	9
3	$3^3 + 1$	28
4	$4^3 + 1$	65
5	$5^3 + 1$	126
8	$8^3 + 1$	513

2. Consider the function machine with the following data:

Input	Rule: y=x+10	Output
0	0 + 10	10
1	1 + 10	11

where y = x + 10.

Copy the input/output table and extend it to show the outputs for the following inputs:

a. $x = 3$ b. $x = 5$ c. $x = 7$ d. $x = 2.5$

e. $x = -\frac{1}{2}$ f. $x = -11\frac{3}{4}$

3. Consider the function machine with the following data:

Input	Rule: y = 2x -3	Output
2	2 ·2 - 3	1
3	2 ·3 - 3	3
5	2 ·5 - 3	7

where y = 2x - 3.

Copy the input/output table and extend it to show the outputs for the following inputs. Plot the points for each of these functions on a coordinate plane.

a. $x = 7$ b. $x = 9$ c. $x = 11$ d. $x = -3.5$

e. $x = \frac{1}{2}$ f. $x = 2\frac{1}{4}$

4. Given the function where the output y is given by $x^3 + 1$. Determine the output for the given inputs.

Input	Rule: $x^3 + 1$	Output
0		
1		
2		
3		
4		
5		
6		
7		
8		

5.

a.

Input	Output
0	1.5
2	3.5
3.75	5.25
8	9.5
100	101.5
2.05	3.55
x	1.5 + x

b.

Input	Output
0	0
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
x	1.5x

6

Input	Output
.7	4
1.7	5
2.7	6
3.4	6.7
5	8.3

Because $y = x + 4$, the students may be encouraged to think about $y - 4 = x$ as the relationship for finding x if you know y.

7. $y = 25n$
   So, if $n = 4$, then $y = 100$
   if $n = 7$, then $y = 175$
   if $n = 11$, then $y = 275$
   if $n = 13$, then $y = 325$
   if $n = 15$, then $y = 375$

8. a

x = # of gallons	y = cost of x galllons
0	10
1	13
2	16
3	19
5	25
10	40
x	10+3x

See last row of the table.

5. a. Use the function rule y = x + 1.5 to make a table with inputs 0, 1, 2, 3, 4 and 5.

   b. Use the function rule y = 1.5x to make a table with inputs 0, 1, 2, 3, 4 and 5.

   c. Plot the points from both of these tables and sketch the two lines.

   d. Discuss what you observe from your graphs and table. Compare how the functions y = x, y = ax and y= x+a are similar or different numerically in your table? Graphically? How does the constant *a* affect each graph?

   e. Work this same problem with the rules y = x +3 and y = 3x. Compare each of these new graphs with your previous solutions.

6. Suppose a function is given by the rule y = x + 4. Fill in the inputs in the table below that would yield the given outputs. In other words, what input x would produce each of the outputs listed in the table below?

Input	Output
	0
	2
	4
	6
	8

7. Miranda sells necklaces for $25 each. Let *n (input)* represent the number of necklaces she sells and *y (output)* represent the amount of money she earns from selling *n* number of necklaces. Create an input/output table to reflect the following information. Then write a rule (function) that explains the relationship in terms of y.

   n = 4 n = 7 n = 11 n = 13 n = 15E.

8. Monica is going to sell lemonade. She spent $10.00 for materials to get started. It then cost her $3.00 a gallon for each gallon of lemonade she made.

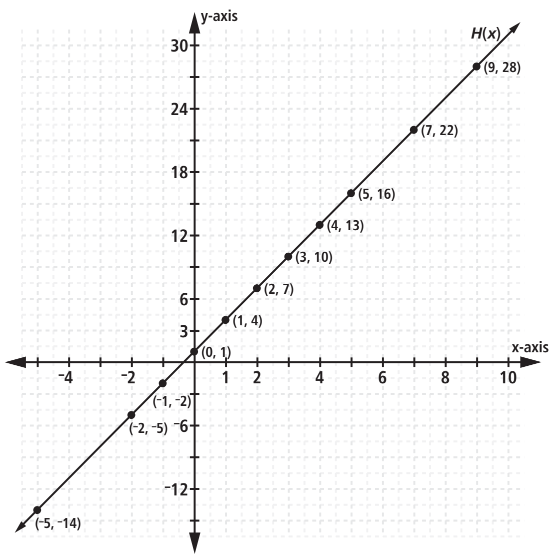
   a. Complete the input/output table to determine *y*, the cost of making *x* gallons of lemonade.

x = # of gallons	y = cost of x gallons
1 0	
1	

9. a. 1
   b. 4
   c. 7
   d. -8 (At this point we have not shown students how to multiply by a negative, so this may be a bit tricky.)
   e. 16
   f. 91
   g.

x	$y = 3x + 1$	y
1	3 (1) + 1	4
2	3 (2) + 1	7
-1	3 (-1) + 1	-2
0	3 (0) + 1	1
5	3 (5) + 1	16
3	3 (3) + 1	10

9. h.



10. See on the next page

2	
3	
5	
10	
x	

b. Write an equation in terms of y that represents the cost of making x gallons of lemonade. table.

9. Consider the function given by the rule $y = 3x + 1$. Compute the outputs corresponding to the following inputs:

a. 0 1 d. -3 -2

b. 1 4 e. 5 16

c. 2 7 f. 30

g. We can think of the above as ordered pairs (x, y). Copy and complete the table below using the function as given.

x	y	$y = 3x + 1$
1	4	3 (1) + 1
2	7	3 (2) + 1
-1	-2	3 (-1) + 1
0	1	3 (0) + 1
5	16	3 (5) + 1
3	10	3 (3) + 1

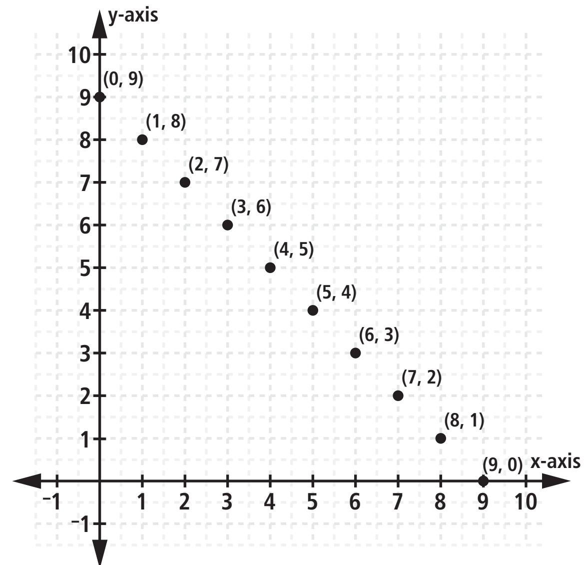
h. Graph the ordered pairs from parts **a** through **g**, and then connect the points to graph the function. See TE, although answers will vary.

10.

a.

x = Time (min.)	$M(x)$ = Number of M&Ms left
0	11
1	10
2	9
3	8
4	7
5	6
6	5
7	4
8	3
9	2
10	1
11	0

b.



c. $M(x) = 11 - x$

d. Terry will run out of M&M's after 11 minutes.

e. [0, 11]. Domain: time in minutes, Range: Number of M & Ms left

Ingenuity:

11. $F(x) = |x| + 2$

Investigation:

12. See on the next page.

10. Terry has 11 M&M's. He eats them very slowly; in fact, he takes 1 minute to eat each one. See TE.

    a. Make a table for the number of M&M's Terry has left after x minutes. Use zero for the starting time.
    b. Graph the points from part **a**.
    c. Find the function M that gives the number, y, of M&M's Terry has left after x minutes.
    d. When will Terry run out of M&M's?
    e. What are the domain and range for the function M?

11. **Ingenuity:**

Write a formula of a function having the following table of values:

Input	Output
-3	5
-2	4
-1	3
0	2
1	1
2	0
3	1
4	2
5	3

12. **Investigation:**

    In this Investigation, we will take a look at the function $F(x) = x^2 + 2x$. This function is called a *quadratic function* because it involves the square of the input and does not involve any higher power of the input.

    a. Make a table of inputs and outputs for the function F(x). What do you notice from looking at the table?
    b. Graph the points from part (a). What do you notice from looking at the graph?
    c. Based on the table and/or the graph, can you find any solutions to the equation $x^2 + 2x = 0$?

Investigation

12.

a. Make a table of inputs and outputs for the function F(x). Be sure to include some positive values and some negative values. What do you notice from looking at the table?

Answers will vary. Here is an example of what students might come up with:

x	F(x)
-3	3
-2	0
-1	-1
0	0
1	3
2	8
3	15

b. Graph the points from part (a). What do you notice from looking at the graph?

We notice that the graph of this function is not a straight line, unlike the graphs of many of the functions we have studied in this section. Instead, the graph of this function appears to have its lowest point at the point (-1, -1). (We explore why this is the case in Math Explorations: Algebra I.) On the right and left sides of that point, the graph ascends, gradually rising faster as we move away from (-1, -1). The result is a U-shaped graph called a [ital] parabola [/ital]. Note that students may connect the dots from their table of values using straight line segments; do not worry about this right now.

c. Based on the table and/or the graph, can you find any solutions to the equation x^2 + 2x = 0?

We are looking for values of x such that F(x) = 0. Looking at our table, we find two such values of x: x = -2 and x = 0. We can also get this information from the graph, provided that we have drawn the graph well enough. The solutions to the equation x^2 + 2x = 0 will be the values of x where the graph of the function F(x) = x^2 + 2x = 0 crosses the line y = 0, which is the x-axis.

1. $x = 11 - 4, x = 7$
2. $x = 2(15) + 3, x = 33$
3. $x = 2(19) = 38$
4. $x = 16 - 15, x = 1$
5. $x = \frac{1}{2}(40) + 3, x = 23$
6. $t = 9$
7. $p = 31$
8. $y = 303$
9. $m = 18$
10. $d = 4$
11. $a = 1$
12. $m = 151$
13. $x = 32$
14. $a = 29$
15. $b = 57$
16. She hasn't passed her birthday yet. After her birthday then the year will count for her.
    $1991 + x = 2005, x = 14$
    $1991 + x = 2006, x = 15$
17. The answer depends on the present year. Take the 21st century year and add 525. $x = PY - 1475$.

REVIEW PROBLEMS

In problems 1–5, translate the sentence into an equation and solve for the unknown variable. Does your answer make sense?

1. A number is 4 less than 11.
2. A number is 3 more than twice 15.
3. A number is double the number 19.
4. A number is 15 less than 16.
5. A number is 3 more than a number that is half of 40.

In problems 6–15, solve the equation. Use either the balance scale or the number line to show how you got the answer.

6. $3 + t = 12$
7. $p - 16 = 15$
8. $y = 20 + 283$
9. $63 + m = 81$
10. $28 + d = 32$
11. $9 = a + 8$
12. $76 = m - 75$
13. $x = 82 - 50$
14. $54 - a = 25$
15. $b - 17 = 40$

16. Delores was born September 4, 1991. How old was she on the 4th of July 2006? How old was she at Christmas of that same year?
17. The great Renaissance artist Michelangelo was born in 1475. If Michelangelo were still alive, how old would he be on his birthday this year? Write a mathematical equation, solve and check.

18. 237 m

19. $31

20. 15

21. 18

23. $y = x^2 + 2$

24. $y = 3x + 3$

18. The tallest mountain in the world with an elevation of 8848 meters above sea level is Mount Everest, a part of the Himalayan Mountains near Nepal and Tibet. The next highest peak, at an elevation of 8611 meters above sea level, is K2, also a part of the Himalayas. How much taller is Mount Everest than K2?

19. Marianne had $43 in bills before she went out to eat. After paying for dinner, she found she had a $10 bill and two $1 bills. Approximately how much did she spend?

In problems 20 and 21, define a variable, set up an equation, and solve.

20. Valerie bought 8 books. She now has 23 books. How many books did she start out with?

21. In 24 years, Victor will be 42 years old. How old is Victor now?

22. $2 + a = 6$ and $b = 8 + a$. Solve for b and a. Use the number line to show your work.

In problems 23 and 24, write a rule for each that gives the output in terms of the input, x

23.

Input	**Output**
2	6
3	11
4	18
6	38
10	102

24.

Input	**Output**
4	15
5	18
6	21
10	33
12	39

25. a)

1	3
2	6
3	9
4	12
5	15
10	30
x	3x

b) $y = 3h$

c) 90 hrs.

25. When Hiroko was writing her new math book, she realized that she could finish 3 pages every hour. Let y = the number of pages Hiroko completed. Let h = the number of hours she worked.

    a. Create an input/output table using the following inputs and find the corresponding outputs.

       $h = 1$ $h = 2$ $h = 3$ $h = 4$ $h = 5$ $h = 10$

    b. Write an equation, in terms of y, to describe the relationship between the number of hours (h) and the total pages completed (y).

    c. Use the equation you created in part b to determine the number of hours Hiroko will have to work to complete a 270 page book.

[redacted]

MATH EXPLORATIONS

Rates, Ratios and Proportions

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Multiplication of fractions is introduced in section 7.1 to be used as a tool to be used when working with proportions and measurement. The operation is developed through the linear and area models so that students are able to make sense of the concept that underlies the procedure. The intent of this section is as an introduction and not as a complete discussion on multiplication of fractions. Section 72 has been added as a way to introduce students to division of fractions through models and motivate making sense of the standard algorithms of fraction division. Section 7.3 uses fractions to represent rates and ratios. Units are emphasized as the concepts of rates and ratios are developed. Proportional reasoning and understanding how rates and ratios can be related to each other in an equation are the main concepts in section 7.4. In addition to observing patterns, several methods of solving proportions are presented including working with equivalent fractions, unit rates, and multiplication across equations.

Section 7.1 - Multiplying Fractions

Big Idea:
Multiplying fractions

Key Objectives:

- Use the linear and area models to understand multiplication of fractions.
- Develop a method for multiplying fractions.

Materials:
Number lines, Grid paper

Pedagogical/Orchestration:
Although the linear model for multiplying fractions is usable, the area model is far superior when both factors are fractions. Encourage your students to have a repertoire of models, and other mathematical structures they feel comfortable with, when they are introduced to new mathematical concepts. Some, like the number line and the area model, are adaptable to many situations through algebra and calculus.

TEKS:
6.1(B); 6.2(C,D); 6.11(A,B,C,D); 6.12(A,B); 6.13(A,B); New: 6.3(A, C, D)

Launch for Section 7.1:
Remind your students that we used the area model to represent the multiplication of whole numbers. Ask them to draw an area model to compute 4 times 6. Once they have the model drawn, ask them how they could shade it to represent $\frac{1}{2}$ times 6. Let students share their drawings with each other, and ask them to figure out what $\frac{1}{2}$ times 6 equals. Make sure that the connection is made that $\frac{1}{2} \times 6 = 3$, and that this is the same as saying $\frac{1}{2}$ of $6 = 3$. Tell students, "In today's lesson we will be using models such as this to multiply fractions." Objective: The students will play this game to practice their skills in finding reciprocals of given fractions and whole numbers.

This would be half a 4-mile jump (or lap) 2 miles long. Multiply $4 \cdot \frac{1}{2} = 2$. Point out that $\frac{1}{2}$ of 4 miles is 2 miles.

Notice that $4 \cdot \frac{1}{2} = \frac{1}{2} \cdot 4$. So, this could also be viewed as 4 hops of length $\frac{1}{2}$

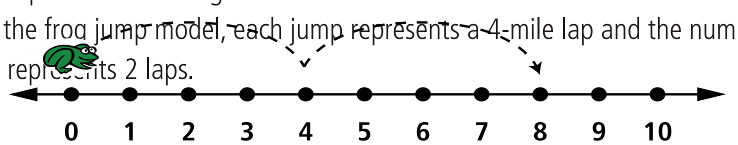
Write this product as $6 \cdot \frac{1}{3}$, which is equal to 2. One third of a jump 6 units long is 2 units long.

7 RATES, RATIOS AND PROPORTIONS

SECTION 7.1 MULTIPLYING FRACTIONS

We used the linear model to understand addition and subtraction of fractions and now use it to understand multiplication of fractions. However, as with multiplication of integers, the area model is also helpful in understanding multiplication of fractions. We will begin by first exploring the linear model.

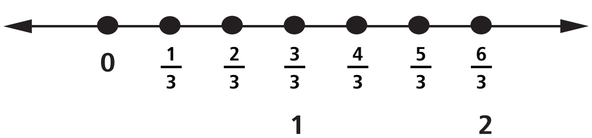
Mary trains at a cross-country facility that has a 4-mile trail. Usually runs one 4-mile lap but decides to go twice as far. She must run 8 miles because $4 \cdot 2 = 8$. Using the frog jump model, each jump represents a 4-mile lap and the number of jumps represents 2 laps.



If Mary runs half a lap, how far will she run? Show what this looks like on the number line. Now write Mary's distance as a multiplication problem.

PROBLEM 1

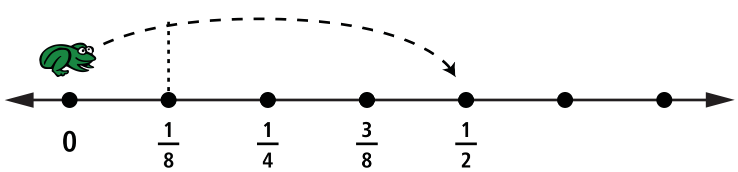
What would $\frac{1}{3}$ of 6 be? In other words, what is the product $\frac{1}{3} \cdot 6$? Remember that we multiply by using the first factor as the length of the jump and the second factor as the number of jumps. Illustrate the process on the number line below and represent the product in words.



Notice that, $\frac{1}{3} \bullet 6 = \frac{1}{3} \bullet \frac{6}{1} = \frac{6}{3} = 2$. In general, when we multiply a fraction $\frac{1}{n}$

Teachers, you might point out that "arithmetic" the adjective is not pronounced like "arithmetic" the noun. Encourage students to look up both words in the dictionary.

The arithmetic statement is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.



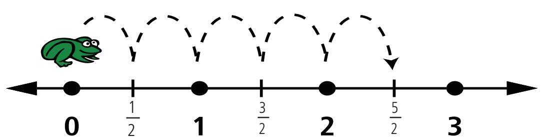
times a number m, or $\frac{1}{n}$ · m, then we have $\frac{m}{n}$. For example, in the area formula for a triangle, we can write both A = $\frac{1}{2}$(bh) or as A = $\frac{(bh)}{2}$. Notice that $\frac{1}{n}$ times m, the product $\frac{m}{n}$ is smaller than m because it is a fraction of m.

EXAMPLE 1

Multiply $\frac{1}{2} \cdot 5$. Write your answer in both improper and mixed fractions.

SOLUTION

Notice that when we multiply $\frac{1}{2} \cdot 5$, we have: $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = (\frac{1}{2})(5) = \frac{5}{2}$

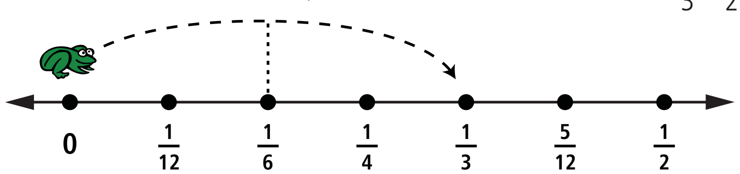


EXAMPLE 2

Jane has $\frac{1}{2}$ a yard of ribbon and needs to cut $\frac{1}{3}$ of its length. To do this, she finds out what $\frac{1}{3}$ of $\frac{1}{2}$ is. Use a number line to show how much ribbon she cuts.

SOLUTION

To find how much ribbon Jane cuts, use the linear model to calculate $\frac{1}{3} \cdot \frac{1}{2}$:



The arithmetic statement is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$. If each jump is $\frac{1}{3}$ yard and the frog makes a $\frac{1}{2}$ of a jump, it travels $\frac{1}{6}$ of a yard.

PROBLEM 2

Show what $\frac{1}{4}$ of the $\frac{1}{2}$ yard of ribbon is on an appropriate number line. Write the corresponding arithmetic statement for this and the amount in yards it equals.

The examples in Problem 2 are all a unit fraction times a unit fraction. The pattern is $\frac{1}{m} \bullet \frac{1}{n} = \frac{1}{mn}$. If your students do not formulate this rule yet, the next model will help them discover it.

This is $\frac{1}{3}$ of $\frac{1}{2}$. The area is $\frac{1}{6}$ because it is one out of six equal pieces of the whole.

EXPLORATION 1

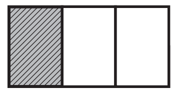
Have groups do one of the following problems and present their work to the rest of the class.

a. The answer is $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$.
b. Translate into the product: $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.

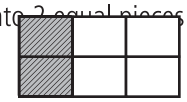
c. Students will probably see from the previous examples in parts a and b that the product of two unit fractions $\frac{1}{m}$ and $\frac{1}{n}$ is equal to $\frac{1}{mn}$.
d. Have students put up representations of these products. Reflect on whether they used the linear or area model. The area model is more transparent and easier to see. One of the best ways to show part (d) is to simply rotate the rectangle so the vertical and horizontal divisions are switched.

What pattern do you see in these problems?

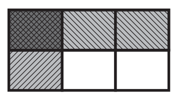
With the linear model it is important to be very exact when drawing the picture. To see the advantage of the area model, look at the problem above: $\frac{1}{3} \cdot \frac{1}{2}$. To begin the process using the area model, draw $\frac{1}{3}$ as a shaded part of the whole rectangle with area 1.



One way to represent $\frac{1}{2}$ of the shaded area is to cut the rectangle representing $\frac{1}{3}$ in half by cutting vertically. But this is the same process as the linear model. Instead, we cut the rectangle into 2 equal pieces by cutting horizontally.



One of the 2 pieces from the second cut is shaded to represent $\frac{1}{2}$ of the original $\frac{1}{3}$ rectangle. What part of the whole rectangle is the double-shaded area?



EXPLORATION 1

a. Translate $\frac{1}{2}$ of $\frac{1}{5}$ into a multiplication problem and draw the corresponding picture to find the product.

b. Translate $\frac{1}{4}$ of $\frac{1}{3}$ into a multiplication problem and draw the corresponding picture to find the product.

c. Make a conjecture about a rule for multiplying unit fractions:

$$\frac{1}{m} \cdot \frac{1}{n} = \underline{\qquad}$$

d. Explain why multiplication of unit fractions is commutative, that is

$$\frac{1}{m} \cdot \frac{1}{n} = \frac{1}{n} \cdot \frac{1}{m}$$

EXPLORATION 2

Have students explore these problems individually and in groups. Also make sure they know why *b* and *d* cannot be zero.

a. $\frac{1}{3}$ Use the area model of rectangles and have $\frac{2}{3}$ vertically and $\frac{1}{2}$ horizontally. Cross hatching should occur in $\frac{1}{3}$ of the rectangle.

b. $\frac{4}{15}$ Use the area model of rectangles and have $\frac{2}{5}$ vertically and $\frac{2}{3}$ horizontally. Cross hatching should occur in $\frac{4}{15}$ of the rectangle.

c. $\frac{12}{35}$ Use the area model of rectangles and have $\frac{3}{5}$ vertically and $\frac{4}{7}$ horizontally. Cross hatching should occur in $\frac{12}{35}$ of the rectangle.

d. Allow students to make a conjecture for the rule: $(\frac{a}{b})(\frac{c}{d}) = \frac{(ac)}{(bd)}$. Reflect with them about (1) what a denominator *bd* represents in the rectangle, and (2) what the numerator *ac* represents in the rectangle. Note that the first product can be simplified but parts (b) and (c) cannot be simplified.

EXPLORATION 3

Multiplying a positive number N by a positive fraction less than one results in product that is less than N. Multiplying a positive number N by a positive fraction greater than one results in product that is greater than N.

EXPLORATION 2

What is the product of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, where *a*, *b*, *c* and *d* are positive integers with *b* and *d* not zero? Use the area model to compute the following products.

a. What is $\frac{1}{2}$ of $\frac{2}{3}$? Use the area model to illustrate and find the answer.
b. What is $\frac{2}{5}$ of $\frac{2}{3}$? Use the area model to illustrate and find the answer.
c. What is $\frac{3}{5}$ of $\frac{4}{7}$? Use the area model to illustrate and find the answer.
d. Make a conjecture for the rule for multiplying two fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \_\_\_\_$$

Summarizing this pattern of multiplying fractions,

RULE 7.1: MULTIPLYING FRACTIONS
The product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where *b* and *d* are non-zero, is $$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \bullet c}{b \bullet d}$$

Multiplying fractions can be very useful. For example, you know how to multiply $\frac{3}{2} \cdot 6$ to obtain the product 9. When we multiply by more than 1 of a number, such as $\frac{3}{2}$ of 6, the resulting product is more than $\frac{2}{2}$ or 1 times the 6. In fact, the product is ½ more than 6 because $\frac{3}{2} = \frac{2}{2} + ½$. Multiplying ½ of 6 yields 3, so the result is 6 + 3 = 9, and a clear increase in the result. When you study ratios and rates in the next section, you will see that your skill in multiplying fractions will provide efficient ways to solve problems.

EXPLORATION 3

Multiply 24 by each fraction from the following 2 groups:
$\{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{5}{12}\}$ and $\{\frac{3}{2}, \frac{4}{3}, \frac{11}{6}, \frac{19}{12}\}$

a. What do you notice about these products?
b. What do the products from each group have in common?
c. Make a rule about multiplying a number N by a fraction.

EXERCISES:

1. It should be relatively easy to convince your students that simplifying before multiplication is easier and simpler than multiplying first.

a. 5	b. 9	c. 20	d. 6	e. 7
f. $22\frac{2}{3}$	g. 12	h. 3 / 32	i. 18	j $\frac{8}{15}$
k. 4 / 63	l. 11 / 24			

2. $\frac{3}{10}$ mile

3. 16 students with younger siblings

4. $\frac{9}{20}$ m^2

5. $\frac{3}{16}$

As you can see from Exploration 3, when two whole numbers, or factors, are multiplied, the product is always greater than or equal to the two factors. However, when a number is multiplied by a fraction less than 1, the product is less than the original number and the number decreased. For example, ¾ of 12 is $\frac{(3\times12)}{(4\times1)} = 9$, so 12 decreased to 9. When a number is multiplied by a fraction greater than 1, the resulting product is greater than the original number so the number increased. For example, $\frac{3}{2}$ of 12 is $\frac{(3\times12)}{(2\times1)} = 18$, and the product is greater than 12. In other words, 12 increased by a factor of $\frac{3}{2}$ is 18.

EXERCISES

1. Compute the following products and simplify if needed:

a. $\frac{1}{2}\cdot 10$	b. $27\cdot\frac{1}{3}$	c. $\frac{1}{5}\cdot 100$
d. $\frac{1}{3}\cdot 18$	e. $\frac{1}{7}\cdot 49$	f. $\frac{2}{3}\cdot 34$
g. $16\cdot\frac{3}{4}$	h. $\frac{3}{8}\cdot\frac{1}{4}$	i. $\frac{3}{5}\cdot 30$
j. $\frac{4}{5}\cdot\frac{2}{3}$	k. $\frac{2}{7}\cdot\frac{2}{9}$	l. $\frac{11}{12}\cdot\frac{2}{3}$

For each problem below, write an equation, solve the problem, and write your answer in a complete sentence.

2. Betty is riding her bike to the library which is $\frac{3}{5}$ of a mile from her house. She rides her bike $\frac{1}{2}$ of the distance before she gets a flat tire. What fraction of a mile did she bike before her bike broke down?

3. During a class discussion, Mr. Garza found out that $\frac{2}{3}$ of his students have younger brothers or sisters. If he has 24 students, how many students in his class have younger siblings?

4. Mr. Flores was setting up a rectangular window box to grow flowers under his kitchen window. It measures $\frac{3}{4}$ of a meter long and $\frac{3}{5}$ of a meter wide. What is the area of the rectangular box?

5. Madison had half of her birthday cake left over after her party. If she gives $\frac{3}{8}$ of the left over cake to her friend, how much cake will her friend get?

6. a. $\frac{3}{20}$

   b. $\frac{2}{5}$

   c. 120 choose soft drinks (80 prefer sugar free)
   50 choose juice
   30 water

7. 5 students are left handed

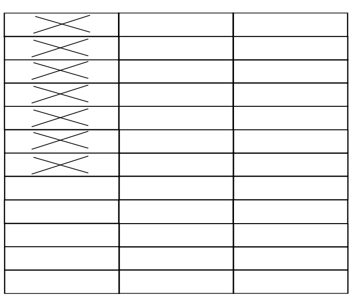
8. $\frac{1}{2}$ ft^2

9. 480 fish were added.

Spiral Review (6.1 B): 10 c. 11. 2•2•3•5•11

Ingenuity:

12. We can solve this problem using the area model for fractions. We know that 1/3 of the cookies were sugar cookies and the rest were chocolate chip cookies, so let's divide a rectangle into three thirds, with the first third representing sugar cookies and the other two representing chocolate chip cookies. We know that 7/12 of the first third have been eaten, so we will divide the entire rectangle into twelfths and mark 7/12 of the first third as having been eaten.



We know that only 1/4 of the cookies are left, so 3/4 of the cookies have been eaten. This is equivalent to 27 thirty-sixths in the above picture. We have already marked 7 thirty-sixths as having been eaten, so we need to mark 20 more. We can do this by marking the top 10 thirty-sixths in each of the second and third columns. Note that this is 10/12 of the chocolate chip cookies, not 10/36 or 20/36 (though it is 20/36 of the whole batch of cookies). So the answer is 10/12, or 5/6.

6. In a recent school survey, $\frac{3}{5}$ of the students reported preferring soft drinks, $\frac{1}{4}$ preferred fruit juice, and the rest preferred drinking water.

   a. What fraction of the students preferred drinking water?

   b. Of those students choosing soft drinks, $\frac{2}{3}$ said they liked sugar-free soft drinks. What fraction of the students preferred sugar-free soft drinks?

   c. If the survey was conducted on 200 students, how many students would have selected each type of drink?

7. While conducting an experiment, Mrs. Ayala found that $\frac{4}{5}$ of her students were right handed and $\frac{1}{6}$ of the students were left handed. One student could use either hand to do many things. If there are 30 students in her class, how many of them are left handed?

8. Raymond wants to cover a window on a motorized child's car with window tinting film. The window measures $\frac{2}{3}$ of a foot in length and $\frac{3}{4}$ of a foot in height. How much tinting film will he use?

9. A pond at the fish hatchery contained 800 fish. They recently added $\frac{3}{5}$ the amount of the original number of fish to the pond. How many fish were added into the pond? What is the total number of fish in the pond now?

Spiral Review

10. Shirley has $\frac{4}{5}$ of her homework finished for tomorrow. Which of the following are not equivalent to $\frac{4}{5}$?

    a. $\frac{8}{10}$ b. 0.80 c. $\frac{12}{18}$ d. 0.8

11. What is the prime factorization of 660?

12. **Ingenuity:**

    Randall made a large batch of cookies. $\frac{1}{3}$ of the cookies were sugar cookies, while the other $\frac{2}{3}$ were chocolate chip cookies. At a party, Randall's guests ate $\frac{7}{12}$ of the sugar cookies, along with most of the chocolate chip cookies. After the party, Randall noticed that he had only $\frac{1}{4}$ of his cookies left. What fraction of the chocolate chip cookies got eaten?

Investigation

13. TE: The purpose of this Investigation is to foreshadow the skill of dividing fractions that have the same denominator.

 (a) we know that 7/2 = 7 x 1/2, so Rikki can make 7 pieces.
 (b) Suppose we were to divide the salt into "handfuls" of 1/4 pound each. Then Umberto has 15 handfuls, and each bag is to contain 3 handfuls. So Umberto can make 15 [divided by] 3 = 5 bags.
 (c) This time, it is helpful to convert the mixed fractions into improper fractions. We know that 6 1/4 = 25/4, and 1 1/4 = 5/4. If we think of the candy bar as being made of 1/4-inch "segments", then Bill has a total of 25 segments, and each piece is to contain 5 segments. So Bill can make a total of 25 [divided by] 5 = 5 pieces.

13. **Investigation:**

Solve each of the following problems. You may find it helpful to draw pictures.

a. Rikki has a bamboo pole of length $\frac{7}{2}$ feet that she wants to divide into $\frac{1}{2}$ foot pieces. How many pieces can she make?

b. Umberto has $\frac{15}{4}$ pounds of salt, and he wants to divide the salt into bags that contain $\frac{3}{4}$ pounds of salt each. How many bags can Umberto fill?

c. Bill has $6\frac{1}{4}$-inch candy bar, which he divides into $1\frac{1}{4}$-inch pieces. How many pieces does he make?

Section 7.2 - Division of Fractions

Big Idea:
Dividing fractions

Key Objectives:
- Use linear and area models to understand the meaning of division of fractions
- Understand division by a fraction and division of a fraction.
- Develop a method for dividing fractions
- Understand the relationship between fractions and division
- Use fractions in daily situations
- Find the reciprocal or multiplicative inverse of a non-zero integer.

Materials:
Number lines, Grid paper, ribbon and scissors for Launch

Pedagogical/Orchestration:
This section introduces division of fractions in several ways with models to represent different situations. A connection is made between multiplication and its relation to division. For example, in 4 divided by ½, we ask what times ½ is equal to 4? We also recall that division of whole numbers could be thought of in several ways. The measurement model is one way to represent 4 divided by ½ when asking, "How many halves are there in 4?" This recalls a problem such as 12 divided by 3 can be thought of as "How many groups of 3 are there in 12?"

Activity:
"Reciprocal Concentration"   Launch Ribbon Cutting activity connects to measurement.

Exercises:
Encourage students to draw models when requested in the exercises or even if not, to better understand what the resulting quotient means in the problem.

Vocabulary:
reciprocal, multiplicative inverse.

TEKS:
7.2(A)(B)(D)(E)(F)(G);   7.5(A);   7.13(C); 8.2(A)(B)(C)(D)

New: 6.2(F);   6.3(B,G)

Launch for Section 7.2:

Tell students, "The big idea for today will be dividing with fractions. Let's go back to the idea of division with whole numbers. What is one way to think about 20 divided by 5?" Students will probably just give the answer, 4, but remind them that performing the operation 20 ÷ 5 is a way to answer questions of the type, "How many groups of 5 are there in 20?" Tell your students, "Now let's relate this idea to fractions. If you are asked to work a problem such as 4 ÷ ½, how can you relate this problem to the meaning of division? In other words, how many half units are in 4? We will do an activity that will help us make sense of this." Put the students in their groups and hand each group a length of ribbon 4 feet long. Tell them it takes a certain length of ribbon to make a bracelet or necklace. Assign each group a different fractional length so that the ribbon is cut into pieces of that size. For instance, the groups could be assigned ½ foot lengths, 1/3 foot lengths, ¼ foot lengths and even ¾ foot lengths and so on. The question is how many bracelets or necklaces can they make of given length form the 4-foot ribbon they were given. Also ask them what division problem they are modeling. For instance, the group assigned ½ foot lengths I modeling 4 ÷ ½. Have a class discussion on the results and write the results for each group on the board. Inform your students, "As you work through today's lesson, think about the meaning of division to help you come up with methods for dividing fractions, and make sure you notice patterns that will make the process easier."

Reciprocal Concentration

Objective: The students will play this game to practice their skills of finding reciprocals of given fractions and whole numbers.

Materials:
Deck of "Reciprocal Concentration Cards" (One deck for each group of 3 to 4 students)

Activity Instructions:
Copy the "Reciprocal Concentration Cards", either on cardstock, or on index cards if you choose to write them by hand. Have each group shuffle their deck very well before playing the game.

Once all cards are shuffled well, each group will arrange the cards FACE DOWN in a five by five square array. Each deck will contain 12 pairs of reciprocals and one card that is an automatic loser.

Players will take turns picking two cards from the square array, trying to find reciprocal pairs. If the two cards chosen are indeed a reciprocal pair, the player will pick up the cards and keep them. If not, the cards are flipped back upside down in their original spot, and it is the next player's turn. If one of the two cards chosen is a "Loser" card, the player loses their turn and the next player gets to choose.

The game ends when all cards have been flipped over. The player with the most reciprocal pairs is the winner of the game. If time permits, the group can reshuffle the cards and play again.

Reciprocal Concentration Cards

$\frac{2}{3}$	$\frac{3}{2}$	$\frac{4}{7}$
$\frac{7}{4}$	$\frac{1}{8}$	8
$\frac{8}{9}$	$\frac{9}{8}$	$\frac{5}{6}$
$\frac{6}{5}$	$\frac{3}{10}$	$\frac{10}{3}$
$\frac{12}{23}$	$\frac{23}{12}$	$\frac{1}{6}$

6	$\frac{4}{9}$	$\frac{9}{4}$
$\frac{7}{15}$	$\frac{15}{7}$	$\frac{2}{19}$
$\frac{19}{2}$	$\frac{8}{7}$	$\frac{7}{8}$
	Loser!	

EXPLORATION 1

It is important for students to make sense of what division of fraction means. The number line gives us one way to give it meaning. Notice here we start with a whole number dividend and a friendly fraction divisor.

PROBLEM 1

$4 \div \frac{1}{2} = 8$ bags

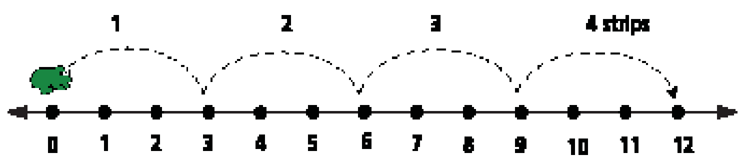
SECTION 7.2 DIVISION OF FRACTIONS

In order to better understand division of fractions, you can use linear and area models to represent the process of division just as with whole numbers. Try the following exploration to see how you can solve and explain the process that you used. You may wish to use words as well as pictures to explain your thinking.

EXPLORATION 1

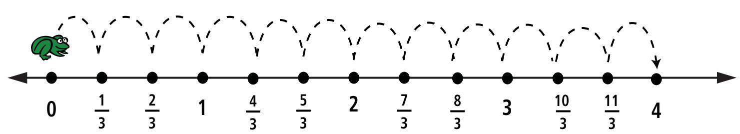
Melinda wants to cut a 4 yard fabric into $\frac{1}{3}$ yard strips. How many strips will she have? Explain how you reached your conclusion.

Just as with a whole number division problem, such as cutting a 12 yard fabric into 3 yard strips, you can use a linear model to show that in this case there are 4 strips possible.



That is, $12 \div 3 = 4$

You can also use a linear model to represent the problem in exploration 1.



We can see that $4 \div \frac{1}{3} = 12$.

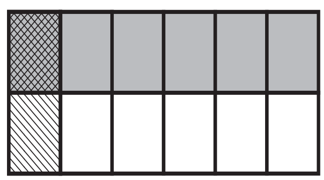
PROBLEM 1

Liz has 4 pounds of jellybeans. She plans to make little party bags containing $\frac{1}{2}$ pound of jellybeans. How many party bags can she make?

This is an important observation about the division and its related multiplication problem. This is still at an observational and exploratory level leading soon to $a \div b = a \cdot \frac{1}{b} = \frac{a}{b}$. This connects back to noting that $\frac{m}{n} = m \div n$ in Chapter 5.

PROBLEM 2

$\frac{1}{2} \div 6 = \frac{1}{12}$

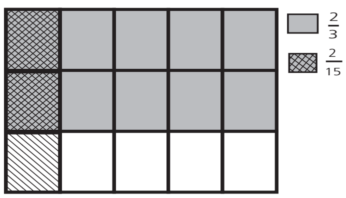


EXPLORATION 2

Chuck has two-thirds of a pan of brownies and shares it evenly among his 5 friends. What fraction of the pan of brownies does each friend receive? Explain how you reached your conclusion.

One way to think about the problem in exploration 2 is as a division of $\frac{2}{3} \div 5$. Another approach is to think of each of Chuck's 5 friends receiving $\frac{1}{5}$ of the brownies or as a multiplication problem, $\frac{2}{3} \cdot \frac{1}{5}$.

An area model of this problem can be represented as:



$\frac{2}{3} \div 5 = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{15}$ or the darker shaded region.

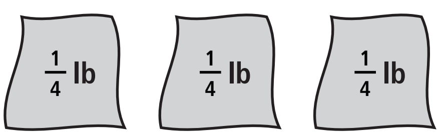
PROBLEM 2

Barbara has $\frac{1}{2}$ of a pan of brownies and shares it evenly among 6 friends. What part of the pan of brownies does each friend receive? Use the area model to show and explain how you reached your conclusion.

Look at a similar problem but with fractional quantities: How many $\frac{1}{4}$ pound bags does it take to pack $\frac{3}{4}$ pounds of sand? In other words, what is $\frac{3}{4} \div \frac{1}{4}$?

28 cups

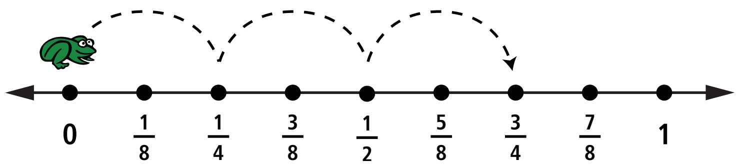
Using a repeated subtraction model, make 3 equal parts. With the first $\frac{1}{4}$ pound bag, $\frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ pounds are left. The second $\frac{1}{4}$ pound leaves $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so the third $\frac{1}{4}$ pound bag leaves no sand.



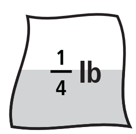
3 represents the number of $\frac{1}{4}$ - pound bags in $\frac{3}{4}$ pounds.

Writing this as a division problem, $\frac{3}{4} \div \frac{1}{4} = 3$. At first, it might be surprising that when dividing two fractions, the answer is an integer, especially when the integer is large compared to the fractions. What does 3 represent in this case?

Another way to think about this problem uses the missing factor method. What number times $\frac{1}{4}$ equals $\frac{3}{4}$? Starting at 0 on the number line, 3 jumps of length $\frac{1}{4}$ equals $\frac{3}{4}$. So, $\frac{3}{4} \div \frac{1}{4} = 3$.



Notice in the earlier example with bags of sand, the quantity of sand exceeded the bag size. A $\frac{3}{4}$ pound bag was being separated into smaller $\frac{1}{4}$ pound bags. The number of bags was $\frac{3}{4} \div \frac{1}{4} = 3$ bags. What if the initial quantity is less than the bag size, like having $\frac{1}{8}$ pound of sand and a bag that holds $\frac{1}{4}$ of a pound? What is $\frac{1}{8} \div \frac{1}{4}$?



It is impossible to use a "repeated-subtraction" model, because there is no way to fill even one $\frac{1}{4}$ pound bag with only $\frac{1}{8}$ pound of sand. You can see that with the $\frac{1}{8}$ pound, only $\frac{1}{2}$ of the $\frac{1}{4}$ pound bag is filled. Therefore, $\frac{1}{8} \div \frac{1}{4} = \frac{1}{2}$.

Using the relationship between fractions and division, that $m \div n$ is the same as $\frac{m}{n}$, rewrite $\frac{1}{8} \div \frac{1}{4}$ as a big fraction, that is $\frac{\frac{1}{8}}{\frac{1}{4}}$. This looks pretty complicated, but luckily it can be simplified.

Writing a division problem such as $4 \div 8$ as a fraction $\frac{4}{8}$ does not appear complicated. However, working with a division problem with two fractions such as

$\frac{1}{8} \div \frac{1}{4}$ can seem more difficult.

Expect them to notice that $\frac{a}{b} \cdot \frac{b}{a} = 1$ provided that each of a and b is not zero. We call $\frac{b}{a}$ the "reciprocal" or "multiplicative inverse" of a/b because their product is 1. Notice that Rule 7.1 is a special case of Rule 7.2.

Recall that simplifying a fraction requires rewriting the fraction as an equivalent fraction. But instead of factoring the numerator and denominator and applying the equivalent fraction property, we create an equivalent fraction. Multiply by 1 but in an appropriate form to both the numerator and denominator. This process will convert the denominator to a very friendly product. Before we explore this process, remember that a non-zero number n divided by n is 1. We also saw from exploration 2 that n divided by n is the same as multiplying n by $\frac{1}{n}$. We have the following result:

$n \div n = 1$ but we know that $n \div n = n \cdot \frac{1}{n}$ so $n \cdot \frac{1}{n} = 1$. We have a special name for $\frac{1}{n}$

RULE 7.1: MULTIPLICATIVE INVERSE
If a is any non-zero number, then the **multiplicative inverse** or **reciprocal** of a is the unit fraction $\frac{1}{a}$. The product $\frac{1}{a} \cdot a$ is $\frac{1}{a} \cdot a = a \cdot \frac{1}{a} = 1$

EXPLORATION 3

Compute each of the following products:

a. $\frac{2}{3} \cdot \frac{3}{2}$ b. $\frac{4}{5} \cdot \frac{5}{4}$ c. $\frac{4}{7} \cdot \frac{7}{4}$ d. $\frac{4}{9} \cdot \frac{9}{5}$

What do you notice?

From the discussion on Exploration 2, we have the following rule:

RULE 7.2: MULTIPLICATIVE IDENTITY
If each of a and b in not zero, then the product $\frac{a}{b} \cdot \frac{b}{a} = 1$. 1 is called the **multiplicative identity**.

Note: In general, if x is a number and $x \neq 0$, then there is a number, denoted as $\frac{1}{x}$, so that the product of x and $\frac{1}{x}$ is 1, that is, $x \cdot \frac{1}{x} = 1$. We call $\frac{1}{x}$ as the reciprocal or multiplicative inverse of x.

PROBLEM 3

a. 2 b. $\frac{5}{3}$ c. $\frac{7}{8}$ d. 1 e. $\frac{1}{14}$

PROBLEM 4

a. $2 \div \frac{1}{4} = \frac{2}{\frac{1}{4}} \cdot \frac{4}{4} = \frac{8}{1} = 8$

b. $3 \div \frac{1}{4} = \frac{3}{\frac{1}{4}} \cdot \frac{4}{4} = \frac{12}{1} = 12$

c. $\frac{1}{2} \div \frac{1}{4} = \frac{\frac{1}{2}}{\frac{1}{4}} \cdot \frac{4}{4} = \frac{\frac{4}{2}}{1} = 2$

d. $\frac{1}{4} \div \frac{1}{2} = \frac{\frac{1}{4}}{\frac{1}{2}} \cdot \frac{2}{2} = \frac{\frac{2}{4}}{1} = \frac{1}{2}$

PROBLEM 5

$$\frac{5}{6} \div \frac{1}{6} = \frac{\frac{5}{6}}{\frac{1}{6}} \cdot \frac{6}{6} = \frac{\frac{30}{6}}{1} = 5$$

We return to the division $\frac{1}{8} \div \frac{1}{4} = \frac{\frac{1}{8}}{\frac{1}{4}}$ expressed as a fraction. If we multiply the numerator and denominator by the reciprocal of $\frac{1}{4}$, namely by 4, then

$$\frac{1}{8} \div \frac{1}{4} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{\frac{1}{8} \cdot \frac{4}{1}}{\frac{1}{4} \cdot \frac{4}{1}} = \frac{\frac{4}{8}}{\frac{4}{4}} = \frac{\frac{4}{8}}{1} = \frac{4}{8} = \frac{1}{2}$$

In general, when the denominator of a fraction is a fraction, multiplying both the numerator and denominator by the reciprocal of the denominator produces a simpler fraction. Another approach to simplify complicated fractions uses the pattern that $m \div n = \frac{m}{n} = m \cdot \frac{1}{n}$. Using this pattern, rewrite $\frac{\frac{1}{8}}{\frac{1}{4}}$ as $\frac{1}{8} \cdot \frac{4}{1}$, because the reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$. Then multiply to find the answer: $\frac{1}{8} \cdot \frac{4}{1} = \frac{4}{8} = \frac{1}{2}$.

PROBLEM 3

Find the multiplicative inverse or reciprocal of the following numbers.

a. $\frac{1}{2}$ b. $\frac{3}{5}$ c. $\frac{8}{7}$ d. 1 e. 14

PROBLEM 4

Compute the following division of fractions using the stacking method from above.

a. $2 \div \frac{1}{4}$ b. $3 \div \frac{1}{4}$ c. $\frac{1}{2} \div \frac{1}{4}$ d. $\frac{1}{4} \div \frac{1}{2}$

PROBLEM 5

Valerie's bird feeder holds $\frac{5}{6}$ of a cup of birdseed. Valerie is filling the bird feeder with a scoop that holds $\frac{1}{6}$ of a cup. How many scoops of birdseed will Valerie put into the feeder? Use the numerical technique from above. Write your answer in simplest form.

EXAMPLE 1

Compute the quotient $\frac{\frac{2}{5}}{\frac{3}{4}}$

Have the students be aware that the multiplication of the dividend is by the reciprocal of the divisor.

EXERCISES

1. a. 8 b. 12 c. $\frac{4}{3} = 1\frac{1}{3}$ d. 2

   e. 18 f. $\frac{1}{18}$ g. $1\frac{1}{2}$ h. $1\frac{1}{3}$

EXAMPLE 1

SOLUTION

We use Rule 7.2 to pick a fraction and then multiply it by the numerator and the denominator to obtain an equivalent fraction so that the denominator simplifies to be the value of 1. This transforms a division of fractions calculation into a multiplication of fractions calculation.

$$\frac{\frac{2}{5}}{\frac{3}{7}} = \frac{\frac{2}{5}\cdot\frac{7}{3}}{\frac{3}{7}\cdot\frac{7}{3}} = \frac{\frac{2}{5}\cdot\frac{7}{3}}{1} = \frac{2}{5}\cdot\frac{7}{3} = \frac{14}{15}$$

In summary, when dividing by a fraction, or simplifying a fraction whose denominator is a fraction, use one of the two following techniques:

Method 1: Write the division problem as a fraction and multiply the numerator and denominator of this fraction by the reciprocal of the denominator. This results in an equivalent fraction with denominator 1:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\cdot\frac{d}{c}}{\frac{c}{d}\cdot\frac{d}{c}} = \frac{\frac{a}{b}\cdot\frac{d}{c}}{1} = \frac{a}{b}\cdot\frac{d}{c}$$

Method 2: Or, because division is equivalent to multiplication by the reciprocal, rewrite the division as multiplication:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c}$$

Either approach will find the quotient. The major point is division transforms into multiplication, not magically, but from a well-motivated reason using a deep understanding of fractions and how they work.

Remember that in $m \div n$, n cannot be zero.

EXERCISES

1. Use a visual model to compute the following quotients. Check your work by using the product of the quotient and the divisor.

   a. $2 \div \frac{1}{4}$ b. $3 \div \frac{1}{4}$ c. $1 \div \frac{3}{4}$ d. $\frac{1}{3} \div \frac{1}{6}$

   e. $3 \div \frac{1}{6}$ f. $\frac{1}{6} \div 3$ g. $\frac{3}{5} \div \frac{2}{5}$ h. $\frac{2}{3} \div \frac{1}{2}$

2. a. $\frac{3}{6} = \frac{1}{2}$ b. $\frac{4}{6} = \frac{2}{3}$ c. $\frac{4}{5}$ d. $\frac{3}{2} = 1\frac{1}{2}$

3. a. 1 b. 1 c. 1 d. 7

4. $14 \div \frac{2}{5} = 35$ pieces

5. $\frac{7}{8} \div \frac{1}{4} = 3\frac{1}{2}$ bags

6. $D = RT$ or $T = D \div R$. So, $\frac{1}{8} \div \frac{3}{4} = \frac{1}{6}$ of an hour.

7. $1\frac{7}{8}$ hours

9. $\frac{2}{3}$

10. 30 cranes

2. Compute the following quotients by using the process developed in Method 1. Check your answer by using Method 2. Simplify your answer if needed.

   a. $\dfrac{\frac{1}{6}}{\frac{1}{3}}$ b. $\dfrac{\frac{1}{2}}{\frac{3}{4}}$ c. $\dfrac{\frac{2}{5}}{\frac{1}{2}}$ d. $\dfrac{\frac{3}{10}}{\frac{1}{5}}$

3. Compute the following quotients by using either Method 1 or Method 2:

   a. $\frac{2}{3} \div \frac{2}{3}$ b. [illegible] c. $\frac{3}{5} \div \frac{3}{5}$ d. $\frac{1}{[illegible]}$

4. Hugo has 14 meters of wire on a roll. He needs to cut the wire into $\frac{2}{5}$ meters lengths. If Hugo cuts the whole roll of wire, many pieces of wire will Hugo have.

5. $\frac{7}{8}$ of a pound of pecans is packaged into $\frac{1}{4}$ pound bags. How many bags of pecans can be packaged?

6. Porky the dachshund walks at a rate of $\frac{3}{4}$ mph. What fraction of an hour does it take Porky to walk $\frac{1}{8}$ of a mile?

7. Oliver walks at a speed of $3\frac{1}{5}$ mph. How long will it take him to walk 6 miles?

8. August biked 10 miles in $\frac{1}{4}$ of an hour. How far can August bike in 1 hour?

9. A recipe calls for $\frac{3}{4}$ cup of flour and you have $\frac{1}{2}$ cup of flour. What part of the recipe can you make?

10. Jacob can fold an origami crane in $\frac{1}{12}$ of an hour. How many origami cranes can Jacob fold in $2\frac{1}{2}$ hours?

11. Stephanie is decorating costumes with $\frac{3}{5}$ yard of ribbon for each costume. How many costumes can Stephanie decorate if she has 3 yards of ribbon?

11. $3 \div \frac{3}{5} = 5$ costumes

Ingenuity

12.

B	B	G
B	B - 4	G + 4

$4 \div \frac{1}{6} = 24$ total students

Investigation

13.

a. E

b. B

c. B

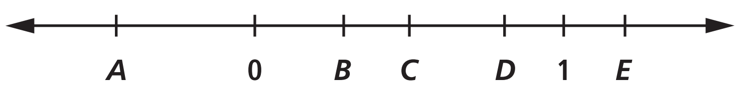
d. F

12. **Ingenuity:**

Two-thirds of Ms. Tate's sixth-grade students are boys. To make the number of boys and girls equal, 4 boys go the other sixth-grade class, and 4 girls come from that class into Ms. Tate's class. Now one-half of her students are girls. How many students are in Ms. Tate's class?

13. **Investigation:**

Consider the following number line.



a. Which point best represents the sum of the fractions B and D? Explain why you think so.

b. Which point best represents the difference, D – C? Explain why you think so.

c. Which point best represents the product of the fractions C and D? Explain why you think so.

d. Which point best represents the quotient D ÷ C? Explain why you think so.

REVIEW EXERCISES

1. a. $\frac{29}{10}$ b. $\frac{24}{35}$ c. 4 d. $\frac{8}{15}$
2. a. $\frac{423}{200}$ b. $\frac{16}{125}$ c. $\frac{22}{45}$ d. $\frac{35}{12}$
3. a. $\frac{5}{24}$ b. $\frac{1}{24}$ c. $\frac{3}{2}$ d. $\frac{1}{96}$
4. a. $\frac{7}{6}$ b. $\frac{19}{48}$ c. $\frac{15}{8}$ d. $\frac{3}{10}$
5. a. $1\frac{3}{5}$ b. $3\frac{1}{8}$ c. $\frac{221}{21}$ d. $\frac{51}{91}$
6. a. $\frac{103}{12}$ b. 2 c. 6 d. $1\frac{19}{24}$

REVIEW EXERCISES

Perform the indicated operations:

1. a. $\frac{2}{5} + \frac{5}{2}$ b. $\frac{7}{5} - \frac{5}{7}$ c. $\frac{18}{5} \cdot \frac{10}{9}$ d. $\frac{4}{9} \div \frac{5}{6}$
2. a. $\frac{6}{25} + \frac{15}{8}$ b. $\frac{6}{25} \div \frac{15}{8}$ c. $\frac{8}{5} - \frac{10}{9}$ d. $\frac{14}{9} \cdot \frac{15}{8}$
3. a. $\frac{1}{8} + \frac{1}{12}$ b. $\frac{1}{8} - \frac{1}{12}$ c. $\frac{1}{8} \div \frac{1}{12}$ d. $\frac{1}{8} \cdot \frac{1}{12}$
4. a. $\frac{7}{15} + \frac{7}{10}$ b. $\frac{17}{24} - \frac{5}{16}$ c. $\frac{21}{20} \cdot \frac{25}{14}$ d. $\frac{21}{16} \div \frac{35}{8}$
5. a. $4\frac{2}{5} + 2\frac{4}{5}$ b. $7\frac{3}{4} - 4\frac{5}{8}$ c. $(2\frac{3}{7})(4\frac{1}{3})$ d. $(2\frac{3}{7}) \div (4\frac{1}{3})$
6. a. $2\frac{5}{6} + 5\frac{3}{4}$ b. $6\frac{2}{5} \div 3\frac{1}{5}$ c. $(1\frac{3}{7})(4\frac{1}{5})$ d. $4\frac{5}{12} - 2\frac{5}{8}$

Section 7.3 - Rates and Ratios

Big Idea:

Understanding and using rates and ratios

Key Objectives:

- Understand the definition of a rate.
- Understand the definition of a ratio.
- Understand the difference between ratios and rates.
- Represent rates and find distances using the distance/rate formula.
- Note the importance of units in representing ratios and rates.
- Use ratios and rates in problem settings.
- Notice scale factor and unit rates.

Materials:

Calculators suggested for Exercises 6 and 7

Pedagogical/Orchestration:

This section explains rates and ratios and the difference between them. It also provides rich and numerous ways to solve problems involving ratios and rates.

Activity:

"Writing Ratios"

Exercises:

Exercises 6, 7, 9, and 10 connect to measurement conversions.

Vocabulary:

rate, unit rate, ratio, (On CD: customary system, metric system)

TEKS:

6.2(C); 6.3(B)(C); 6.4(A) New: 6.3(A,B,C,G); 6.4(B,C,D,E); 6.5(A, C)

Launch for Section 7.3:

Write the following on the board and ask your students what they have in common:

10 miles/hour $70.00 per 2 days $\frac{240 \text{ miles}}{8 \text{ gallons}}$ 6 candies for every 3 students

Let students discuss their thoughts, and if need be rewrite all of the rates using the fraction bar, such as $\frac{10 \text{ miles}}{1 \text{ hour}}$. Let students know that these are all special ratios called rates. The two values being compared in a rate have different units and describe how one value changes in relation to another. Tell students, "In this section, you will be learning numerous ways to solve problems involving ratios and rates."

Continue the launch by gathering classroom data that include:

1. How many students in the class?
2. How many students wear glasses?
3. How many students have brothers or sisters?
4. How many students have pets?
5. How many students play a musical instrument?
6. How many people play chess?

Or any other data that may be appropriate for your class.

Writing Ratios Activity

Practice writing ratios in three different forms by answering the comparisons below using the data set provided. Remember to simplify the fraction form of each ratio as needed.

100 17 43 0

15 1 36 50

50 45 8 16

Comparison	"to"	":"	Fraction Form
A. Even numbers to odd numbers			
B. Prime numbers to composite numbers			
C. One-digit numbers to two-digit numbers			
D. Multiples of 3 to numbers divisible by 2			
E. Square numbers to all numbers			

A response to the question of interpreting the simplified fraction as ratio is to think of sampling the population. In that sample of 10 students you would expect 7 students to live within 2 miles of campus.

Emphasize that there are not suddenly just 7 students who live within 2 miles and 3 who don't. That is, these numbers are part of a ratio, rather than absolute numbers. The equivalent ratio follows from the equivalent fractions

$$\frac{280 \text{ students}}{400 \text{ students}} = \frac{7}{10}$$

SECTION 7.3 RATES AND RATIOS

Fractions are often used to compare quantities. For example, Miller Middle School has 400 students, and 280 of them live within 2 miles of the campus. Simplifying, $\frac{280 \text{ students}}{400 \text{ students}} = \frac{7}{10}$. Notice that both units are "students" and the fraction simplifies. How can you interpret the meaning of the simplified fraction?

In Miller Middle School, 7 out of every 10 students live within 2 miles of the school. Converting the fraction $\frac{7}{10}$ into a percent, 70% of the students live within 2 miles. The fractional form of this comparison is called a ratio. A **ratio** is a division comparison of two quantities with or without the same units.

Ratios can be written in the form of first one quantity, then a colon followed by a second quantity. Ratios can also be written using the word "to" in place of the colon as well as in fraction form. In this example, the unit of measure is the number of students.

Because there are 280 students who live within 2 miles, write

280 students : _______

Compared to these 280 students within 2 miles, there are 400 total students. So the ratio of students who live within 2 miles to total number of students is

280 students who live within 2 miles : 400 total students

Notice that there is more than one way to write a ratio. A ratio that relates quantity x to quantity y, can be written as:

1. x to y
2. x : y
3. $\frac{x}{y}$

Ignoring for a moment the units and using only the numbers, write the ratio as 280 : 400. Just as with fractions, we can simplify this to 7 : 10. Always remember what kinds of things are being compared. In this problem, what does the ratio 7 : 3 describe?

It is useful to observe that ratios may relate part to whole and in other instances ratios may relate part to part as in the example above with 7 : 10 and 7 : 3.

The word rate is usually used to distinguish an even more specific kind of ratio: one involving change, such as $\frac{\text{dollars}}{\text{hour}}$, $\frac{\text{miles}}{\text{hour}}$ and $\frac{\text{miles}}{\text{gallon}}$. On the other hand, suppose you wish to compare the genders in a class with 12 boys and 18 girls. Comparing the number of girls to boys in this class gives the ratio $\frac{\text{18 girls}}{\text{12 boys}}$. Simplifying this fraction, the ratio of girls to boys in the class is $\frac{3}{2}$. Although it is mathematically correct to say the rate of girls per boy in the class is $\frac{3}{2}$, we rarely use the word rate in that comparison because the units are different and the ratio does not involve change.

Note that 30/5 is being simplified to 6 dollar /1 hour and then discuss the 6 (dollar/hour) as another way to write this where the number 6 has a unit rate of (dollars/hour).

Going from general rates to unit rates can be very useful. Because the unit rate is in reference to one unit, then this can be used as a factor for a multiple or for parts of a unit. Have students note that some unit rates could be fractional. For example if a person buys 3 melons for $2 then we can view this as 3 melons/$2 or as 3/2 (melons/dollar). Other rate units can be: miles per hour (mph or mi/hr), feet per second, miles per gallon, births per year, number of people per square mile, etc.

PROBLEM 1

Diana sent 300 messages ÷ 3.5 hrs. = 60 messages each hour.

EXPLORATION 1

There are 6 possible rates.

(150 miles)/(3 hours) = 50 miles/hour = 50 miles per hour = number of miles driven in 1 hour

(3 hours)/(150 miles) = 1/50 hour/mi = a little more than 1 minute per mile = how long it takes to drive 1 mile

(150 mi.)/(5 gal.) = 30 mi/gal = 30 miles per gallon = number of miles driven on 1 gallon of gasoline

(5 gal)/(150 mi.) = 1/30 gal/mi = about 0.03 gallons per miles = number of gallons it took to drive 1 mile

(3 hours)/(5 gal.) = 3/5 hr/gal = more than half an hour per gallon = how long he drove on 1 gallon of gasoline

(5 gal)/(3 hours) = 5/3 gal/hr = almost 2 gallons per hour = no. of gallons used while driving 1 hour

Rates are special ratios that compare different units. Suppose you earn 30 dollars for doing 5 hours of yard work and mowing the lawn. You know that $30 \div 5 = 6$ indicates how much money you earned per hour. Using fractions, this calculation looks like $\frac{30}{5} = 6$. However, it is usually helpful to write this problem using the units that describe each quantity. So the calculation becomes:

$$\frac{30 \text{ dollars}}{5 \text{ hours}} = \frac{6 \text{ dollars}}{1 \text{ hour}} = 6\,\frac{\text{dollars}}{\text{hour}}.$$

You read "$6\ \frac{\text{dollars}}{\text{hour}}$" as "six dollars per hour." The answer explains exactly how many dollars you earned each hour. This quantity is an example of a rate. A **rate** is defined as a division comparison between two quantities, usually with two different units, like dollars and hours. What are some other rates that you have worked with or know about?

The simplified fractional answer in the example is called a **unit rate** because it represents a number or quantity per 1 unit, or hour in this case. The units may be written in fractional form, like $\frac{\text{dollars}}{\text{hour}}$, $\frac{\text{miles}}{\text{hour}}$ or $\frac{\text{miles}}{\text{gallon}}$, and is usually read as "dollars per hour", "miles per hour", or "miles per gallon".

PROBLEM 1

Diana sent 300 text messages in 5 hours. Find the number of messages Diana sent each hour.

EXPLORATION 1

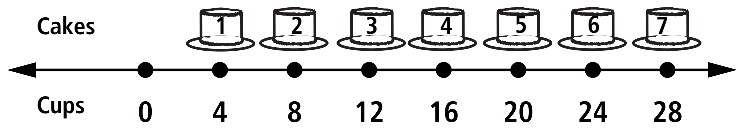
Juan drove 150 miles in 3 hours and used 5 gallons of gasoline. Make as many rates using these quantities and their units as possible. Explain what each unit fraction means.

EXAMPLE 1

Sandra's bakery uses 4 cups of flour per cake when she bakes. How many cups of flour will she use when she bakes 7 cakes for a customer? 28 cups

SOLUTION

The unit rate for amount of flour per cake is given as $\frac{4\text{ cups}}{1\text{ cake}} = 4\ \frac{\text{cups}}{\text{cake}}$. To find how much flour is used to bake 7 cakes, we first use a visual representation.



This process can be written as follows:

$$4\ \frac{\text{cups}}{\text{cake}} \cdot (7\text{ cakes}) = 28\ \frac{\text{cups}}{\text{cake}} \cdot \text{cake} = 28\text{ cups}$$

We can view $\frac{\text{cups}}{\text{cakes}} \cdot \text{cakes} = \text{cups},$ just as when we multiply fractions such as $\frac{5}{3} \cdot 3 = 5.$

The unit of "cake" in the numerator and denominator simplify to 1 and the answer is in cups. This way of keeping track of the units is very useful in application problems, especially in science.

EXAMPLE 2

Phil mixed one teaspoon of chocolate with one cup of milk, and Sam mixed three teaspoons of chocolate with one quart of milk.

a. Which mixture would taste more "chocolaty"? Recall that 2 cups = 1 pint and 2 pints = 1 quart. Justify your answer mathematically.

b. How much more chocolate per cup would the stronger mixture have?

SOLUTION

a. The ratio of chocolate to milk in Phil's mixture is 1 teaspoon of chocolate per 1 cup of milk. In order to compare this to Sam's mixture, we must use the same units. Using the fact that 1 quart = 2 pints and 1 pint = 2 cups, convert units from quarts to cups and multiply by the conversion factors $\left(\frac{1\,quart}{1\,\text{pint}}\right)\left(\frac{1\,\text{pint}}{2\,cups}\right) = \frac{1\,quart}{4\,cups}$.

So the ratio of chocolate to milk in Sam's mixture is

$$\left(\frac{3\,tsp\ chocolate}{1\,quart\ milk}\right)\left(\frac{1\,quart}{2\,\text{pints}}\right)\left(\frac{1\,\text{pint}}{2\,cups}\right) = \frac{3\,tsp\ chocolate}{4\,cups\ milk} = \frac{\frac{3}{4}\,tsp\ chocolate}{cup}$$

We see that Phil's drink will seem more "chocolaty" because it has a greater ratio of chocolate to milk.

b. Phil's drink had ¼ teaspoon more chocolate per cup than Sam's.

EXAMPLE 3

Samantha and Georgeanne went bowling. Samantha made two strikes every seven times she tried, and Georgeanne made one strike every four times she tried.

a. Which bowler would be more likely to make a strike?
b. Explain your answer in a) by comparing the ratio of strikes to attempts for each bowler.
c. If each bowler tried 20 times, on average how many strikes would each be predicted to make?

SOLUTION

a. In order to compare the bowlers, compute the ratio of strikes to attempts.

b. For Samantha the ratio is 2 strikes in 7 attempts, or 2:7 = $\frac{2}{7}$

For Georgeanne the ratio is 1 strike in 4 attempts, or 1:4= $\frac{1}{4}$

Converting these ratios to fractions with like denominators:

$\frac{1}{4} = \left(\frac{1}{4} \times \frac{7}{7}\right) = \frac{7}{28}$

$\frac{2}{7} = \left(\frac{2}{7} \times \frac{4}{4}\right) = \frac{8}{28}$

So, on average, Samantha is more like to make a strike than Georgeanne, with a difference of 1 strike in 28 attempts.

c. To find how many strikes each bowler might make in 20 attempts, multiply:

For Samantha: $(\frac{2\,strikes}{7\,attempts})$(20 attempts) = $\left(\frac{40}{7}\right)$ strikes, more than 8 but less than 9 strikes.

For Geogeanne: $\left(\frac{1\,strike}{4\,attempts}\right)$ (20 attempts) = 5 strikes.

PROBLEM 2

Rate is given by miles per hour in this case. We take the distance, 56 miles, and divide by the time, $3\frac{1}{2}$ hours. The result is $56 \div 3\frac{1}{2} = 56 \div \frac{7}{2} = 56 \cdot \frac{2}{7} = 16$ miles per hour. In an hour and a half, Norman must have traveled 16 mph $\cdot\ 1\frac{1}{2}$ hours = 16 miles/hour $\cdot\ \frac{3}{2}$ hours = 24 miles.

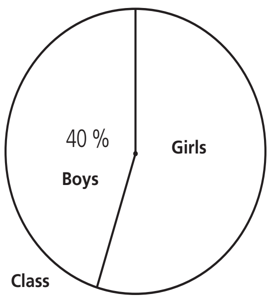
A common question to ask is, "where did the 150% come from?" But, if you lead your students to see the *ratio* of girls to boys is 60:40 reduced to 3:2 which is 1.5 or 150%.

Another approach would be to let x = number of strikes Samantha would make in 20 attempts. Then $\frac{x}{20} = \frac{2}{7}$. Multiplying both sides of this equation by 20 to solve for x, $\frac{x}{20} \times 20$ = x = $\frac{2}{7} \times 20 = \frac{40}{7}$ strikes, which agrees with the results of d.

PROBLEM 2

Norman rode his bike for $3\frac{1}{2}$ hours and traveled 56 miles. What was his average rate, or speed? Approximately, how far did he travel in the first hour and a half?

When a comparison uses percents, it is based on the ratio of an amount to an original or base amount. For instance, "40% of the class is boys" can be thought of as the ratio of boys to the class. 40% boys is equivalent to 40/100, the fractional part of the class that is boys. What percent of this class must be girls? How



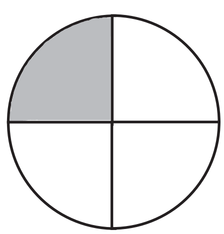
did you decide the percent girls must be 60%? One way to think of this is to consider the whole class as 100% and removing 40% leaves 60%. What then is the ratio of boys to girls? ($\frac{40}{60} = \frac{.4}{.6} = \frac{2}{3}$) What is the ratio of girls to boys? ($\frac{60}{40} = \frac{.6}{.4} = \frac{3}{2}$). In this class, the number of girls is 150% the number of boys. In this case, the number of boys is the original, or base, amount. In both cases, the original, or base, amount is 100%.

EXAMPLE 4

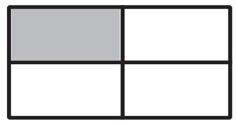
In a class of 40 students at Miller Junior High School, 25% of the students take band. How many students take band?

SOLUTION

One approach is to create a circle graph and cut it into four equal pieces, shown below. In this case, the whole circle represents forty students. Because 25% is the same as ¼ of the circle, then ¼ of 40 is 10. That means that 25% of the class of 40 is really 10 out of the 40 students in the class.



Another approach is to create an area model with 25% of it shaded. It is easy to see that this is ¼ of the whole area so represents 10 of the 40 students.



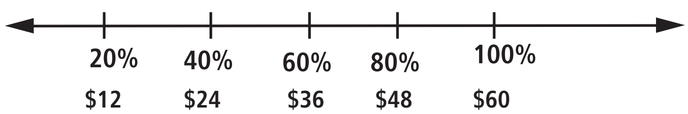
EXAMPLE 5

Frankie spent $12 to see a movie. That amounted to 20% of her monthly allowance. What is Frankie's monthly allowance from her parents?

SOLUTION

20% = 1/5. Because $12 is 1/5 of Frankie's total allowance, the total must be 5 times $12 because 5 times 1/5 is 5/5 = 1 or all of the total allowance. 5 x 12 = 60, so Frankie's monthly allowance is $60.

Another way to visualize the whole allowance from 20% of it is to look at a number line model to see five 20% increments to 100% with $12 for each 20% summing to 5 x 12 = 60. The same answer as above.

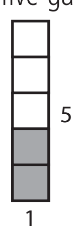


PROBLEM 3

Mr. Graham needs fifteen yards of fabric, which is 25% of afabric bolt, to make curtains for his school's summer musical. He will use the other 75% of the bolt to make costumes. How many yards of material are in the bolt?

PROBLEM 4

The shaded part of the rectangular grid represents a five-gallon water jug that is 20% full. How many gallons are in the water jug?



EXPLORATION 3

a. 4, 8, 20 liters d. $y = 4x$ e. $y = 4(42)=168$ L

f. $244 = 4x$, $x =61$ min g. $95 = 4x$, $x = 95/4 = 23\ \frac{3}{4}$ min

Building a table is one helpful strategy for observing patterns related to ratios and rates.

EXPLORATION 2

Gloria works as a computer consultant for the Bayou Company and earns $612 for working 36 hours. If she charges a fixed amount per hour, how much will she earn working 18 hours? 9 hours? 1 hour? 4 hours?

EXPLORATION 3

Ms. Jones starts fill a very large empty tank with water at the rate of 4 liters per minute.

a. How much water is in the tank after 1 minute? 2 minutes? 5 minutes?
b. Make a table with filling time as input minutes and amount of tank filled as outputs liters of water.
c. Draw a graph that includes at least 5 points.
d. Write an equation that relates how much water is in the tank after x minutes.
e. How much water will be in the tank after 42 minutes?
f. When will there be 244 liters in the tank?
g. When will there be 95 liters in the tank?

EXAMPLE 6

Lisa and Adam both jog everyday. Lisa jogs an average of 2200 meters in 40 minutes and Adam jogs an average of 1500 meters in 30 minutes.

a. Who jogs faster? Explain.
b. On average, how far does Lisa jog in 12 minutes?
c. At the same rate, how much further would Lisa jog in 12 minutes than Adam?
d. Maintaining his pace, how long does it take Adam to jog 600 meters?
e. Find a rate that is exactly midway between the jogging rates of Lisa and Adam.

SOLUTION

a. Lisa jogs at a rate of $\frac{2200\, meters}{40\, \min utes} = 55\ \frac{meters}{\min ute}$. Adam jogs at a rate of $\frac{1500\, meters}{30\, \min utes} = 50\ \frac{meters}{\min ute}$. So Lisa jogs faster than Adam.

b. On average, Lisa would jog (55 $\frac{meters}{\min ute}$) •12 minutes = 660 meters.

c. Adam would jog (50 $\frac{meters}{\min ute}$)•12 minutes = 600 meters. So Lisa would go 60 meters further in 12 minutes than Adam.

d. Since Adam jogs 50 $\frac{meters}{\min ute}$, we let x = the number of minutes he takes to jog 600 meters. Then (50 $\frac{meters}{\min ute}$) • x minutes = 600 meters. 50 x = 600, x = 12 minutes. (which is what we already found above)

e. A rate midway between Lisa's rate and Adam's rate is 52.5 $\frac{meters}{\min ute}$. This is the average of the two rates, $\frac{(55+50)}{2}$ = 52.5.

Example 6 is an example of the rate formula that assumes that something is traveling at a constant rate. The distance traveled is equal to the constant rate multiplied by the time traveled: $d = r \cdot t$ or $d = rt$. The constant rate is often written as the unit rate. Typical units for *r* are:

miles per hour ($\frac{\text{mi}}{\text{hr}}$) = miles traveled in one hour

feet per second ($\frac{\text{ft}}{\text{sec}}$) = feet traveled in one second

meters per minute ($\frac{\text{m}}{\text{min}}$) = meters traveled in one minute

EXAMPLE 7

Class members are surveyed to find out what is their favorite type of drink, soda or lemonade. Forty percent of the class prefers soda, and sixty percent prefers lemonade. What is the ratio of students who prefer soda to students who prefer tea?

SOLUTION

If the class has T students, then the number of students who prefer soda is 0.40T, and the number who prefer lemonade is 0.60T. So the ratio is $\frac{(0.40\,T)}{(0.60\,T)} = \frac{0.40}{0.60} = \frac{4}{6} = \frac{2}{3}$.

EXAMPLE 8

Two quantities x and y are related as you can see in the table below:

x	y
1	½
2	1
3	$\frac{3}{2}$
4	2
5	$\frac{5}{2}$
6	3

a. Write an equation that gives y in terms of x

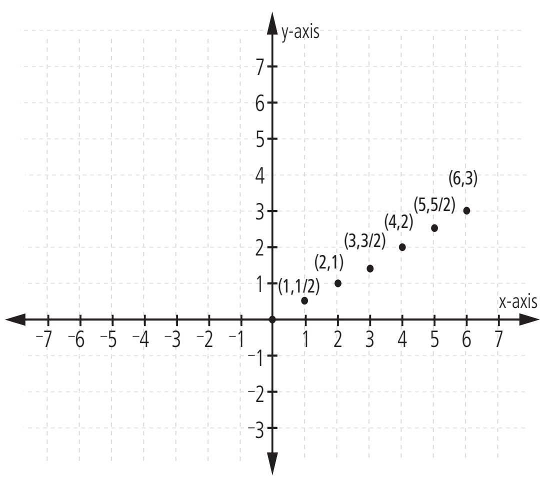
b. Graph the equation

c. Examine the ratio of y to x using two different (x, y) pairs. What do you notice?

SOLUTION

a. The ratio of y to x is ½ to 1. So y = ½ x.

b. The graph of the equation plots the six ordered pairs (x, y)



EXERCISES

1. a. 6:4 or 3:2 b. 4:6 or 2:3 c. 4:3 d. 3:4
   e. 6:3 or 2:1 f. 3:6 or 1:2

2. a. 10
   b. 3:5

3. a. 1:5
   b. 9:20
   c. 4:7
   d. 7:9
   e. 9:4

c. The ratio is ½.

PROBLEM 5

The ratio of two quantities, p and q, are the same as the equation $q = 3p$. Make a table with at least 5 entries that relate p and q.

EXERCISES

Answer the following questions using the fraction form of the ratios. Simplify your ratios if possible.

1. Carmen packed her suitcase with 6 shirts, 4 pairs of pants and 3 belts.
   a. What is the ratio of shirts to pants?
   b. What is the ratio of pants to shirts?
   c. What is the ratio of pants to belts?
   d. What is the ratio of belts to pants?
   e. What is the ratio of shirts to belts?
   f. What is the ratio of belts to shirts?
2. Ms. Acosta decided to reward $\frac{2}{5}$ of her class with a free homework pass. There are 25 students in her class.
   a. How many students received a free homework pass?
   b. What ratio of students did not received a free homework pass?
3. A bag contains red, blue and green color tiles. It has 35% red tiles, 20% blue tiles, and 45% green tiles.
   a. What is the ratio of blue tiles to the total number of tiles?
   b. What is the ratio of green tiles to the total number of tiles?
   c. What is the ratio of blue tiles to red tiles? 3:2
   d. What is the ratio of red tiles to green tiles? 2:5
   e. What is the ratio of green tiles to blue tiles? 5:3

4. 10: 8 or 5:4

5. The word "question" refers to the first sentence, rather than the exercise as a whole.
   a. The ratio of vowels to consonants in the first sentence is $\frac{19}{29}$ = 19:29 = 19 to 29.
   b. The ratio of vowels to consonants in the second sentence is $\frac{16}{23}$ = 16:23 = 16 to 23.
   c. Both were between 60 to 70%. So, the number of vowels are usually less than the number of consonants.
   d. Not really.
   e. There are more vowels than consonants.
   f. There are fewer vowels than consonants.

4. Sarah and Max each purchased donuts for their math team. Sarah bought three glazed donuts, two jelly-filled donuts, and three powdered donuts. Max purchased five glazed donuts, eight jelly-filled, and six powdered donuts.

a. What is the ratio of glazed donuts to the total for Max's team? For Sarah's team?

b. If one of Sarah's team members chose a donut at random from Sarah's donuts, and one of Max's team selected a donut at random from his donuts, explain who would be more likely to obtain a glazed donut.

5. a. What is the ratio of vowels to consonants in this question?

b. Now examine the ratio in this statement as well.

c. Conjecture what might be an expected ratio of vowels to consonants in most English sentences. Check with several other sentences and observe the ratios you find. Express the ratio of vowels : consonants.

d. Is it close to 1?

e. What does it mean if the ratio is greater than 1?

f. Less than 1?

g. Based on the ratio you found in Part a, if you read a book with 25,000 letters, about how many of these letters would you expect to be vowels?

6. Use the following table that gives the ratio of two quantities x to y

x	y
1	3
2	6
3	9
4	12
5	15

8. a. 16 cups per gallon
   b. 12 dozen per box
   c. $3 per box

9. a. 8800 yds.
   b. 26400 feet

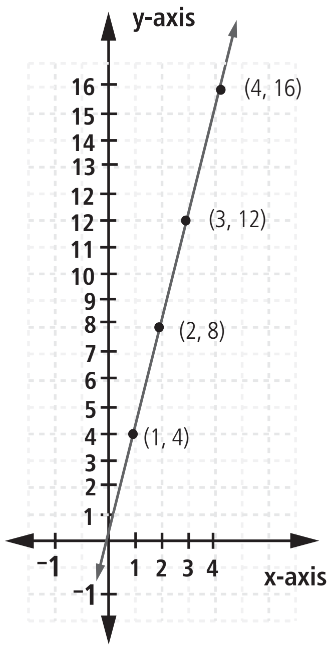
10. 27 hot dogs

a. Write an equation that relates x and y. Explain your answer.

b. Graph the equation on a coordinate system.

c. How are any two points (x, y) on the graph related? Explain your answer.

7. Use the graph below that relates two quantities x and y to write an equation that relates the ratio of y to x.



8. Solve the following unit rates.

a. A gallon of milk has 128 ounces. A cup holds 8 ounces. What is the number of cups per gallon?

b. There are 12 eggs per dozen. A box holds 144 eggs. What is the number of dozens per box?

c. You can buy 20 boxes of crayons for $60. What is the price per box?

9. Fineas is riding his bike at 5 miles per hour.

a. How many yards will he travel in one hour? (Hint: 1 mi. = 1760 yds)

b. How many feet will he travel in one hour? (Hint 1 mi. = 5280 ft)

10. Luis eats 45 hot dogs in 5 hours. How many hot dogs will he eat in 3 hours?

11. 5 feet in one hour.
    60 feet in 12 hours.

12. He can swim for 28 laps

14. a. \$5, \$7.50, \$15 d. y = 2.5x e. y = 2.5(24)=\$60 f. 100 = 2.5x , x = 40

15. a. (18)(150) = 2700 inches of ribbon b. 18x = 4680 , x = 260 arrangements

Spiral Review (7.2 E):	**Spiral Review (6.2 B)**:
16. 5 + 3 (11 - 2) ÷ 16 5 + 3 (9) ÷ 16 $6\frac{11}{16}$	17. Combined weight = 97.1 grams; Weight difference = 6.5 grams

11. Robert climbs 120 feet in one day. Assuming he climbs at a constant rate with no rest, how many feet does he climb in one hour? How many feet in 12 hours? (Hint: 1 day = 24 hrs)

12. Stephen swims 7 laps every 15 minutes. Complete the table to find the total number of laps he can swim in one hour. Represent this data on a graph.

Laps	7	14		
Time in minutes	15	30		

13. A store sells organic sugar at the rate of $2.50 per pound.
    a. How much does 2 pounds cost? 3 pounds? 6 pounds?
    b. Make a table with input as pounds of sugar and outputs as cost in dollars.
    c. Draw a graph using at least 5 points.
    d. Write an equation for cost as a function of amount of sugar.
    e. What is the cost of 24 pounds of sugar?
    f. How much sugar can you buy for $100?

14. A floral shop uses 18 inches of ribbon to make each flower arrangement.
    a. How much ribbon do you need to make 150 arrangements?
    b. How many arrangements can you make with 4680 inches of ribbon?

15. John can finish a job in 45 minutes. Working with Sue, they can finish the job in 30 minutes.
    a. Who works faster, John or Sue? Explain
    b. How long would it take Sue to finish the job alone?

Spiral Review:

16. Simplify the expression $5 + 3\ (11 - 2) \div 4^2$

17. The three baby robins in the nest outside Greg's window weigh 32.2 grams, 35.7 grams, and 29.2 grams. What is the combined weight of the three baby birds? What is the difference in weight between the largest and the smallest baby bird?

Ingenuity: 18. There are several ways to approach this problem. One way is to keep track of when each snail finishes a lap:
Sally finishes 1 lap after 4 hours; Sam finishes 1 lap after 5 hours; Sally finishes 2 laps after 8 hours; Sam finishes 2 laps after 10 hours; Sally finishes 3 laps after 12 hours; Sam finishes 3 laps after 15 hours; Sally finishes 4 laps after 16 hours; Sam finishes 4 laps after 20 hours; Sally finishes 5 laps after 20 hours
Notice that, at the 20-hour mark, Sally has finished exactly one more lap than Sam. So the answer is 20 hours.

Another way to approach this problem is to realize that Sally finishes 1/4 of a lap every hour, whereas Sam finishes 1/5 of a lap every hour. So Sally's lead over Sam grows by a total of 1/4 - 1/5 = 1/20 of a lap every hour. Given this, it is easy to see that Sally's lead over Sam will grow to a full lap after 20 hours.

Investigation:
19. TE: The purpose of this investigation is to give students some experience with unit conversions involving rates. Students will be required to do conversions like these in their high school science courses, especially physics.

(a) In one hour, the car travels 65 miles. Since each mile consists of 1600 meters, the car travels 65 x 1600 = 104000 meters in one hour.

(b) In one minute, the car travels 1/60 of the distance it covers in an hour. So it travels 104000 [divided by] 60 [approximately equals] 1733 meters per minute.

(c) In one second, the car travels 1/60 of the distance it covers in one minute. So it travels 1733 [divided by] 60 [approximately equals] 29 meters per second.

(d) We will go through the steps we did in parts (a) through (c) in reverse order to answer this question. If light travels 300 million meters per second, then it travels 60 times as far in one minute, and 60 times as far as that in one hour. So light travels 300 x 60 x 60 = 1080000 million, or 1.08 trillion meters per hour. Since there are 1600 meters in a mile, light travels 1080000 [divided by] 1600 = 675 million miles per hour.

18. **Ingenuity:**

    Sam, a snail, makes one lap around a circular track every five hours. Another snail, Sally, makes one lap around the same track every four hours. If Sam and Sally start at the same point on the track, how long does it take before Sally has completed one more lap than Sam?

19. **Investigation:**

    A car travels 65 miles per hour on a highway. Based on this information answer the following questions:

    a. How many meters does the car travel per hour? (There are 1600 meters in a mile.)

    b. How many meters does the car travel per minute?

    c. What is the car's speed in meters per second?

    d. The speed of light is approximately 300 million meters per second. How many miles per hour is this?

Section 7.4 - Proportions

Big Idea:
Understanding and using proportional thinking to solve problems

Key Objectives:
- Understand the definition and use of a proportion.
- Represent proportional relationships using tables.
- Represent proportional relationships using unit rates.
- Solve proportions in the context of real-life situations.

Materials:
Two-column tables from CD, Maps

Pedagogical/Orchestration:
- This section involves developing proportions through patterns in tables and unit rates. Proportions are perhaps the handiest mathematical tool that middle school students should learn to use. This section is rich with situations that call for proportional thinking and mathematics.
- Be aware of the prerequisites needed for students to be successful in understanding ratios, such as knowing the proper way to set up a proportion, paying particular attention to unit alignment, and solve for the unknown. Guide the students through the mini-lesson included before the examples to help students understand the Big Idea.

Activity:
"M&Ms in a Bag"

Exercises:
Exercise 9 uses the proportional reasoning that the number of fish tagged out of the population will equal the number of tagged fish caught out of the sample. In other words, $\frac{30}{p} = \frac{3}{50}$. The following website/activity can help students understand this type of population problem: http://illuminations.nctm.org/LessonDetail.aspx?id=L721

Vocabulary:
proportions

TEKS:
6.2(C); 6.3(A,C); 6.4(A); 6.5; 6.11(A,B,C); New: 6.3(B, G); 6.4(A, B, C, D, E, G, H); 6.5(A, B, C)

Launch for Section 7.4:

Guide students through the following activity.

M&Ms in a Bag Activity:

Materials:

M&Ms (yellow, red, blue, green; at least 10 of each color)
Colors or markers

Instructions:

Teacher arranges students in groups of 3 or 4. Each group receives a bag of about 20 M&Ms. Teacher sets up this activity by choosing how many M&Ms of each color to use. One suggestion is to place the following amount of M&Ms in the bag: 4 yellow, 7 blue, 3 red and leave green as unknown. Students will answer the following questions about the M&Ms in the bag:

Set up each ratio and simplify as needed:

1. What is the ratio of yellow M&Ms in the bag?
2. What is the ratio of blue M&Ms in the bag?
3. What is the ratio of red M&Ms in the bag?
4. What is the ratio of green M&Ms in the bag?

If one M&Ms of each color is removed from the bag, what will be the new ratios of each color or M&Ms in the bag

5. yellow:
6. blue:
7. red:
8. green:

Consider the fun size M&M pack and examine the ratio of color M&M to the whole pack, and making the ratio in large pack comparable (heading towards the idea of equivalent fractions). Explore M&M color ratio in larger size bags like the family size M&M bag.

Lead your students in a discussion of how they might solve this equation. The discussion might include the following: Multiply both sides of the equation by 2. Or multiply the right side by 1 in the form of $\frac{2}{2}$ to produce the equation $\frac{100}{2} = \frac{x}{2}$ and then note that the numerators must be equal, so x = 100. Some students might know that the actual distance x is equal to 100 miles using arithmetic, not algebra.

Sometimes, the first ratio in the proportion compares different units. If the first ratio compares inches to miles, then the second ratio also compares inches to miles: $\frac{1 \text{ in}}{50 \text{ mi}} = \frac{2 \text{ in}}{x \text{ mi}}$. If the first ratio compares miles to inches, then the second ratio also compares miles to inches: $\frac{50 \text{ mi}}{1 \text{ in}} = \frac{x \text{ mi}}{2 \text{ in}}$. It does not matter which quantity, miles or inches, is in the numerator. The important thing is that the ratios are equivalent as fractions.

This is a very important point in understanding proportions. Make sure your students see how varied a correct proportion can be and what the big mistake is – not to match the units in the fractions' numerators and denominators.

SECTION 7.4 PROPORTIONS

When you look at a map of Texas, you know that the actual state is much larger than the map. For example, 1 inch can represent 50 miles, according to a scale designation on the map legend. That means that the ratio of the map distance to the actual distance is 1 inch to 50 miles. This ratio is written 1 inch : 50 miles or $\frac{1 \text{ in}}{50 \text{ mi}}$, as in Sections 7.3.

Using this information, what actual distance does 2 inches represent? This time, writing the information as a ratio of actual distance to the map distance, the fraction is $\frac{x \text{ mi}}{2 \text{ in}}$, where x is the actual distance represented by 2 inches on the map. Using the scale of 50 miles to 1 inch from the map, combine the two ratios in the equation $\frac{50 \text{ mi}}{1 \text{ in}} = \frac{x \text{ mi}}{2 \text{ in}}$. We will explore a way to solve equations like this for the unknown x later in this section.

DEFINITION 7.3: PROPORTION
A **proportion** is an equation of ratios in the form $\frac{a}{b} = \frac{c}{d}$, where b and d are not equal to zero.

In a proportion, each side of the equation is a ratio. Sometimes, a proportion can compare two different types of the same units, like inches to inches and miles to miles, as long as both ratios are equivalent as fractions: $\frac{x \text{ mi}}{2 \text{ in}} = \frac{50 \text{ mi}}{1 \text{ in}}$ or $\frac{2 \text{in}}{1 \text{in}} = \frac{x \text{ mi}}{50 \text{ mi}}$

EXAMPLE 1

A colony of leafcutter ants cuts up 4 leaves in 7 minutes. Now, write a ratio that corresponds to this relationship. Write a proportion that corresponds to the following relationship: How many leaves does the colony cut in 35 minutes?

SOLUTION

We first use a ratio of leaves to time, $\frac{4 \text{ leaves}}{7 \text{ min}}$. Then use the variable L to represent the number of leaves cut by the leafcutter ant in 35 minutes. The second ratio looks like $\frac{\text{L leaves}}{35 \text{ min}}$. The proportion that we obtain when the two ratios are set equal to each other looks like this:

$$\frac{4 \text{ leaves}}{7 \text{ min}} = \frac{\text{L leaves}}{35 \text{ min}}$$

We look at several ways that you can use to solve this problem.

Tabular Method:

Construct a table to record the time and the number of leaves cut.

Time in minutes	Number of leaves cut
0	0
7	4
14	8
21	12
28	16
35	20
42	24
49	28
56	32
63	36

From the table you can see that 35 minutes corresponds to 20 leaves cut by the leafcutter ant.

Unit Rate Method:

Set up a proportion that compares the ratio of leaves to minutes. Because the ants cut 4 leaves in 7 minutes, using division, the ants must cut $\frac{4}{7}$ of a leaf in 1 minute. This is the unit rate or the number of leaves cut per minute. If the ants keep cutting at this rate, they will cut 35 times this number of leaves in 35 minutes. Call the number of leaves cut in 35 minutes x. Then

$$x = \frac{4 \text{ leaves}}{7 \text{ min utes}} \cdot 35 \text{ min} = \frac{140}{7} \text{ leaves} = 20 \text{ leaves}$$

Proportion Method:

Set up a proportion by comparing amounts for the two different times.

The ants cut 4 leaves in 7 minutes. How many leaves L will the ants cut in 35 minutes?

$$\frac{\text{L leaves}}{35 \text{ min}} = \frac{4 \text{ leaves}}{7 \text{ min utes}}$$

PROBLEM 1: Allow students to solve in methods including table, proportion set up with equivalent fraction, solve equation, unit rate and solve.

Solution

Alberta remembers knitting 24 socks for 3 aliens and wants to find how many socks S she needs for 5 aliens:

$$\frac{24 \text{ socks}}{3 \text{ aliens}} = \frac{S \text{ socks}}{5 \text{ aliens}}.$$

Multiply by both sides of the equation by 5 and simplify

$$5 \cdot \frac{24}{3} = 5 \cdot \frac{S}{5}$$

$$\frac{120}{3} = \frac{5S}{5}$$

$$40 = S$$

So, Alberta needs to knit a total of 40 socks for the 5 aliens.

PROBLEM 2:

Let x = distance between city A and mountain B.

a. Then $\frac{x \text{ miles}}{7 \text{ inches}} = \frac{120 \text{ miles}}{3 \text{ inches}}$ Multiply by 7 inches on both sides of the equation.

$x \text{ miles} = \frac{120 \text{ miles}}{3 \text{ inches}} \cdot 7 \text{ inches} = \frac{40 \text{ miles}}{\text{inches}} \cdot 7 \text{ inches} = 280 \text{ miles}.$

b. The unit rate is obtained by using $\frac{120 \text{ miles}}{3 \text{ inches}}$ which simplifies to $\frac{40 \text{ miles}}{\text{inches}}$

c. Distance between A and B = $\frac{40 \text{ miles}}{\text{inches}} \cdot 7 \text{ inches} = 280 \text{ miles}.$

To solve, multiply both sides of the equation by the denominator 35.

$$35 \text{ min} \cdot \frac{L \text{ leaves}}{35 \text{ min}} = \frac{4 \text{ leaves}}{7 \text{ min utes}} \cdot 35 \text{ min}$$

$$L = \frac{4}{7} \cdot 35 = 20 \text{ leaves.}$$

This proportion method involves the rate of change in the form of speed, the rate of leaves cut per unit time or minute. This is a rate of change like miles per hour or mph.

PROBLEM 1

Alberta likes to knit socks for her grandson's collection of toy aliens, but she has forgotten how many legs each alien has. She remembers knitting 24 socks for 3 aliens. Assuming that the aliens all have the same number of legs, how many socks should she knit for 5 aliens?

PROBLEM 2

In an old map, the map scale has become unreadable. We know that two locations are 120 miles apart and we use a ruler to determine that they are 3 inches apart on the map.

a. City A is 7 inches from mountain B on the map. Use proportion to calculate the distance between them.

b. What is the unit rate in miles per inch?

c. Use the unit rate of miles per inch from part b to calculate the distance in miles from city A to mountain B.

Here is another example that illustrates different approaches to solve proportional reasoning problems.

EXPLORATION 1

Materials: You will need a map of any region that contains a legend with the distance scale and a ruler or tape measure in the same unit system as the map.

Step 1: Find the legend in the map and write a ratio that relates the map measure to the actual measure.

Step 2: Use a measuring instrument to measure the straight-line distance between two major cities on the map.

Step 3: Determine the actual straight-line distance between the cities using proportions.

Step 4: Repeat Steps 2 and 3 with two other cities.

What are the actual straight-line distances between the cities that you chose?

EXAMPLE 2

Set up the following problem using a proportion.
3 bags of chips cost \$2.79. How much do 7 bags of chips cost?

SOLUTION

In the chips problem, any one of these proportions will correctly determine *x*, the cost of 7 bags of chips. The possible proportions are indicated:

a. $\frac{\text{bags of chips}}{\text{bags of chips}} = \frac{\text{cost}}{\text{cost}}$ or $\frac{3}{7} = \frac{2.79}{x}$

c. $\frac{\text{cost}}{\text{bags of chips}} = \frac{\text{cost}}{\text{bags of chips}}$ or $\frac{x}{7} = \frac{2.79}{3}$ or

b. $\frac{\text{bags of chips}}{\text{cost}} = \frac{\text{bags of chips}}{\text{cost}}$ or $\frac{3}{2.79} = \frac{7}{x}$

d. $\frac{\text{cost}}{\text{cost}} = \frac{\text{bags of chips}}{\text{bags of chips}}$ or $\frac{x}{2.79} = \frac{7}{3}$.

Use one of the proportions to solve for *x*. Verify that the cost for 7 bags of chips is the same when each proportion is solved.

Setting up the correct proportion is often the hardest part of solving a proportion problem. Some proportions are easier to solve than others because of the way they are set up. Which of the proportions above was easiest to solve and why?

There are several ways in which you can solve the problem. If we use part (c) then $\frac{x}{7} = \frac{2.79}{3}$ and $x = \frac{2.79}{3} \cdot 7 = 6.51$. From our work, we see that 7 bags of chips cost \$6.51.

EXAMPLE 3

John lives 3 miles from school, and takes his bicycle each day.

a. It takes him 40 minutes to get to school. What is his average rate in miles per hour?

b. If John biked twice as fast, how long would it take him to make the trip?

SOLUTION

John's average rate is 3 miles/40 minutes. To convert miles per minute to miles per hour, use the fact that 60 minutes = 1 hour. Multiply by the factor $\frac{60\,\min utes}{1\,hour}$, to obtain

$$\frac{3\,miles}{40\,\cancel{\min.}} \times \frac{60\,\cancel{\min.}}{1\,hour} = \frac{\frac{30\times 60}{40}\,miles}{hour} = \frac{\frac{9}{2}\,miles}{hour}$$

If John doubled his rate, he would go $2\bullet\frac{9}{2}$ = 9 mph. Remember that distance = rate • time. Let t = time in hours that it takes for John to make one trip, to find 3 miles = (9 $\frac{miles}{hour}$) • t hours.

Simplifying, 3 = 9t and t = $\frac{3}{9}$ = $\frac{1}{3}$ hour.

Alternatively, convert the time to minutes: ($\frac{1}{3}$)hour • ($\frac{60\,\min utes}{1\,hour}$) = ($\frac{60}{3}$) minutes = 20 minutes.

EXAMPLE 4

Sally walked up a hill at a rate of 2 miles per hour, then walked back down at a rate of 4 miles per hour.

a. Was her average rate of walking for the trip less than 3 miles per hour, equal to 3 miles per hour, or greater than 3 miles per hour?

b. What was her average rate of speed for the entire hike?

SOLUTION

a. Since Sally walked the same distance up as down, she spent twice as long walking up the hill as walking down the hill. Because Sally walked more time at the slower speed, her average rate for the entire walk should be closer to 2 miles per hour than to 4 miles per hour. Thus, her average rate is less than 3 miles per hour.

b. The question is how can we calculate her average rate? Let D = distance up the hill. Then D is also the distance down the hill. If Sally spends t hours going up the hill, then she will spend $\frac{t}{2}$ hours doing down the hill.

Use the fact that distance = rate • time. Going up the hill, D = 2t. Going down the hill, D = 4 ($\frac{t}{2}$). For the entire trip, let r = average rate. Then 2D = r (t+$\frac{t}{2}$)

2D = r($\frac{3}{2}$t).

EXPLORATION 2

3. Answers will vary. For example, If $1 = 11.78 pesos then the proportion that we can use to determine how much pesos $50 is worth is:
P/50 = 11.78/1. In this case P = 50(11.78) = 589 pesos.

4. D/100 = 1/11.78 so D = 100/11.78 = $8.49

Because 2D = 4t, substitute to replace D with t and eliminate a variable.

$4t = r\left(\frac{3}{2}t\right)$

$4 = r\left(\frac{3}{2}\right)$

$8 = 3r$

$r = \frac{8}{3} = 2\frac{2}{3}$ mph

PROBLEM 3

a. Calculate the unit rate of cost per bag of chips in Example 2 and use it to calculate the cost of 7 bags of chips.

b. Write an equation for the cost function for this problem.

c. Use it to calculate the cost of 20 bags of chips.

EXPLORATION 2

Converting your Dollars Activity:

The currency in the United States is in dollars and cents. Do you know the currency of our neighbors Mexico? (Peso) Canada? (Canadian dollar and cents) What is the currency in England? (pounds) France?(Euro) Japan? (yen) China? (yuan)

1. Find the currency for six different countries. You may need to use a reference or some outside source.

2. Determine what \$1 US is worth in the currency of each of the six countries. You may need to use the internet for the most current exchange rate.

3. Determine what \$50 is worth in each of the six currencies.

4. Using the currency of one of your countries, determine what 100 of that currency is worth in dollars.

EXPLORATION 3

1. Ben and Jerry are scooping ice cream into a large bowl. They started together, but Jerry scoops faster. While Ben scoops 4 ice cream scoops, Jerry scoops 12 ice cream scoops.

EXERCISES

1. Bobby threw back 24 shells

a. When Ben has scooped 20 ice cream scoops, how many ice cream scoops has Jerry scooped?

b. Write in words a relationship between the number of ice cream scoops Jerry scoops with the number of ice cream scoops Ben scoops.

c. Write an equation for the number of ice cream scoops, J, that Jerry scoops in terms of the number of scoops, B, that Ben scoops.

d. Write an equation for the number of ice cream scoops, B, that Ben scoops in terms of the number of scoops, J, that Jerry scoops.

2. On another day, Ben and Jerry are again scooping ice cream into a large bowl. They scoop equally fast, but Jerry started earlier. When Ben has scooped 4 ice cream scoops, Jerry has scooped 12 ice cream scoops.

a. When Ben has scooped 20 ice cream scoops, how many ice cream scoops has Jerry scooped?

b. Write in words a relationship between the number of ice cream scoops Jerry scoops with the number of ice cream scoops Ben scoops.

c. Write an equation for the number of ice cream scoops, J, that Jerry scoops in terms of the number of scoops, B, that Ben scoops.

d. Write an equation for the number of ice cream scoops, B, that Ben scoops in terms of the number of scoops, J, that Jerry scoops.

3. Compare the relationship found in problem 1 above between Jerry and Ben's scoops with the relationship found in problem 2. Describe the difference(s) in the two relationships.

EXERCISES

Solve the following proportions using ratio tables or by setting up equations. Simplify your answers when possible.

1. Little Bobby was collecting seashells one afternoon. For every 6 shells he found, he would throw two back. After collecting shells for 1 hour, he had a total of 48 shells in his basket. How many shells did he throw back during that hour?

2. \$9

3. a. 70
   b. 112

4. a. $x = 1$ b. $x = 8$ c. $x = 54$ d. $x = 1$

5. A minimum of 7 vans

6. 4 cakes

2. The candy store had a sale of lollipops last week. You could buy 8 lollipops for \$2. At this price, how much would you pay for 36 lollipops?

3. A florist makes balloon bouquets with red, blue, and yellow balloons. The ratio of red to yellow balloons is 3:5 and the ratio of blue to yellow is 8:5. If a bouquet contains 42 red balloons:
   a. How many yellow balloons does each bouquet contain?
   b. How many blue balloons does each bouquet contain?

4. Solve the following equations.
   a. $\frac{3}{4} = \frac{x}{16}$ b. $\frac{x}{4} = \frac{36}{18}$ c. $\frac{7}{9} = \frac{42}{x}$ d. $\frac{35}{x} = \frac{105}{3}$

5. Vans hold 12 students. What is the minimum number of vans that are needed to take 78 students to the lake during summer camp?

6. If 6 large cakes can feed 150 guests, how many cakes will be needed to feed 100 guests?

7. Andrew makes three hits out of every 10 times he bats, and Sam makes 4 hits out of every 13 times he bats.
   a. Who is more likely to get a hit when at bat?
   b. If Sam bats 100 times, how many hits might he expect to get? If Andrew bats 100 times, how many hits might he expect to get?
   c. If Andrew gets 84 hits this season, about how many times did he bat?

8. Sam surveyed 100 randomly selected women to find out their favorite color. He found that 35 women liked red, 25 liked blue, and 40 liked green. Susan survey 120 randomly selected men and found that 40 liked red, 25 liked blue, and 55 liked green.
   a. Based on these surveys, is it more likely that a woman or man would prefer red? Blue? Green? Explain.
   b. If two women are picked at random from Sam's group, how likely is it that they both like red?
   c. If two women are picked at random from Sam's group, how likely is it that at least one of the women likes red?

9. 210 total beads

10. 10 cups

11.
b. $d = 18x$ c. $d = 18(6.5) = 117$ d. $63 = 18x$, $x = \frac{7}{2} = 3.5$

12. 5 minutes

13. 10 baskets

14. 20 days

15. 144 seats have designs

16. a. $x/6 = 133/42$, $x = 19$ b. \$7 c. $P = 7x$ d. $54 = 7x$
e. $100 = 7x$, $x = 14\ 2/7$. So he can buy 14 tickets with \$2 change.

9. Marcia has a bag of purple and orange crystal beads. The ratio of purple beads to the total number of beads is 5:7. If she has 60 orange beads, how many total beads are in the bag?

10. Grandma Janie's recipe for sugar cookies calls for 4 cups of sugar for every 72 cookies. How much sugar is needed to make 180 cookies?

11. Josh's car uses 18 miles per gallon.

    a. Use a proportion to calculate how many gallons he needs to travel 126 miles.
    b. Write an equation for the distance traveled if he uses x gallons. Let d stand for the distance.
    c. How far can he travel if he has 6.5 gallons of gasoline?
    d. How many gallons does it take to travel 63 miles?

12. Melissa can type 144 words in 3 minutes. If she keeps the same speed, how many minutes will it take Melissa to type 240 words?

13. Diane has made 40 baskets out of 120 attempts. How many baskets can Diane's coach expect her to make if she attempts 30 baskets in a game?

14. Roxy purchased a book with 380 pages. If she reads 19 pages each day, how many days will it take her to read the entire book?

15. Emilio noticed that every 12 seats out of 75 had designs. If the entire arena has 900 seats, how many seats have designs?

16. Evan buys 6 movie tickets for $42.
    a. How many movie tickets can Evan buy with $133?
    b. How much does each ticket cost?
    c. Write an equation for the price P of x tickets.
    d. How many tickets can he buy with $100?

Spiral Review:

17. After one hour, 25% of 6th grade students at Greathouse elementary had finished the STAAR test. What percent had not finished the test? Represent the answer as a decimal, fraction, and percent?

Spiral Review (6.3 B):
17. 75% had not finished. (.75, 75/100 = 3/4)

Spiral Review (6.3 B):
18. $3.46

19. **Ingenuity:**
We know that the entire trip takes 72 minutes (the entire 12:00 hour, plus 12 additional minutes). So we know that it takes Mitchell 72 minutes to drive 80 miles, which is a rate of 9/10 minutes per mile. Therefore, the amount of time it takes Mitchell to reach San Marcos is
9/10 minutes per mile x 30 miles = 27 minutes.
So Mitchell passes through San Marcos at 12:20 P.M.

20. **Investigation:**

[Foreshadowing probability - I admit I don't really know how much background the students have in this, or whether they'd have any intuition about the subject. They might be able to intuit what to do just from the way the question is set up.]

TE: The purpose of this Investigation is to foreshadow the idea of probability, which we explore in more depth in Chapter 10. The topic of probability is very challenging for students, partly because of the heavy emphasis on ratios and proportional thinking, and partly because we are making a transition from thinking about things that are definite, as we have for most of this book, to things that involve chance and uncertainty. The problems here may make good fodder for classroom experimentation and discussion.

(a) How many of the cards in the deck are spades?
TE: Since the cards in the deck are equally divided into four suits, the number of spades is 52 [divided by] 4 = 13.

(b) Suppose we pull 20 cards from the deck without looking at them. About how many of these cards would you expect to be hearts? Explain your answer.

TE: If we randomly pull 20 cards from the deck, we should expect the 20 cards to be divided roughly equally into the four suits, since there is no reason to expect that one suit will be more common than the others in the sample we pull. So we should expect about 5 hearts. Note that this does not mean that we [ital] will [/ital] get five hearts; we may get a little more or a little less, or we may even get a lot more or a lot less. However, if we do this experiment many times, we should expect to get about 5 hearts on average.

(c) Suppose we pull a card from the deck without looking first, then look at the card and record the result. Suppose we repeat this process 280 times. About how many times should we expect to draw clubs? Explain your answer.

TE: We will draw clubs about 1/4 of the time, so we should expect to get about 70 clubs. Again, note that it is actually not very likely that we will get exactly 70, but it is likely that we will get around 70 clubs.

18. Brandy bought lunch for herself and for two of her friends. Brandy's lunch cost \$6.53 and her friends' lunches cost \$5.75 and \$4.26. If tax was already included in the cost. What is the change Brandy received if she paid with a \$20.00 bill?

19. **Ingenuity:**

    Mitchell is driving from Austin to San Antonio, a total distance of 80 miles. He leaves Austin at 11:53 A.M. and reaches San Antonio at 1:05 P.M. If Mitchell drives the same speed for the entire trip, at what time does he pass through San Marcos, which is 30 miles from Austin?

20. **Investigation:**

    In a standard deck of playing cards, there are 52 cards. These cards are divided equally into four suits: spades, hearts, diamonds, and clubs.

    a. How many of the cards in the deck are spades?
    b. Suppose we pull 20 cards from the deck without looking at them. About how many of these cards would you expect to be hearts? Explain.
    c. Suppose we pull a card from the deck without looking first, then look at the card and record the result. Suppose we repeat this process 280 times. About how many times should we expect to draw clubs? Explain your answer.

1.	a.	11	b.	$\frac{5}{8}$	c.	$\frac{1}{6}$	d.	12	e.	$\frac{1}{2}$	f.	$5\frac{5}{8}$
2.	a.	16	b.	20	c.	2	d.	1	e.	6	f.	6

3. 5 : 6

4. 4 : 3

5. 11 : 19

6. 19 : 8

7.	a.	14	b.	12	c.	24	d.	9	e.	8	f.	21

REVIEW PROBLEMS

1. Compute the following products. Simplify if needed.

   a. $\frac{1}{3} \cdot 33$ b. $\frac{5}{6} \cdot \frac{6}{8}$ c. $\frac{2}{3} \cdot \frac{1}{4}$

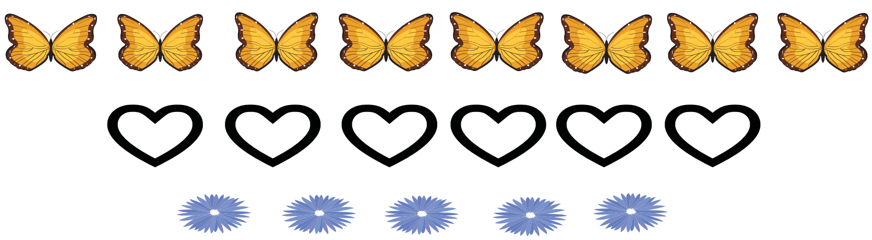
   d. $\frac{3}{5} \cdot 20$ e. $\frac{8}{12} \cdot \frac{3}{4}$ f. $15 \cdot \frac{3}{8}$

2. Compute the following quotients.

   a. $4 \div \frac{1}{4}$ b. $5 \div \frac{1}{4}$ c. $\frac{1}{2} \div \frac{1}{4}$

   d. $\frac{1}{4} \div \frac{1}{4}$ e. $\frac{3}{4} \div \frac{1}{8}$ f. $3 \div \frac{1}{2}$

Use the picture to answer the following ratios problems. Write the ratios as fractions. Simplify if needed.



3. What is the ratio of flowers to hearts?
4. What is the ratio of butterflies to hearts?
5. What is the ratio of hearts and flowers to all?
6. What is the ratio of all to butterflies?
7. Solve the following proportions. (Hint: Use what you know about simplifying fractions to help you.)

   a. $\frac{2}{3} = \frac{x}{21}$ b. $\frac{5}{x} = \frac{20}{48}$ c. $\frac{6}{16} = \frac{9}{x}$

8. 14 people per table

9. 55 students per bus

10. 12 students with glasses

11. 2 out of 15 of the girls like lighter shades of purple better.

12. a. 1 : 4 b. 2:3

13. 12 minutes

14. 85 texts per day.

15. 98 miles

d. $\frac{15}{27} = \frac{5}{x}$ e. $\frac{x}{10} = \frac{20}{25}$ f. $\frac{6}{x} = \frac{4}{14}$

Set up unit rates for the given data. Solve.

8. Twelve tables will seat 168 people. What is the number of people per table?

9. Six buses hold 330 students. What is the number of students per bus?

Solve the following word problems using what you know about ratios and proportions.

10. Three eighths of the students wear glasses. If there are 32 students, how many wear glasses?

11. A survey shows that $\frac{2}{5}$ of girls like purple as their favorite color. $\frac{1}{3}$ of the girls who like purple prefer the lighter shades of purple. What fraction of the girls like lighter shades of purple?

12. Mrs. Fields boxed a dozen cookies. It contained 5 chocolate chip cookies, 3 peanut butter cookies, 2 oatmeal cookies, and 2 sugar cookies.

    a. What ratio of the cookies are peanut butter cookies?

    b. What ratio of the cookies are NOT oatmeal or sugar?

13. Sergio is in a hot dog eating contest. He ate 2 hot dogs in 3 minutes. If he continues eating at this rate, how many minutes will it take him to eat 8 hot dogs?

14. Lorianne checked the number of texts she sent in one week and found it to be 595. If she sent the same number of texts each day of the week, how many texts did she send each day?

15. Jerry's car averages 14 miles per gallon when he drives around the city. If he has 7 gallons of gas in his car, how many miles will her be able to travel around his city?

16. 120 animals

17. 84 marbles

18. 16 days

19. 12 scoops

16. At the animal shelter, 6 workers care for 180 animals. How many animals can be cared for by 4 workers?

17. April found a box of red and blue marbles. The ratio of blue marbles to all the marbles is 4:7. If the box has 48 blue marbles, how many total marbles are in the box?

18. The latest book by my favorite author has 240 pages. If I read 15 pages each day, how many days will it take me to read the entire book?

19. Nicole scoops out $\frac{3}{4}$ cups of potting soil from a big bag containing 9 cups of soil. How many scoops will she have?

[redacted]

MATH EXPLORATIONS

Measurement

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Chapter 8 is organized around introducing and understanding measurement in both customary and metric units. A measurement book created over the course of this chapter is a way to organize all the measurement units and for students to display their artistic talents. Whenever possible, students are encouraged to feel the weights and temperatures as well as see lengths and volumes to have references in real-world settings. Section 8.1 develops pictorial models for units of length measurement and methods for converting units within either metric or customary units. Decimals and powers of ten are emphasized in metric conversions along the number line. Dimensional analysis is introduced as a powerful method for converting units. The process used in dimensional analysis reinforces proportions and multiplication of fractions that were introduced in the previous chapter. Section 8.2 focuses on understanding units of capacity and volume measures. Conversions of metric units use the number line and decimals that were introduced in the Length measurement. Dimensional analysis and proportions are used to perform customary conversions. Section 8.3 extends conversions of units to weight and mass measurements. Finally, section 8.4 examines elapsed time and temperature units in Celsius and Fahrenheit. Benchmark values are used for reasonable temperatures.

Section 8.1 – Measurement Concepts - Length

Big Idea:
Understanding units of measuring length (customary and metric)

Key Objectives:
- Developing pictorial models of each unit of length measurements
- Convert with the customary units of length measurement
- Convert with the metric units of length measurement

Materials:
Rulers, colors, construction paper, measuring tape, yard stick, meter stick.**Pedagogical/Orchestration:**

- Conversions in the customary units will be done through setting up proportions.
- Conversions in the metric units can be done using proportions or examining the patterns in powers of ten.

Activity:

"Measurement Book"

Vocabulary:
Length, inch, foot, yard, mile, meter, centimeter, millimeter, kilometer, customary units, metric units, dimensional analysis

TEKS:
6.8(A)(B)(D); 6.11(A-D); 6.12(A) New: 6.4(G, H); 6.5(A, C)

Launch for Section 8.1:
Pose the following questions to the class and record their answers:
- "If you were going to measure the length of the desktop, how would you describe the units you would use?"

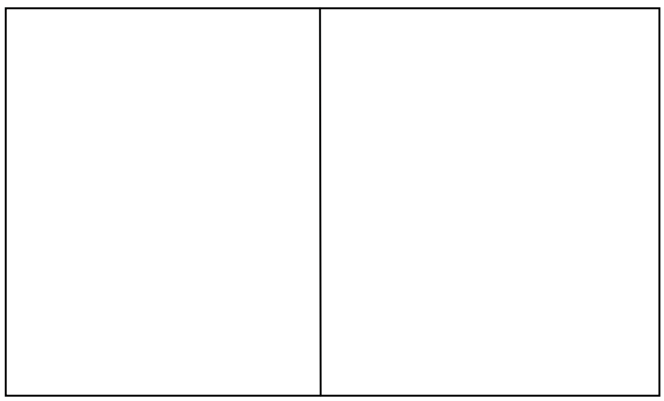
Measurement Book: (To be constructed over the course of the chapter):

Materials:

Construction paper, marker, colored pencils, rulers, mathematics chart

Activity Instructions:

1. Before beginning this measurement book activity, prepare the blank books as follows: Staple one page of construction paper as the cover with 3 or 4 white pages included within.
2. Students will design the cover of their books and create a table of contents. Table of contents can be done during the process or at the conclusion of the chapter.
3. Direct students into designing the length pages as follows:



Customary Length (left page) Metric Length (right page).

Each page contains:
Definition of Length; Student illustrations for each unit of measure with label and abbreviations; Conversion facts within the system; Example of converting using proportions
Variations include:
Illustrations and labels on the left page and conversions with practice on the right page for each system.]

4. The contents of the book include: Length (customary and metric), Weight/Mass (customary and metric), Capacity (customary and metric), Time, Temperature
5. A variation for capacity is Big "G".

Have students discuss what would be important in measuring. Important aspects of measurement to include are: what attribute are we measuring (e.g. weight versus height); choice of appropriate units (e.g. tons versus miles); numerical value.

EXPLORATION

Discuss with students what are some possible units; Inches, feet, centimeter, and meter are appropriate. Miles, kilometers, millimeters are not as appropriate but could certainly be used. Why would they not be as appropriate?

Inches, feet, and miles are the common units that students should know. You may wish to mention that there are other units that students may see in other disciplines such as light years in science, nautical miles in sailing and navigation, furlongs in horse racing, etc. Talk about human benchmarks and what is approximately an inch-between two joints on your finger, a yard-measuring from finger tip to nose, a foot is approximately the length of an adult foot

Students may mention rulers, yardsticks, odometer on a car.

MEASUREMENT 8

SECTION 8.1 LENGTH

What do you notice is common to the following questions?

- How far is your home from school?
- How tall are you?
- What is the record high temperature in San Marcos for today?
- How much water do you consume in one day?
- How much did you weigh when you were born?
- How much time until lunch?

You probably observed that they all involve measurements of some type.

Whether you are measuring distance, height, temperature, capacity, weight, or time there are several important concepts to keep in mind. First, you must determine what is being measured. For example, if you want to know how tall you were when you were born, you would recognize that you are referring to measuring a length and not weight, capacity, or time. Second, you must determine what unit or possibly units are appropriate for measuring the height of a baby. And finally, you must get the actual numerical value for the measurement in the unit chosen.

Throughout this chapter, we will look at two systems of measurement: **customary units** and **metric units**. Customary units are what we in the US use most often though the rest of the world, with a few exceptions, uses metric units.

EXPLORATION

What are the customary units you use for lengths and distances? How do you measure length? Use an appropriate measuring devise such as a ruler, meter stick, measuring tape, or other available devices to measure five different objects using five different units. Explain why you chose the particular unit.

Other 1's are 12 inches/1 foot, 1mile/1760 yards, 5280 feet/1 mile etc.

Inches may be a good choice for the height of a newborn baby, but not as good a choice for a running race distance. Here are some useful conversions in customary units:

LENGTH
Customary

1 mile = 1760 yards
1 mile = 5280 feet
1 yard = 3 feet
1 foot = 12 inches

If a person runs 8 miles, what is this distance in yards? One way to approach this problem is to use the concept of proportions from the last chapter. Let x represent this distance in yards. One form of this proportion could look like this:

$$\frac{x}{8\,\text{miles}} = \frac{1760\,\text{yds}}{1\,\text{mile}}$$

Solving for x, we then have, x = $8 \cdot \frac{1760}{1}$ = 8 · 1760 = 14,080 yards. Another way to think of this problem is to use the idea of unit rates and notice that there are 1760 yards per mile or 1760 $\frac{yds}{mile}$. If there are 1760 yards in each mile then in 8 miles there must be 8 · 1760 = 14,080 yards. We see there are two ways to approach this problem of converting 8 miles to yards.

We introduce another powerful process for converting units called **dimensional analysis**. This process is based on equivalent forms of measurement. Recall that a fraction of the form $\frac{n}{n}$ = 1, with n a non-zero number. You would easily recognize that $\frac{15}{15}$ = 1. Now consider $\frac{1\text{ foot}}{12\text{ inches}}$. Because 1 foot = 12 inches, then a non-zero quantity in one unit divided by an equal non-zero quantity in another unit must equal 1. What is important here is that the units must be indicated every time. Remember, we are not claiming that in general, $\frac{1}{12}$ = 1. Units are important!

Create other 1's using the equivalent forms above.

PROBLEM 1

a. 2 ft b. 144 inches c. 15,840 ft d. $3\frac{7}{9}$ e. 4 mi f. 54 ft

Some typical equivalents with the corresponding conversion rates equal to 1 are:

3 feet = 1 yard which means $\frac{3\text{ ft}}{1\text{yd}} = 1 = \frac{1\text{ yd}}{3\text{ ft}}$

and 60 minutes = 1 hour means $\frac{60\text{ min}}{1\text{ hr}} = 1 = \frac{1\text{ hr}}{60\text{ min}}$

Rewriting a measurement from one unit to another is much easier using dimensional analysis.

EXAMPLE 1

Convert 31,680 inches to feet.

SOLUTION

Inches and feet are related by the equivalent forms 1 foot = 12 inches. Use this equivalence to write the number 1 in two ways, $\frac{1\text{ foot}}{12\text{ inches}}$ and $\frac{12\text{ inches}}{1\text{ foot}}$. The problem is to convert 31,680 inches to feet. Use the $\frac{1\text{ foot}}{12\text{ inches}}$ form of 1, where the numerator contains the feet unit to which you want to convert. Set up the problem: 31,680 inches $\cdot \frac{1\text{ foot}}{12\text{ inches}}$.

Multiplying the numerical part of the fractions gives us $\frac{31680}{12} = 2640$.

What unit is this? Notice that we have inches $\cdot \frac{\text{feet}}{\text{inches}}$. Treating these expressions as we would fractions, $\frac{\text{inches} \times \text{feet}}{\text{inches}}$ simplifies to feet. The units for 2640 is feet. Therefore, 31,680 inches = 2640 feet.

PROBLEM 1

Convert the following measurements. Make sure to set up equivalent units correctly.

a. 24 inches = _____ ft

b. 12 ft = _____ inches

c. 3 mi = _____ ft

d. 136 inches = _____ yards

e. 7040 yd = _____ mi

f. 18 yd = _____ ft

PROBLEM 2

a. $2640 \text{ feet} \cdot \dfrac{1 \text{ mile}}{5280 \text{ feet}} = \dfrac{1}{2} \text{ mi}$

PROBLEM 2

Convert 2640 feet to miles.

Another system for measuring length is the metric system. This system is a base 10 or decimal system. Your knowledge of decimals will be very useful in the metric system. The base unit of length in the metric system is the meter. Prefixes are used with the base to create larger and smaller units.

Prefixes commonly used in order from larger units to smaller units are:

Kilo	Hecto	Deka	(base)	Deci	Centi	Milli
k	h	da		d	c	m

The unit to the left of a given unit is 10 times larger than the one to its right.

In the metric system, the base unit for length is given by **meter.** The table of units from larger units to smaller looks like the following:

Metric Units for Length

Kilometer km	Hectometer hm	Dekameter dam	Meter (base) m	Decimeter dm	Centimeter cm	Millimeter mm

The chart above also includes the abbreviation used for the units.

For example, 10 millimeters = 1 centimeter, 10 centimeters = 1 decimeter, 10 decimeter = 1 meter, and so on.

Here are some ways to connect metric units of measure to common references.

A meter is a little longer than a yard.

A decimeter is about the width of a hand.

A centimeter is about the width across ones fingernail.

A millimeter is about the thickness of ones fingernail.

You can think of a kilometer as about the length of 10 football fields. Use 1 km = 1000 m to convert 17 km to meters.

By dimensional analysis, use $\frac{1000m}{1km} = 1$ and multiply this by 17 km.

$$17 \text{ km} \times \frac{1000m}{1km} = 17 \times 1000 \text{ m} = 17000 \text{ m}.$$

How does this differ from the problem to convert 17 m to kilometers?

PROBLEM 3

12 m

PROBLEM 4

a. 32.5 m b. 400,000 cm c. 0.14 km d. 0.032 m e. 500,000 cm

EXERCISES

1. a. 120

I) By Proportion $\frac{x \text{ dm}}{12 \text{ m}} = \frac{10 \text{ dm}}{1 \text{ m}}$ so, $x = 12 \text{ m} \cdot \frac{10 \text{ dm}}{1 \text{ m}} = 120 \text{ dm}$

ii) $\frac{10 \text{ dm}}{1 \text{ m}}$ is a unit rate per meter. For 12 meters we multiply $12 \cdot \frac{10}{1} = 120$ m

iii) Dimensional Analysis $12 \text{ m} \cdot \frac{10 \text{ dm}}{1 \text{ m}} = 120 \text{ dm}$

Similar solutions for parts b., c., and d.

b. 1200

c. 12000

d. 0.012

Notice how this is related to the place value chart and converting from km to m.

| Thousands | Hundreds | Tens | Ones | . | Tenths | Hundredths | Thousandths

The metric system is based on our decimal system. Can you see that just as our number system gets larger by a factor of 10 as our units increase, the metric system increases by a factor of 10 as we move to the left on the prefixes?

Some useful metric conversions include:

LENGTH
Metric

1 kilometer = 1000 meters
1 meter = 100 centimeters
1 centimeter = 10 millimeters

PROBLEM 3

Henry measured the length of his patio and found it to be 1200 cm long. He went to purchase an outdoor rug to cover the length of the patio. He noticed it was sold in meters. How long is his patio in meters? **12 m**

PROBLEM 4

Convert the following metric measurements. Make sure you set up equivalent units correctly:

a. 3250 cm = ______ m
b. 4000 m = ______cm
c. 140,000 mm = ______ km
d. 32 mm = ______ m
e. 5 km = ______cm

EXERCISES

1. Convert 12 meters to the indicated units using each of the following methods: a) proportions, b) unit rate, and c) dimensional analysis:
   a. decimeters **120**
   b. centimeters **1200**
   c. millimeters **12000**
   d. kilometers **0.012**
2. Convert 12 feet to the indicated units using each of the following methods:

2. a. 144 in
   b. 4 yrds
   c. 1/440 miles

3. 71 inches

4. She walked 4 miles total. So, we convert 4 miles to yards as follows:
   4 miles = (4 . 5280) feet = $\frac{1}{3}$ (4 . 5280) yards = 7040 yards

5. 2.538 km

6. The distance between the towns in centimeters is 800,000 centimeters. We converted 8 kilometers to centimeters as follows: 8 km = 8000 m = 800,000 cm

7. 138 cm = 1.38 m. No, the tablecloth won't be long enough for the entire table because it is 1 m = 100 cm in length while the table is 138 cm long. So, it would be 38 cm short.

8. 3.5 mi
   Converting feet to miles. 18,480 miles = 18,480 miles x $\frac{1 \text{ mile}}{5280 \text{ feet}}$ = 3.5 miles

9. a. approx. 1 : 2 b. 4.6 cm : 9.4 cm, which also approx. 1:2. The ratios are the same.

10. 0.0034 km

i) proportions, ii) unit rate, iii) dimensional analysis:

a. inches b. yards c. miles

3. Coach Rodriguez is 5 feet 11 inches tall. How tall is he in inches?

4. Kassandra walked 1 mile on Monday, 1 mile on Tuesday and 2 miles on Wednesday. How many yards did she walk?

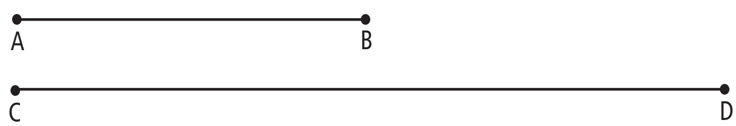
5. Luis lives 2538 meters away from Bob. How many kilometers does Luis live away from Bob?

6. The distance from one town to another is 8 kilometers. How many centimeters is this distance?

7. The length of a table is 138 centimeters. The length of a tablecloth is 1 meter long. Will the tablecloth be long enough to cover the entire top of the table? Explain your reasoning.

8. Emily ran a total of 18,480 feet in one week. How many total miles is this?

9. Measure the lengths of the two line segments AB and CD using inches and then centimeters.



a. Determine the ratio of AB to CD using customary units.

b. Determine the ratio of AB to CD using metric units. Compare the two ratios. What do you notice? Explain any difference or similarities in the two ratios.

10. Convert 3400 mm to kilometers.

Ingenuity

11. The mark shown in the picture is a sixteenth mark, so one of the larger marks immediately to the left or right of it must be an eighth mark. The one on the left is smaller than the one on the right, so the one on the left must be an eighth mark. By the same reasoning, one of the marks on the right or left that is larger than the eighth mark must be a quarter mark; it must be the one on the right, because it is smaller than the one on the left. This means that the mark on the far left must either be a whole-inch mark or a half-inch mark; however, we cannot see enough of the ruler to know which it is. If it is a whole-inch mark, then it must be the 7″ mark, and the mark shown must be the 7 3/16″ mark. If it is the 7 1/2″ mark, then the mark shown must be the 7 11/16″ mark.

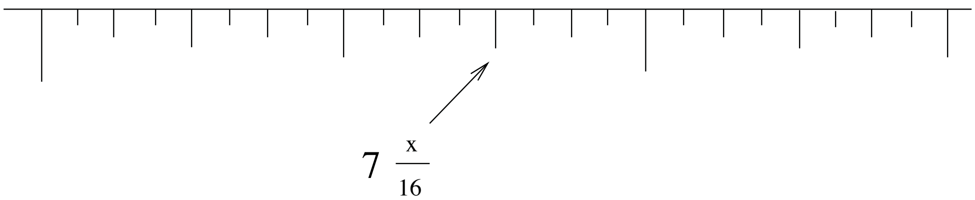
Investigation

12. Here is a list, with notes, in order from longest to shortest.

Parsec - a unit of length used to measure distances in astronomy. Equal to about 3.26 light-years.
Light-year - the distance light travels in one year. Equal to about 300 million meters.
League - a unit used to describe the length a person or horse can walk in an hour. Definitions vary, but the most commonly accepted definition was that a league is 3 miles.
Nautical mile - the distance one would have to walk to cover one minute of a degree of latitude, if walking along a meridian. A little more than one mile.
Furlong - said to be the distance a team of oxen could plough without resting. Equal to one-eighth of a mile.
Micron - a micrometer, or one millionth of a meter.
Angstrom - a unit used to describe sizes of atoms and chemical bonds. Equal to one ten-billionth of a meter.

11. **Ingenuity:**

Shown below is a small part of the edge of a ruler. If the line indicated below is the $7\frac{x}{16}$ inch mark on the ruler, what are the possible values of x?



12. **Investigation:**

Look up the following units of length and put them in order from longest to shortest. Explain where each unit of length is (or was) used in real life.

angstrom, furlong, league, light-year, micron, nautical mile, parsec

Section 8.2 - Measurement Concepts - Capacity and Volume

Big Idea:
Understanding units of measuring capacity and volume (customary and metric)

Key Objectives:
- Developing pictorial models of each unit of capacity/volume measurements
- Convert with the customary units of capacity/volume measurement
- Convert with the metric units of capacity/volume measurement

Materials:
Measuring cups; 1, 2, and 3 liter soda containers; 1 pint, quart, gallon milk containers.

Pedagogical/Orchestration:
- Conversions in the customary units will be done through setting up proportions.
- Conversions in the metric units can be done using proportions or examining the patterns in powers of ten.

Activities:
"Measurement Book", "Big G, Mr. Gallon Man."

Internet Resource:
http://www.theteacherwebsite.com/mrgallonmanproject-tools.pdf

Vocabulary:
Capacity, volume, liter, milliliter, gallon, quart, pint, cup, fluid ounces

TEKS:
6.8(A,B,D); 6.11(A,B,C,D); 6.12(A) New: 6.4(G, H); 6.5(A, C)

Launch for Section 8.2:

Pose the following questions to the class and record their answers:

- "If you were going to measure the capacity of a car gas tank, how would you describe the units you would use?"
- "What about the amount of milk in your cereal?"
- "What about the amount of soda in a big plastic container?"
- "What about the amount of space in a rectangular room?"

Measurement Book Activity:

(Build on the length pages to now include capacity/volume.):

Big "G": Draw big G. Inside the G write 4 Qs; inside each Q write 2 Ps; inside each P write two Cs as shown. Use images of the gallon man template.

Here’s an internet resource for Mr. Gallon Man Project

http://www.theteacherwebsite.com/mrgallonmanproject-tools.pdf

Do the Mr. G activity available on the internet resource.

http://www.theteacherwebsite.com/mrgallonmanproject-tools.pdf

SECTION 8.2 CAPACITY AND VOLUME

The measure of the space in a container is called the **volume** of the container. We will study volumes of familiar shapes such as cubes more carefully in chapter 9. You will see that cubic units are used to measure volume. In this section, we will explore the capacity of a container. **Capacity** refers to how much liquid a container can hold. Our examples are often in liquid form because it will easily conform to any shaped container. The customary units for capacity in decreasing order are gallons, quarts, pints, cups, and fluid ounces.

EXPLORATION:

Examine the household items that you brought from home to determine which of the following customary units would be most useful for measuring their capacity:

Gallons, quarts, pints, cups and fluid ounces.

Explain why you chose those units.

Record your prediction for the capacities of each item that you brought using two different units. Use a measuring cup and water to record and confirm your predictions.

We summarize the relationships among the customary units as follows:

CAPACITY
Customary

1 gallon	=	4 quarts (qt.)
1 quart	=	2 pints (pt.)
1 pint	=	2 cups
1 cup	=	8 fluid ounces (fl. oz.)

EXAMPLE 1

Isabelle is making soup that requires 3 quarts of water. She only has a 1-cup measuring cup. How many cups of water will Isabelle need in order to make this soup?

PROBLEM 1

$3\frac{1}{8}$ quarts

Kiloliters, hectoliters, decaliters are larger units. Deciliters, centiliters, and milliliters are smaller units.

SOLUTION

Recall that quarts can be converted to pints using $\frac{2\text{ pints}}{1\text{ quart}}$. Pints can then be converted to cups using $\frac{2\text{ cups}}{1\text{ pint}}$. Using dimensional analysis, you can set up the problem as follows:

$3\text{ qt} \cdot \frac{2\text{ pt}}{1\text{ qt}} \cdot \frac{2\text{ cups}}{1\text{ pint}} = (3 \cdot 2 \cdot 2)\text{ cups} = 12\text{ cups}.$

PROBLEM 1

Convert 100 fluid ounces to quarts.

EXAMPLE 2

Determine what fractional part

a. 1 quart is to 1 gallon

b. 1 pint is to 1 gallon

SOLUTION

a. 1 gallon = 4 quarts
   1 quart then must equal $\frac{1}{4}$ of a gallon.

b. 2 pints = 1 quart. Therefore, 1 pint = $\frac{1}{2}$ quart.
   4 quarts = 1 gallon. Therefore, 1 quart = $\frac{1}{4}$ gallon.
   1 pint = $\frac{1}{2}$ of $\frac{1}{4}$ = $\frac{1}{8}$ gallon. We can use dimensional analysis to see this: $1\text{ pt} \cdot \frac{1\text{ qt}}{2\text{ pt}} \cdot \frac{1\text{ gal}}{4\text{ qt}} = \frac{1}{8}\text{ gal}.$

The metric system uses **liters** as a base for capacity. Just as with length measures, the prefixes will create larger or smaller units based on liters.

Write three other units that are larger than liters and three other units that are smaller.

The units with liter (L) as base:

Metric unit for Capacity

Kiloliter kl	Hectoliter hl	Dekaliter dal	Liter (base) l	Deciliter dl	Centiliter cl	Milliliter ml

PROBLEM 2

0.350 L

PROBLEM 3

a. 1 cup is $\frac{1}{16}$ of a gallon b. 1 mL is $\frac{1}{1000}$ of a L

EXERCISES

1. a. 3000mL
   b. 5.340 L
   c. 40 oz
   d. 80 fl. oz
   e. 160 fl. oz
   f. 640 fl. oz

2. 5500 ml

The units that are used most often are the liter and milliliter. Notice that the relationship between the two units is 1 liter = 1000 milliliters, which can also be written as, 1 milliliter = $\frac{1}{1000}$ liter.

EXAMPLE 3

A camel drinks 20 liters of water a day. How many milliliters does this equal?

SOLUTION

1 liter is equal to 1000 milliliters. 20 liters must equal 20000 milliliters. Using dimensional analysis, we can set up the problem as follows: 20 liter $\cdot \frac{1000 \text{ mL}}{1 \text{ liter}}$

PROBLEM 2

Convert 350 ml to liters.

PROBLEM 3

Determine what fractional part

a. 1 cup is to 1 gallon

b. 1 milliliter is to 1 liter

EXERCISES

1. Convert each of the following:
   a. 3 liters = ______________milliliters
   b. 5340 ml = ______________liters
   c. 5 cups = ______________ fluid ounces
   d. 5 pints = _______________fluid ounces
   e. 5 quarts = ______________fluid ounces
   f. 5 gallons = _____________ fluid ounces

2. Jeremy measured the amount of sports drink in the cooler and found it contained 6500 ml. If the container originally held 12 L, how much sports drink had been dispensed?

3. 41 qt = $10\frac{1}{4}$ gal. Yes, 41 qt will fill the 10 gallon container because 41 qt is more than 10 gallons.

4. 8 qt

5. 32 qt

6. 48

7. 192 cups

8. 7000mL

9. Amy can make 2 cheesecakes. She won't have any sour cream left for the stroganoff.

10. 1 gal = 128 fl oz. Nathan drinks 64 fl oz less than a gallon.

11. **Ingenuity:**
By following the steps shown below, Dee can put exactly 5 gallons of water in the 7-gallon bucket. At each stage, we show how many gallons of water are in each bucket.

Action	**Gallons in small bucket**	**Gallons in large bucket**
Fill the small bucket	3	0
Empty the small bucket into the large one	0	3
Fill the small bucket	3	3
Empty the small bucket into the large one	0	6
Fill the small bucket	3	6
Fill the large bucket using the small bucket	2	7
Empty the large bucket	2	0
Empty the small bucket into the large one	0	2
Fill the small bucket	3	2
Empty the small bucket into the large one	0	5

3. A container can hold 10 gallons of water. There are 41 quarts of water in another container. Will 41 quarts of water fill the 10-gallon container? Explain.

4. 16 pints of paint was left over. How many quarts of paint is this?

5. How many quarts of ice cream are in an 8-gallon container?

6. Jeremy's mom bought 6 gallons of iced tea for a party. She stored all the tea in pint-sized containers. How many containers would she need?

7. Mr. Reyna bought 12 gallons of punch for the Math Camp reception. How many cups will he be able to serve?

8. After a strong storm, Lisa found 7 liters of rain water had been trapped in an empty bucket. How many milliliters of rainwater did she have?

9. Amy is using a cheesecake recipe that calls for 3 cups of sour cream. She has 3 pints of sour cream. How many cheesecakes can she make? How much, if any, sour cream is left for the stroganoff?

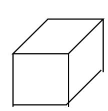
10. Nathan drinks 64 fluid ounces of water after playing badminton. Is this amount more or less than one gallon? How much more or less?

11. **Ingenuity:**

    Dee has two buckets; she knows that the buckets can hold exactly 3 gallons and 7 gallons of water, respectively. She takes the buckets to a stream and wants to measure exactly 5 gallons of water. The buckets do not have any intermediate markings that would allow Dee to know when she has filled a bucket with a certain amount of water. How could Dee put exactly 5 gallons of water in the 7-gallon bucket?

12. **Investigation:**

    One milliliter is equal to one cubic centimeter. That is, if we made a cube with dimensions 1 cm x 1 cm x 1 cm, as shown below, and filled it with water, it would take 1 mL of water to fill the cube.



12. **Investigation:**

(a) One decimeter is equal to 10 centimeters, so a box with dimensions 1 dm x 1 cm x 1 cm is made of ten 1 cm cubes. So it would take 10 mL of water to fill this box.

(b) The box described can be made using ten 1 dm x 1 cm x 1 cm boxes, placed next to one another. So it would take 10 x 10 = 100 mL of water to fill this box.

(c) The cube described can be made using ten 1 dm x 1 dm x 1 cm boxes, stacked on top of each other. So it would take 100 x 10 = 1000 mL of water to fill this box. This is equal to 1 L. So 1 L is the same volume as the volume of a one-decimeter cube.

a. Suppose we made a rectangular box with dimensions 1 dm x 1 cm x 1 cm. How many milliliters of water would it take to fill this box?

b. Suppose we made a rectangular box with dimensions 1 dm x 1 dm x 1 cm. How many milliliters of water would it take to fill this box?

c. Suppose we made a cube with dimensions 1 dm x 1 dm x 1 dm. How many milliliters of water would it take to fill this cube? How many liters is this?

Section 8.3 - Weight and Mass

Big Idea:

Understanding units of measuring weight and mass

Key Objectives:

- Developing pictorial models of each unit of weight/mass measurements
- Convert with the customary units of weight/mass measurement
- Convert with the metric units of weight/mass measurement

Materials:

Construction paper, marker, colored pencils, rulers, mathematics chart, bathroom scale, postage or kitchen scale using metric, if possible.

Pedagogical/Orchestration:

- Conversions in the customary units will be done through setting up proportions.
- Conversions in the metric units can be done using proportions or examining the patterns in powers of ten.

Activities:

"Measurement Book"

Internet Resources:

Use the following resources as a review for sections 8.1-8.3:

Matching game to review customary units: http://www.quia.com/jg/65838.html

Rags to riches game to review metric units: http://www.quia.com/rr/96635.html

Vocabulary:

Weight, mass, grams, pounds, ounces, tons

TEKS:

6.8(A,B,D); 6.11(A,B,C,D); 6.12(A) New: 6.4(G, H); 6.5(A, C)

Launch for Section 8.3:

Pose the following questions to the class and record their answers:

- "If you were going to measure the amount of steak to buy at the store, what units would you use?"
- "What about the weight of your family car?"
- "What about the amount of metal used in a ring?"
- "What about your weight?"

What is mass? What is weight? In our class, we will use the concept of weight to be a measure of mass. The discussion should note that while weight is dependent on gravity, mass is not. This will help integrate science concepts.

Measurement Book Activity:

(Build on the other pages to now include weight and mass.)

Students may wish to explore the distinction between mass and weight in their science class. Have students recall units of measure for weight.

Have some everyday objects that give a sense of ounces –can of vegetables, pounds-spaghetti noodles, picture of objects measured in tons such as blue whale, elephant, car.

The ounce, as a unit, is used in several contexts and the numerical equivalences are different. Recall that fluid ounces are used in capacity with 8 fluid ounces = 1 cup. In weight, ounces and pounds are related but not always in the same way. In avoirdupois ounces, the most common ounces, 16 ounces (oz) = 1 pound (lb). Students may want to investigate different ounces and find that there are: international troy ounces, used to measure precious metals such as gold (1 troy pound = 12 troy ounces); apothecaries ounces; Maria Theresa ounces and metric ounces. In other words, ounces are not standardized.

SECTION 8.3 WEIGHT AND MASS

How heavy is an object? This question refers to the **weight** of an object. We often refer to the **mass** of an object in terms of weight but we should note that there is a scientific distinction.

The customary units for weight in decreasing order are **tons**, **pounds**, and **ounces**.

EXPLORATION:
Examine the items that you brought from home and determine the mass and weight of each item. Feel how heavy each item weighs. Find two items in the classroom and make an educated guess about their weights. (Teachers should provide a scale, both a bathroom scale and a smaller food scale would allow a good range of weight measures.)
Here are some useful relationships among the customary:

MASS AND WEIGHT Customary
1 ton = 2000 pounds (lbs.) 1 pound = 16 ounces (avoirdupois) (oz.)

EXAMPLE 1

A female African elephant weighs approximately 7900 pounds. Convert the weight to tons using two different methods, for example the dimensional analysis method, unit rate method, or the proportions method.

SOLUTION

We use the equivalence of 1 ton = 2000 lbs and write 1 in the form of the fraction $\frac{1 \text{ ton}}{2000 \text{ lbs.}}$. The problem can then be solved using 7900 lbs $\cdot$ $\frac{1 \text{ ton}}{2000 \text{ lbs.}}$ = $\frac{7900}{2000}$ tons. Doing the division $\frac{79}{20}$ = 3.95 tons or a little less than 4 tons.

The above uses the dimensional analysis method. In the unit rate method we see that 2000 pounds is equivalent to 1 ton or $\frac{1 \text{ ton}}{2000 \text{ lbs.}} = 0.0005 \frac{\text{ton}}{\text{pound}}$. Then in 7900 pounds, there must be 7900 $\cdot$ 0.0005 = 3.95 tons.

Finally, the proportion method introduces the variable x to represent the unknown number of tons 7900 pounds represents. The proportion can be written as

$$\frac{x}{7900 \text{ pounds}} = \frac{1 \text{ ton}}{2000 \text{ pounds}}$$

PROBLEM 1

$7900 \text{ lbs} \cdot \dfrac{16 \text{ oz}}{1 \text{ lb}} = 126{,}400 \text{ oz}$

or by proportions, we have $\dfrac{x}{7900 \text{ lbs}} = \dfrac{16 \text{ oz}}{1 \text{ lb}}$

$x = 7900 \text{ lbs} \cdot \dfrac{16 \text{ oz}}{1 \text{ lb}} = 126{,}400 \text{ oz}$

Students should note kilograms, hectograms, decagrams for larger units and decigrams, centigrams, milligrams for smaller units.

Solving for x, we have $x = 7900 \cdot \frac{1}{2000} = \frac{7900}{2000} = 3.95$ tons.

PROBLEM 1

Convert the weight of 7900 pounds to ounces.

The metric system uses **grams** as its base for weight measure. The abbreviation for grams is simply the letter g. The prefixes will give us larger and smaller units. What units would be larger weight measures? What units would be the smaller weight measures?

In the table below are the metric units for weight with gram as base:

Metric unit for Weight

Kilogram kg	Hectogram hg	Dekagram Dg	Gram base g	Decigram dg	Centigram cg	Milligram mg

Do you know what weighs 1 gram (g)? A cubic centimeter of water weighs 1 gram. A paper clip is approximately 1 gram. In other words, a gram weighs very little. 450 grams is approximately equal to one pound. 1 kilogram is approximately equal to 2.2 pounds.

EXAMPLE 2

A female Asian elephant weighs approximately 3000 kilograms. Convert the weight to grams.

SOLUTION

We use the equivalence, 1 kilogram = 1000 grams and write 1 in the form of the fraction $\frac{1000\text{ g}}{1\text{ kg}}$, a unit rate, because we are changing to grams. The problem can be solved using $3000\text{ kg} \cdot \frac{1000\text{ g}}{1\text{ kg}} = 3000 \cdot 1000\text{ g} = 3{,}000{,}000\text{ g}$.

PROBLEM 2

$$300 \text{ mg} \cdot \frac{1 \text{ g}}{1000 \text{ mg}} = \frac{3}{10} \text{ g} = 0.3 \text{ g}$$

1. Answers will vary
2. 124 oz
3. 4000 g
4. 550,000 mg
5. 144 oz
6. 7000 g
7. 1.25 lb
8. Mr. Reyes can lift more weight becuase is can lift 72 kg = 72,000 grams, while Mr. Saenz can lift 68, 790 grams
9. a. 0.1 tons
   b. 3200 oz
10. 3 tons = 6000 lbs. So the car weighs more than the elephant since 6000 lbs < 8000 lbs.

PROBLEM 2

Convert 300 milligrams to grams.

EXERCISES

1. Give an example of an object that can be measured in each of the following units:
   a. grams b. liters
   c. millimeters d. kiloliters
2. Ms. Voigt's baby weighed 7 pounds and 12 ounces when the baby was born. How many ounces did the baby weigh?
3. A bag of potatoes weighs 8 kilograms. How many grams of potatoes would half the bag weigh?
4. Isaac is baking a cake that requires 550 grams of sugar. How many milligrams of sugar does the cake require?
5. Jacob bought a watermelon that weighed 9 pounds. How many ounces does this equal?
6. Stephen was carrying his violin case onto his flight to Oregon. The flight attendant said the case weighed 7 kilograms. How many grams does Stephen's violin case weigh?
7. Aaron buys 20 ounces of sliced baked ham from the deli counter. How many pounds of ham did Aaron buy?
8. Mr. Reyes can lift 72 kilograms of weight. Mr. Saenz can lift 68,790 grams of weight. Who can lift more weight, Mr. Saenz or Mr. Reyes? Explain your answer.
9. David can lift 200 pounds of weight.
   a. How many tons can he lift?
   b. How many ounces can he lift?
10. A young elephant weighs 3 tons. A car weighs 8000 pounds. Which weighs more: the car or the elephant? Explain your reasoning.

11. **Ingenuity:**

Yes; in fact, Roy can do it with only two weighings. Label the coins with the letters of the alphabet from A to I. Then on the first weighing, Roy should place coins A, B, and C in one pan, and D, E, and F in the other pan. If the first pan is lighter, then A, B, or C must be the counterfeit coin. If the other pan is lighter, then D, E, or F must be the counterfeit coin. If both pans weigh the same, then G, H, or I must be the counterfeit coin. Now that Roy has narrowed his search down to three coins, he can place two of them on the balance scale (one on each pan) and see if one of them is lighter. If neither is lighter, then the coin he did not place on the scale is the counterfeit coin.

12. **Investigation:**

(a) One kilogram weighs the most on Jupiter, followed (not very closely) by Neptune and Earth. One kilogram weighs the least on Mercury and Mars. The planets where one kilogram weighs the least seem to be among the smallest planets in our solar system (in fact, if Pluto had still been in our solar system, we would have seen that one kilogram weighs even less there). One kilogram weighs by far the most on Jupiter, which is the largest planet in our solar system. However, Saturn is also very large, and one kilogram does not weigh nearly as much there. So this theory seems to help explain the differences somewhat, but does not explain them completely. (The problem is that when we talk about the size of a planet, we are usually thinking about the radius of the planet - the size that we see. The mass of a planet is also very important, and Jupiter is much more massive than Saturn.)

(b) One kilogram on earth weighs 2.20 lb., so 5 kilograms will weigh 5 x 2.20 lb. = 11.0 lb. On Jupiter, 5 kilograms would weigh 5 x 5.21 lb. = 26.05 lb.

(c) If a man on Earth weighs 150 pounds, then his mass is 150 [divided by] 2.2 [approximately equals] 68.2 kg. Then on Mars, he would weigh about 68.2 x 0.83 = 56.6 lb.

11. **Ingenuity:**

Roy has nine coins, one of which is counterfeit and slightly lighter than the others. Roy doesn't remember which coin is the counterfeit coin, but he has a balance scale that he can use to test the coins. Roy can put some of his coins on the scale, and it will tell him whether one side is lighter than the other. Is it possible for Roy to figure out which coin is the counterfeit coin by using his balance scale three or fewer times?

12. **Investigation:**

Recall that mass and weight are two different concepts. The mass of an object is the same no matter where the object is, but the weight of an object depends on how strong gravity is where the object is located. The following chart shows the weight of a one-kilogram mass on each of the eight planets in our solar system.

Planet	Weight of 1 kg
Mercury	0.83 lb.
Venus	2.00 lb.
Earth	2.20 lb.
Mars	0.83 lb.
Jupiter	5.21 lb.
Saturn	2.02 lb.
Uranus	1.96 lb.
Neptune	2.48 lb.

a. On which planets does one kilogram weigh the most? On which planets does one kilogram weigh the least? Why do you think this is?

b. How much does a 5-kilogram mass weigh on Earth? How much would it weigh if it were transported to Jupiter?

c. A man on Earth weighs 150 pounds. How much would he weigh if he took a trip to Mars?

Section 8.4 - Time and Temperature

Big Idea:
Understanding units of measuring time and temperature

Key Objectives:
- Developing pictorial models of each unit of time and temperature measurements
- Convert units of time using proportions.
- Use elapsed time

Materials:
Analog clock, thermometer with both Celsius and Fahrenheit

Pedagogical/Orchestration:
- Conversions in the customary units will be done through setting up proportions.
- Conversions in the metric units can be done using proportions or examining the patterns in powers of ten.

Activities:
"Measurement Book"

Vocabulary:
Fahrenheit, Celsius, degrees, hours, minutes, seconds, a.m., p.m.

TEKS:
6.8(A,B,D); 6.11(A,B,C,D); 6.12(A) New: 6.4(G, H); 6.5(A, C)

Launch for Section 8.4:

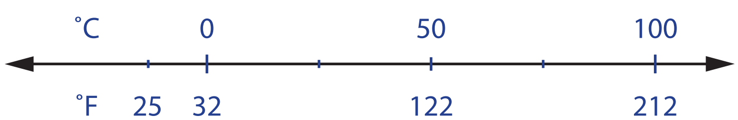
Pose the following questions to the class and record their answers:

- "If you were going to measure the temprature, what units would you use?"
- "What about when you tell the time?"
- "What are different situation which would require you to measure time?"
- "When might you be interested in measuring the temprature?"

Students should be able to come up with different situations from their daily lives which would make measuring the temperature and time necessary. For example, when their mom is baking a cake measuring both temperature and time is important.

Measurement Book Activity:

(Build on the length pages to now include Time and Temperature.)



Students should mention year, days, months, weeks, days, hours, minutes, and seconds. Other larger units include century and millenium

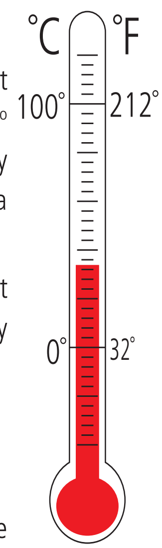
SECTION 8.4 TIME AND TEMPERATURE

Temperature

Does it make sense to say the temperature on a hot day in Texas is 32 degrees? Does 100 degrees make sense? Actually, the answer could be yes or no. How can that be? As you have seen in the previous sections, whenever you measure, you must be careful to always include the units along with the numerical value. For example, a pencil of length 12 inches is very different from a pencil of length 12 feet so saying a pencil is of length 12 is not enough.

The two common units of temperature measure are the **Fahrenheit** and **Celsius** units. Fahrenheit is associated with the customary system while the Celsius is associated with the metric system.

You did an investigation in Section 1.1 regarding the thermometer as part of a number line. Let us recall some important aspects of the two units. The freezing point of water is 32° Fahrenheit and 0° Celsius. The boiling point of water is 212° Fahrenheit and 100° Celsius.



Now 32° in Fahrenheit is not a reasonable temperature on a hot day in Texas. Look on the thermometer to the right. Does 32° Celsius make sense for a hot day? 32° Celsius is approximately equal to 90° Fahrenheit, which is a reasonable temperature in a Texas summer.

What is a reasonable springtime temperature in Texas? What about a cold winter day in Chicago? What is the average body temperature of a healthy person?

Time

What units of time are most familiar to you? What units of time would be appropriate for each of the following instances?

a. What was the runner's time in a 100 meter run?

b. How long is your summer vacation?

c. How much time until dinner?

d. How much time will pass before you turn 21 years old?

PROBLEM 1

a. $\frac{1}{7}$ of a week b. $\frac{1}{365}$ of a year c. $\frac{1}{24}$ of a day d. $\frac{1}{3600}$ of an hour

We do not specify a customary or metric system of time measurement. However, some countries use a 24-hour reading of time while other countries use a 12-hour reading and use a.m. and p.m. to distinguish the morning time from the afternoon time. Generally, a.m. time goes from 12:00 midnight until 12:00 noon and p.m. goes from 12:00 noon until 12:00 midnight. In this book, we will use the 12-hour clock.

Familiar time equivalences include:

TIME		
1 year	=	365 days
1 year	=	12 months
1 year	=	52 weeks
1 week	=	7 days
1 hour	=	60 minutes
1 minute	=	60 seconds

PROBLEM 1

What fraction is:

a. One day of a week?
b. One day of a year?
c. One hour of a day?
d. One second of an hour?

EXAMPLE 1

12 hours and 20 minutes is equal to how many minutes?

SOLUTION

12 hours $\cdot \dfrac{60 \text{ min utes}}{1 \text{ hour}} = 12 \cdot 60$ minutes $= 720$ minutes. So, 12 hours $= 720$ minutes and 12 hours and 20 minutes $= 720$ minutes $+ 20$ minutes $= 740$ minutes.

EXAMPLE 2

100 minutes is equal to how many hours?

SOLUTION

100 minutes $\cdot \dfrac{1 \text{ hour}}{60 \text{ min utes}} = \dfrac{100}{60}$ hours $= \dfrac{5}{3}$ hours $= 1\dfrac{2}{3}$ hours.

PROBLEM 2

a. $\frac{1}{3}$ of an hour b. $\frac{1}{6}$ of an hour c. $\frac{1}{12}$ of an hour

PROBLEM 2

a. 20 minutes is equal to what fraction of an hour?

b. 15 minutes is equal to what fraction of an hour?

c. 5 minutes is equal to what fraction of an hour?

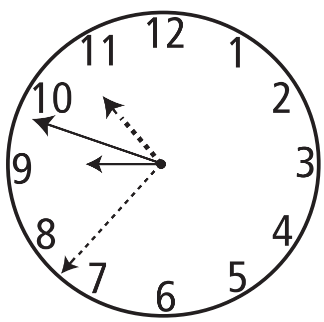
EXAMPLE 3

Ms. Lott's class begins math class at 9:47 a.m. The math classes are 50 minutes long. At what time will Ms. Lott's math class end?

SOLUTION

The time 9:47 is really 9 hours + 47 minutes. The elapsed time is 50 minutes so we add 50 minutes to this time. If we add the minutes, we have 47 + 50 = 97 minutes. 97 minutes = 60 minutes + 37 minutes = 1 hour + 37 minutes.

We can view 9 hours+ (47 + 50) minutes = 9 hours + 1 hour + 37 minutes = 10 hours + 37 minutes or 10:37 a.m.



EXAMPLE 4

Mr. Mungia left McAllen, Texas at 11:28 a.m. on a flight to Chicago, Illinois. His flight arrived in Chicago at 3:05 p.m. How long was his trip?

SOLUTION

One way to answer this question is to add on the time in easy increments. One hour into the trip, the time was 12:28 p.m. Two hours into the trip, the time was 1:28 p.m. Three hours into the trip, the time was 2:28 p.m. Adding another hour will go beyond Mr. Mungia's arrival time so we can add a half an hour or 30 minutes.

EXERCISES

1. a. 5 hours and 10 minutes b. 4 hours and 38 minutes

2. a. 1 hour and 39 minutes b. 1 hour and 35 minutes

3. 18° F

4. 7 hours and 10 minutes

5. 9 hours

Adding 30 minutes to 2:28 p.m. will make the time 2:58 p.m. So far 3 hours and 30 minutes have elapsed. If we add the minutes from 2:58 p.m. to 3:05 p.m., you will see that 7 minutes have elapsed. Ms. Mr. Mungia's trip must have taken 3 hours and 37 minutes.

Another way to answer this question is to subtract the 11 hours and 28 minutes from noon or 12 hours. Then add 3 hours and 5 minutes to that answer. 12 hours – 11 hours and 28 minutes is the same as regrouping the 12 hours into (11 hours + 60 minutes) – (11 hours + 28 minutes) = 60 – 28 = 32 minutes. Add 3 hours and 5 minutes to 32 minutes and we have a total of 3 hours and 37 minutes.

EXERCISES

1. Add

   a.
   ```
     1 hour 38 minutes
   + 3 hours 32 minutes
   ____________________
   ```
   b.
   ```
     3 hour 51 minutes
   +0 hour 47 minutes
   ____________________
   ```

2. Subtract

   a.
   ```
     3 hrs 38 min
   - 1 hr  59 min
   ______________
   ```
   b.
   ```
     5 hrs 16 min
   - 3 hrs 41 min
   ______________
   ```

3. On a hot summer day in McAllen, Texas, the temperature at 10:00 a.m. is 85° F. On a cold winter day in McAllen, the temperature at 10 a.m. is 67° F. Find the difference in the temperature.

4. Josephine left McAllen at 7:35 a.m. and flew to Houston. She then changed planes at 10:30 a.m. and flew to St Louis. She arrived in St. Louis at 2:45 p.m. How long had Josephine been traveling?

5. Ms. Badgett leaves her house at 7:05 a.m. to go to school. She arrives at school at 7:25 a.m. to prepare for the day. She leaves work at 3:45 p.m. and goes home. If she arrives home at 4:05 p.m., how much time has she been away from home that day?

6. 6 hours and 35 minutes

7. 10:44 am

8. 14 - 24 songs

9. 1960 hours

10. 1440 minutes per day, 86400 seconds per day

12. **Investigation:**
Our seasons on earth are determined by the earth's position in its orbit, which takes 365.25 days to complete. If every fourth year is a leap year, then every four calendar years will have 365 x 4 + 1 = 1461 days, which is the same as 365.25 x 4, the number of days in four earth years. If we did not make this adjustment, then our seasons would eventually fall out of alignment with our calendar. Each earth year would take about 1/4 day longer than the corresponding calendar year, so after about years, our seasons would be off by about 25 days.

6. A plane arrived in Austin airport at 11:48 p.m. The same plane left the airport at 6:23 a.m. the next morning. How long was the airplane on the ground at the Austin airport?

7. Nama arrived in San Marcos at 2:14 p.m. She had travelled 3 hours and 30 minutes non-stop from Waco. At what time did she leave Waco?

8. A CD plays for 72 minutes. If each song ranges from 3 minutes to 5 minutes, what is the range of the number of songs that can be on the CD?

9. Mr. Reyna works 40 hours a week. He has three weeks of vacation in a year. How many hours does Mr. Reyna work in a year?

10. Find the number of minutes in one day. Find the number of seconds in one day.

11. **Ingenuity:**

    Now that you have learned about various units in this chapter, create a new unit of measurement of your own. Describe what it can be used to measure, and explain the process of converting it into other appropriate customary or metric units.

12. **Investigation:**

    The earth year lasts approximately 365.25 days; this is the time it takes for the earth to make a full revolution around the sun. Explain why, based on this knowledge, it is logical to have a leap year every fourth year that is one day longer than the typical year.

1.	a. 4 ft	b. 7 lbs	c. 20 qt	d. 9 qt.	e. 480 min.	f. 14,000 lbs.
	g. 6 1/4 ft.	h. 256 oz.	i. .1875 lbs.	j. 12 min	k. 4 mi.	l. 48 oz.
	m. 3/4 T	n. 5.8125 lbs.				

2.	a. 2500	b. 7000	c. 14,000	d. 1.8	e. .175	f. 16,000
	g. 16,000	h. .028	i. .254	j. 4.5	k. .000128	l. 9,000,000

3. 2:45 p.m..

4. 28° F

5. 11:33 a.m.

REVIEW PROBLEMS

1. Convert the following customary units of measure. Refer to the conversion chart.

a.	48 in. = _________ ft.	h.	16 lbs. = _________ oz.
b.	112 0z. = _________ lbs.	i.	3 oz. = _________ lbs.
c.	5 gal. = _________ qt.	j.	720 sec. = _________ min.
d.	18 pt. = _________ qt.	k.	7040 yd. = _________ mi.
e.	8 hrs. = _________ min.	l.	3 pts. = _________ fl.oz.
f.	7 T = _________ lbs.	m.	1500 lbs = _________ T
g.	75 in. = _________ ft.	n.	93 oz. = _________ lbs.

2. Convert the following metric units of measure. Refer to the conversion chart.

a.	250 cm = _________ mm	g.	16 kg = _________ g
b.	7 L = _________ ml	h.	28 m = _________ km
c.	14 km = _________ m	i.	254 ml = _________ L
d.	1800 mg = _________ g	j.	45 mm = _________ cm
e.	175 mm = _________ m	k.	128 mg = _________ kg
f.	16 g = _________ mg	l.	9 km = _________ mm

Solve the following conversions. Remember to label your answers carefully.

3. Frankie left her house at 11:30 a.m. She drove for 3 hrs and 15 min. What time was it when she arrived at her destination?
4. The average temperature for July in Texas is 97 °F while the average temperature in Washington State in July is 69 °F. What is the difference in the average temperatures of the two states?
5. Dr. Shima arrived in New York City at 3:53 p.m. His trip lasted 4 hrs and 20 min. At what time did his trip begin?

6. 141 oz
7. 366 in
8. .65 lbs.
9. 32 pt.
10. 48 cups
11. 2320 oz
12. 1.574 km
13. Yes, mom will have enough fabric. 1 yard = 3 feet. So, 3 yards = 9 feet.
14. .638 g
15. Melissa walked farther
16. 272 cups

Solve the following conversions. Set up proportions whenever possible. Remember to label your answers carefully.

6. Mrs. Preston's baby weighed 8 lbs. 13 oz. when he was born. How many ounces did he weigh at birth?
7. The distance from the front door to the mailbox is 28 feet. What is the distance in inches?
8. A baby hippo weighs 1300 pounds. How many tons does the baby hippo weigh?
9. A pitcher of lemonade holds 4 gallons. How many pints are contained in the pitcher?
10. A recipe calls for 4 cups of flour. How many cups of flour will be needed to repeat this recipe 12 times?
11. Jon can lift 145 pounds. How many ounces can he lift?
12. The library is 1574 meters away from the corner store. If Julie wants to walk from the library to the corner store, how many kilometers will she walk?
13. The new dining room table is 8 feet long. If Mom buys 3 yards of fabric to make a table cloth, will she have enough fabric? What is the difference between the two amounts?
14. A bunch of grapes weigh 638 milligrams. How many grams would the bunch of grapes weigh?
15. Sandra walked 18 yards while Melissa walked 55 feet. Which girl walked farther?
16. For a wedding, the caterer made 68 quarts of gravy for the meal. He needs to store the gravy in cups. How many cups of gravy will he have?

[redacted]

MATH EXPLORATIONS

Geometry

CH. 9 GEOMETRY 357

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Section 9.1 begins with the definition of an angle, the notation for angles, measuring angles using the protractor and identifying and classifying different types of angles. The relationship of perpendicularity is included in this section. Section 9.2 classifies and constructs triangles by the length of sides and also by types of angles. Polygons are discussed more generally in Section 9.3 with quadrilateral and polygons with 4 or more sides. Definition of parallel lines is included in this section. Section 9.4 uses properties of two-dimensional figures to arrive at area formulas for commonly occurring figures as well as explore the area and perimeter of other less common shapes. Section 9.5 develops the formula for the circumference and area of circles and how the constant π plays a role in these formulas. Finally, section 9.6 introduces the vocabulary associated with various 3-dimensional figures including prisms, pyramids and spheres. Volumes of rectangular prisms and their units are included in this section.

Section 9.1 – Measuring Angles

Big Idea:

Students will learn to construct and measure angles.

Key Objectives:

- Measure angles using a protractor.
- Understand how angles are made, using rays and a vertex.
- Learn how to notate angles symbolically.
- Recognize special angles including 180º, 90º, and 45º.
- Define and identify complementary and supplementary angles.

Materials:

Protractor of various sizes, straight edge, rope or yarn

Pedagogical/Orchestration:

- Chapter 9 is a short course in geometry. It contains much more information than the average academic year's requirements. You can treat it like a mathematical cafeteria, choosing what you need and like.
- Working with protractors and straight edges is a useful, too, and, for most students, an interesting skill. Be firm with your students about accuracy and good work habits when drawing and measuring angles.

Activity:

"What's Your Angle?", "Measuring Angles", "Protractor Practice", "Angles All Around Us"

Vocabulary:

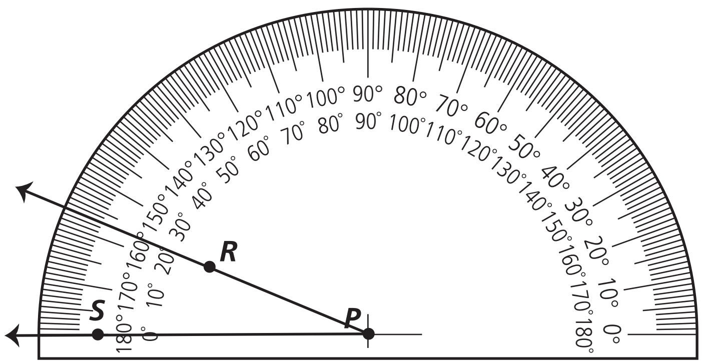
angle, ray, vertex, size of angle, protractor, degrees, straight angle, supplementary, right angle, perpendicular, complementary, acute angle, obtuse angle, measure of angle, supplement, complement

TEKS:

6.6(A); 6.8(C) New: 6.8(A); 6.10(A)

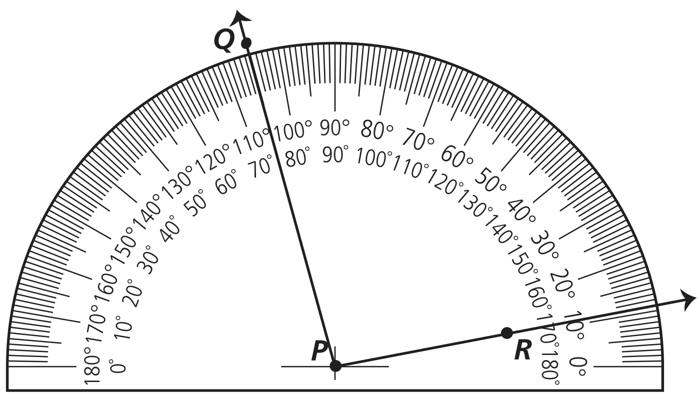
Protractor Practice

1.



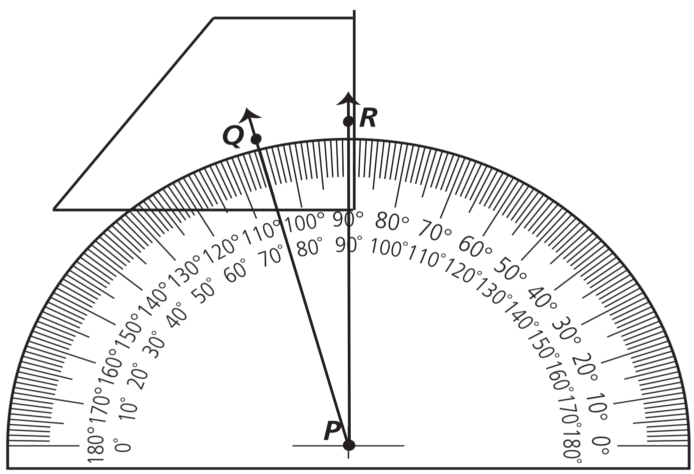
m (∠ SPR) = ___________

2.



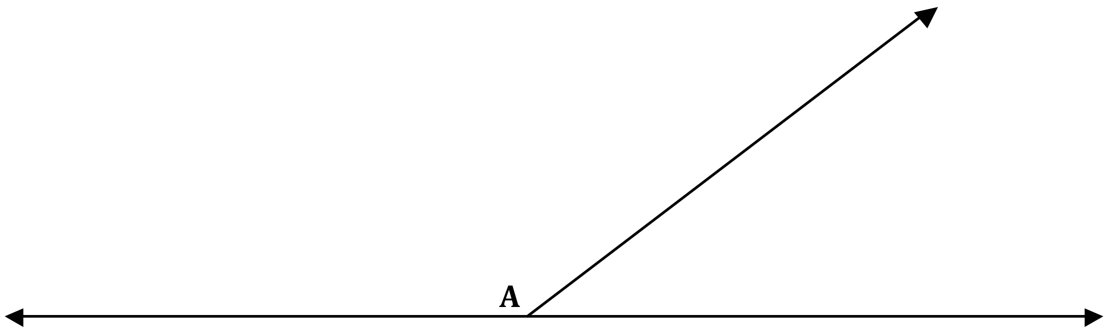
m (∠ QPR) = ___________

3.



m (∠ QPR) = ___________

4. What is the measure of A? (Students will need a protractor.)



F. 36° G. 42° H. 142° * J. 156°

Launch for Section 9.1:

As we begin the geometry chapter, this Launch is meant as a review of terminology in relation to angles. As a Launch you may simply brainstorm what the students know about angles, or if you are feeling adventurous, you may take the students outside and review the terms by creating "human angles." Have the students choose one person to be the vertex, and split the remaining students to form the two rays. Ask students which student will be the only person that is a member of both rays. (The vertex). Have the students stand in a straight line with each ray on either side of the vertex. They may hold on to a rope or yarn to represent the line. Tell them this is called a straight angle. Review other terms by having the students move to form a right angle, an obtuse angle, and an acute angle. You may also ask them to make a 45 degree angle and a 180 degree angle to further test their understanding. Once the students are settled back in class, tell them that they did a great job making human angles, and now they will be learning how to make precisely measured angles using a protractor, and will be learning about some special relationships between certain angles.

Measuring Angles Lesson:

Classifying Angles:

A ray is part of a line. Rays have one endpoint and continue forever in one direction. Angles are formed when two rays meet at a vertex. Make the following chart on the board or on a large poster board for all students to help fill in:

Angle Type	Right Angle	Acute Angle	Obtuse Angle	Straight Angle
Draw				
Define				

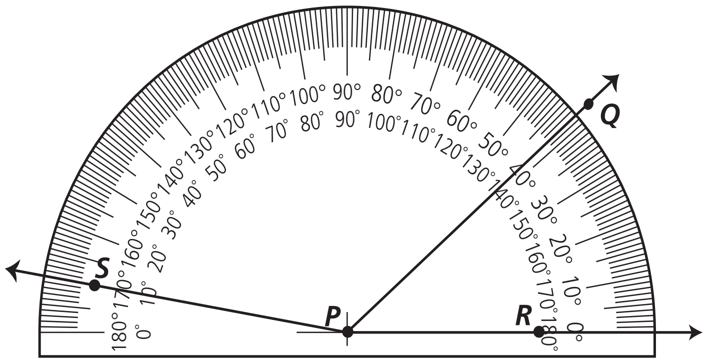
Measuring Angles:

Angles are measured as part of a circle. From its center radii divide a circle into 360 equal parts. The measure of the angle between two adjacent radii is one degree. A full circle contains 360 degrees. The instrument used to measure angles is called a protractor. To measure an angle, place the vertex of the angle on the origin (0) and line up one of the angles with the straight edge or base line of your protractor. How do you measure an angle whose rays are not on the baseline?

To identify angles letters are used to identify the 2 rays and the vertex that forms the angle. The middle letter identifies the vertex. (example: $\angle XYZ$, X and Z are the rays and Y is the vertex).

Have students fill in the Lesson Review.

4.

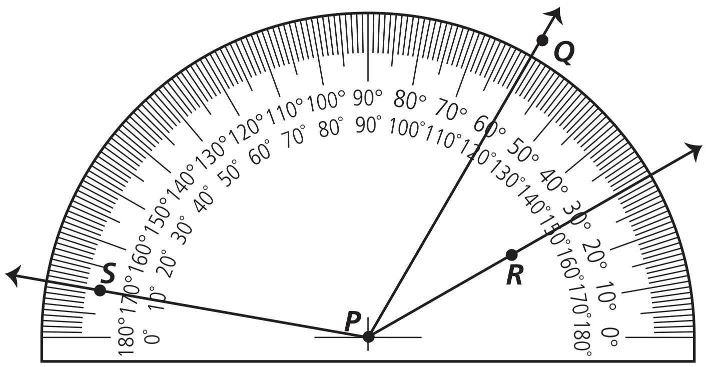


m (∠ SPQ) = ___________

m (∠ QPR) = ___________

m (∠ SPR) = ___________

5.

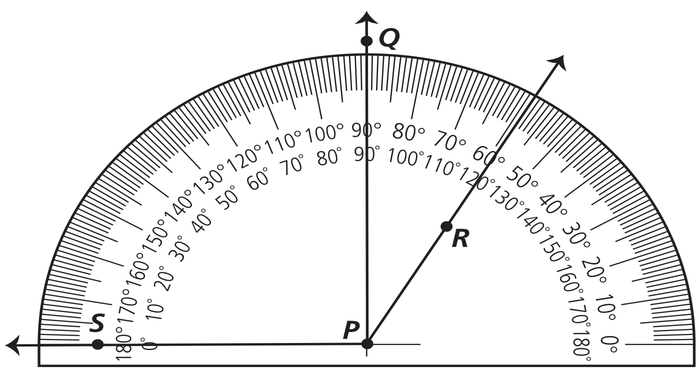


m (∠ SPQ) = ___________

m (∠ QPR) = ___________

m (∠ SPR) = ___________

6.



m (∠ SPQ) = ___________

m (∠ QPR) = ___________

m (∠ SPR) = ___________

Lesson Review

Identify two complementary angles ____________________, ________________________

Identify two supplementary angles______________________, _____________________

A circle measures ____________.

A ___________________________________ is an instrument used to measure angles.

An angle is made up of two _______________________.

They share a common____________________.

An ACUTE angle measures ____________.

An OBTUSE angle measures __________.

A RIGHT angle measures __________.

Practice reading and writing angle measures using the model below:

What is the measure of ∠UST? __________________

What is the measure of ∠RSV? __________________

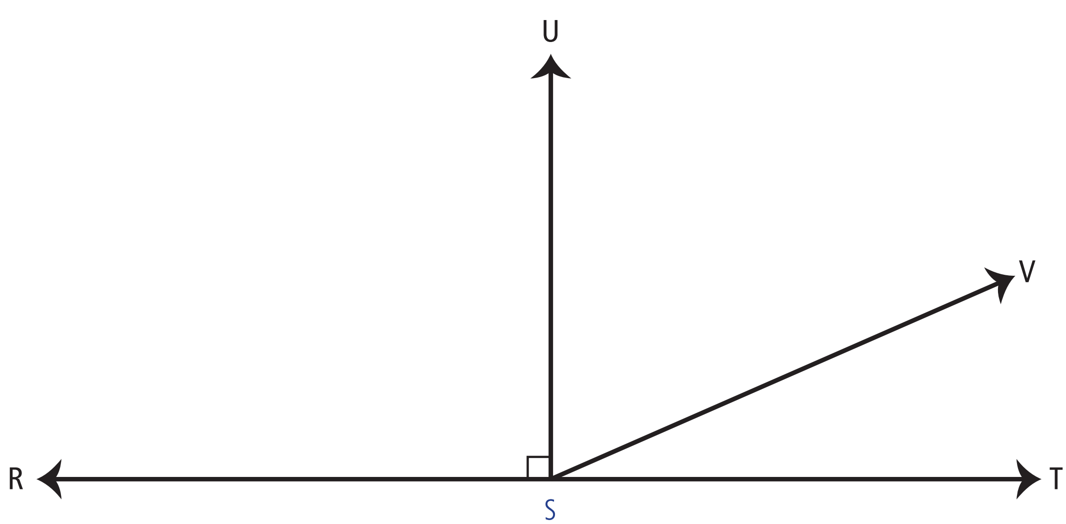
What is the measure of ∠USV? __________________

Identify two ACUTE angles__________________, ______________________

Identify a STRAIGHT angle ________________________

Identify an OBTUSE angle _______________________

Identify two RIGHT angles ____________________, ________________________



Angles All Around Us

Find 3 different types of angles in your household and determine the angles function and explain how dysfunctional it would be if it were any other type of angle.

Angle 1: ____________________

Description: ____

____

Function: ____

____

Angle 2: ____________________

Description: ____

____

Function: ____

Angle 3: ____________________

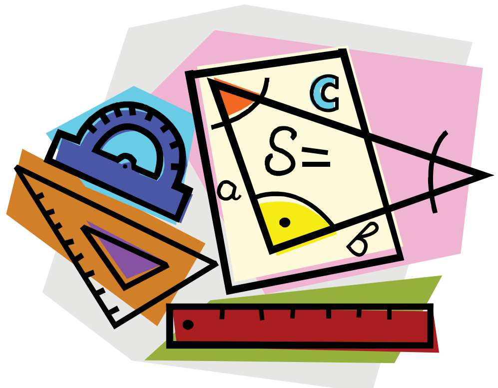
Description: ____

____

Function: ____

____

What's Your Angle?



Objective: To have the students use their protractors and their prior knowledge about angles to construct a picture with exact angle measurements.

Materials:
Protractor
White, unlined paper
Markers, crayons or colored pencils

Activity Instructions:
With their protractors, your students create a design or drawing that contains EXACTLY the number of each type of angle listed below:

10 Right Angles
15 Acute Angles
8 Obtuse Angles

Once the students have created their drawing, they label their angles appropriately with an R for right angles, and A for acute angles, and an O for obtuse angles.

For a challenge, you can increase any number of the angles, or you could add a set number of straight angles to the list.

You might show struggling students some pictures in books or magazines and have them count the number of angles in the pictures.

Review lines, (recalling number lines) and then discuss rays in terms of number line and represent greater than and less than. The notation for rays are similar to "half" the line. Have students also recognize how rays and lines are different from line segments. You may also have students draw and lable examples of lines, sements, and rays.

GEOMETRY 9

SECTION 9.1 MEASURING ANGLES

How would you answer the question, “What is an angle?”

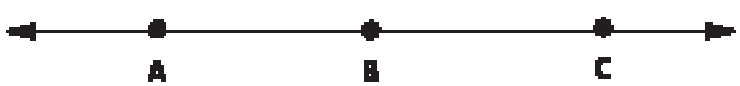
You have probably seen angles in many places in everyday life. Can you name a few of these places? In this section, you will learn what angles are, how to construct angles and how to measure the size of an angle.

Activity: Lines, line segments, and rays

Locate three points on the line below:

Label the left point A, the middle point B, and the right point C. Typically, two points on the line are used to identify a line. For example, if points A and B are used, then we use the notation, AB. You can also use points B and C, in which case the notation would be like BC.

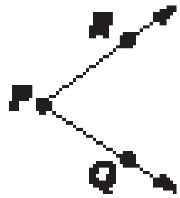
A **line segment** is a part of a line that includes two endpoints and all the points in between the points. For example, the line segment with end points A and B is written, $\overline{AB}$. Identify other line segments on this line that involve using a pair of the given points A, B, C.



A **ray** is also a part of a line that has a starting point and continues forever in only one direction. One ray on the line above is the ray that has starting point B and goes in the direction of C. We write $\overrightarrow{BC}$ for the ray BC. If we want to describe the ray starting at B that goes in the direction of A, what notation could you use? What other rays can you describe on the line above?

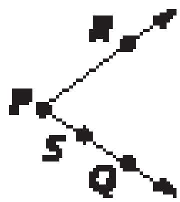
Notice that while line segments AB and BA describe the same line segment, the rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are very different parts of the line.

To construct an angle, first draw two rays from a common point P.



In the figure above there are two rays, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Both of these rays begin at point *P*, and pass through the points *Q* and *R* respectively.

If another point S is between points P and Q, then the ray $\overrightarrow{PS}$ is the same as ray $\overrightarrow{PQ}$. However, the ray $\overrightarrow{SP}$ is a different ray, because it begins at point *S*. In fact, $\overrightarrow{SQ}$ is a third ray that is different from all of the rays mentioned so far.



We can now answer our original question: "What is an angle?"

DEFINITION 9.1: ANGLE
An **angle** is formed when two rays share a common vertex.

The common endpoint *P* on rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is called the **vertex** of the angle. In the diagram, these rays form an angle called angle *QPR* written $\angle QPR$. The symbol "$\angle$" is the math symbol for the word *angle*. To name an angle, you can do the following three steps in order:

1. Write the name of one of the non-vertex points on one of the rays.
2. Write the name of the vertex.
3. Write the name of a non-vertex point on the other ray.

There can be many ways to name the same angle because there are many choices of points on the two rays in steps 1 and 3. You could also label this angle $\angle RPQ$. As long as the middle point identifies the vertex of the angle, the order of the first and last points does not matter.

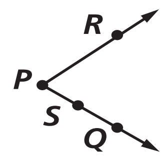
EXAMPLE

Draw an angle $\angle XYZ$. Identify the rays and identify the vertex of this angle.

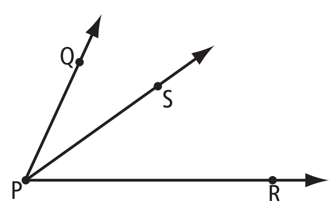
SOLUTION

$\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are the rays, and Y is the vertex of the angle. We use different letters to label different angles, like $\angle QPR$ or $\angle XYZ$.

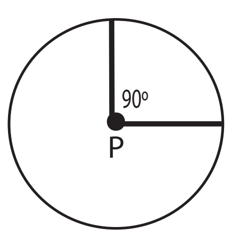
Sometimes the single letter of the vertex is used to name an angle when it will not lead to any confusion. For example, in the figure below, $\angle QPR$ can be called $\angle P$ without any confusion.



However, if more than 2 rays emanate from one common vertex, then it would not be clear to which angle we are referring. For example, in the figure below, we would not use a single letter notation for an angle and instead use the three letter notation.



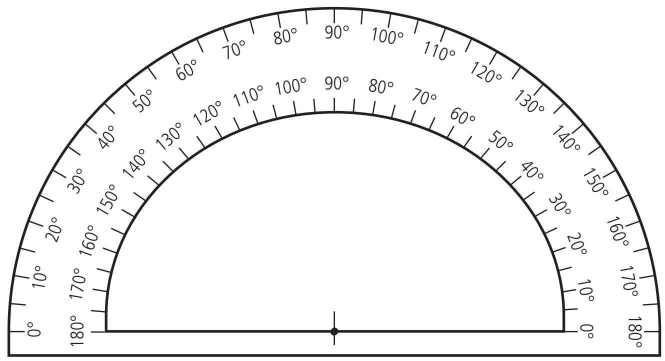
Once you understand the definition of an angle, the next step is to measure the size of the angle. Degrees are commonly used to measure angles. One degree is written $1°$ is the angle formed by $\frac{1}{360}$th of a full revolution around a circle.
An angle that makes a full revolution has measure 360^o. One fourth of a revolution would have measure 90^o and could look like this



Students may be interested to know that there are other units besides degrees to measure angles. For example radians are often used where 1 revolution = 2 π radians. Grads, grade, or gradian uses 1 revolution = 400 grads. We will not work with these units here but by the time students are in trigonometry and calculus, radians will be used more frequently than degrees. Students may want to research the history of angle measures.

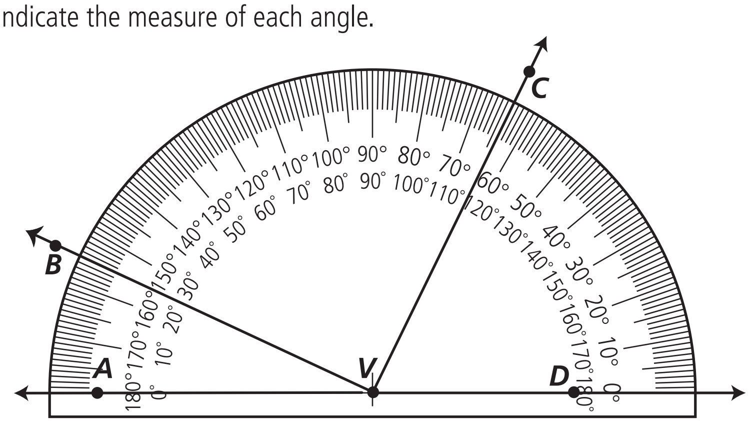
A **protractor** is an instrument used to measure angles. Most protractors use degree units to measure the angles.

Protractors have degree markings along the outside of the curved edge. To measure an angle, place the vertex at the center of the semi-circle so that one ray passes through 0° or 180° and the other ray passes through a mark on the curved edge. If necessary, extend the other ray so that it falls on a mark along the curved edge. The degree at this mark is the measure of the angle or its supplement, which we will define later in this section.



If two rays with a common endpoint form a straight line, the angle they form has a measure of 180 degrees, or 180°. This is called a **straight** angle. Angles that have a measure between 0° and 90° are called **acute** angles. Angles that have a measure greater than 90° but less than 180° are called **obtuse** angles.

Angles that measure exactly 90° are called **right** angles. Using a protractor, construct and label a straight angle, a right angle, an acute angle and an obtuse angle and indicate the measure of each angle.



Identify the straight angle, an acute angle, an obtuse angle, and a right angle from the figure above.

Answers will vary. ∠ DVA (straight), ∠ DVC (acute), ∠ DVB (obtuse), ∠ CVB (right)

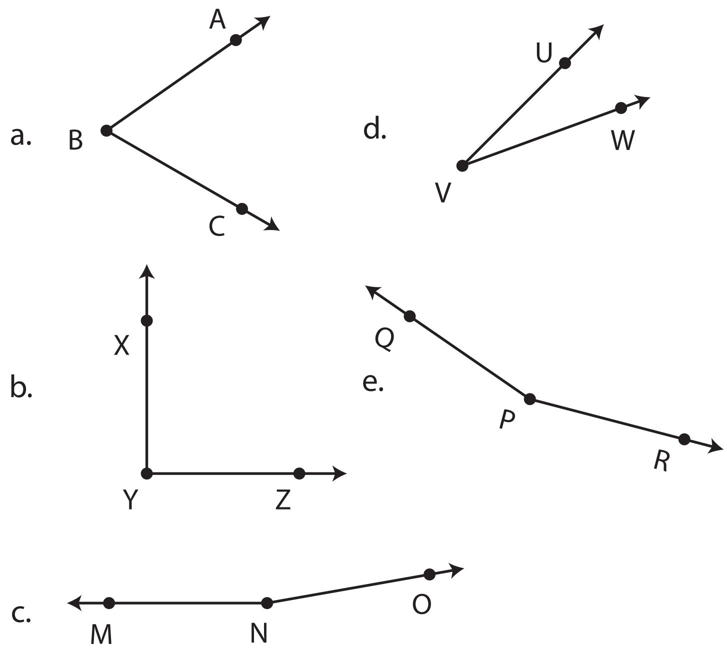
EXPLORATION 1

1. a. ∠ABC has measure 65°. (acute)
   b. ∠XYZ has measure 90°. (right)
   c. ∠MNO has measure 170°. (obtuse)
   d. ∠UVW has measure 25°. (acute)
   e. ∠QPR has measure 160°. (obtuse)
2. ∠QPR or ∠RPQ or ∠P, etc.
3. see #1 for answers

Discuss with the students the process they used to draw their angles. Some students may count along the protractor from 0° to the angle measure. Other students may discover that if the protractor is situated with one ray at 0°, it is enough to look at the protractor reading situated at the other ray. The students must be careful, however, to read the appropriate numbers whether on the inside or the outside of the protractor. A discussion here may be appropriate as to the error in reading the wrong number and whether that angle measure makes sense. In addition, the students may detect if they misread the protractor numbers by recalling how "big" or "small" an obtuse or acute angle looks. The students may also notice ∠PRQ that the sum of the inside and outside numbers is interestingly a constant of 180°. This can be a useful observation that foreshadows the supplementary concept that comes up at the end of this section.

EXPLORATION 1

1. Measure each of the angles below with your protractor:



2. Consider the rays and points above. Name each angle in two ways.

3. Classsify each of the angles as acute, obtuse, right, or straight.

EXPLORATION 2

1. Divide a straight angle into two parts. Describe how you constructed it to another student or your teacher. Measure each angle using your protractor.

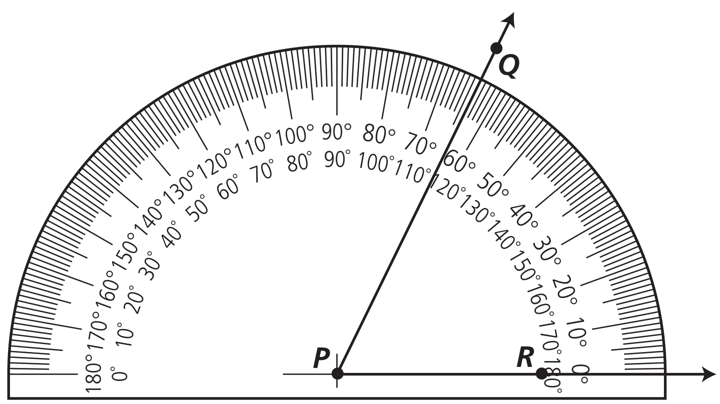
2. Use your protractor to draw rays making the following angles: 35°, 80° and 100°.

PROBLEMS 1 and 2

Answers will vary. Check that the measure of angle is 45°; orientation of angle may also vary.

How did you construct an angle? Here is one approach to construct an angle with a given measure such as 64º.

1. Draw a initial ray and label it *PR*. The initial ray is usually, but not necessarily, horizontal.
2. Place the center of the semi-circle of the protractor on top of the point *P*, with the ray passing through 0°.
3. Find the place along the curved edge of the protractor that corresponds to the degree measure you are constructing and mark it with a new point *Q*.
4. Draw a line connecting the point *P*, which is the vertex of the angle, to the new point *Q* with a straight edge to obtain an angle of 64º.



PROBLEM 1

Use a protractor to construct an angle starting at 0° with measure 45°.

PROBLEM 2

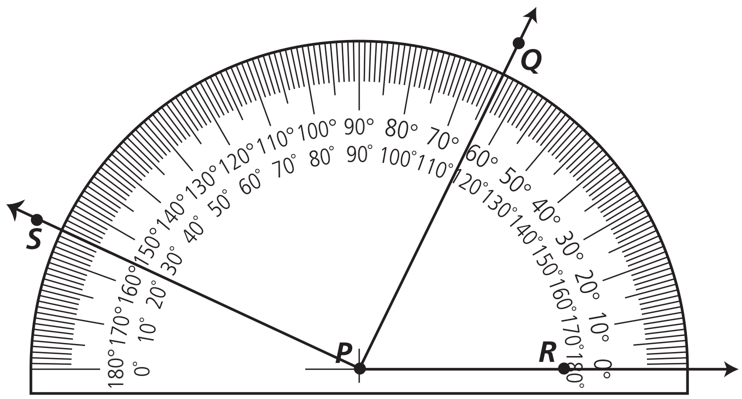
Use a protractor to construct an angle with one ray at the 20° mark and with measure 45°.

PROBLEM 3

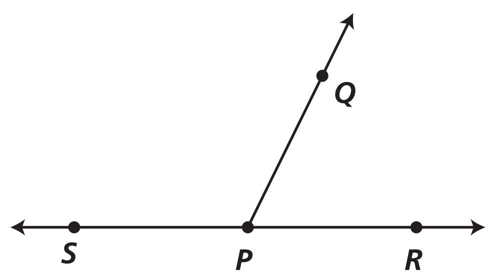
Consider the rays on the protractor below. Measure the angles, ∠QPS and ∠SPR.

This is a historical question that your students can solve most easily on the internet. You might team with the history or science teacher in a joint assignment of these questions.

In ancient Mesopotamia, where writing was invented about 4000 years ago, they counted about 360 days in a year. They added several "leap days" to set the seasons to come out right. In the course of a year, the stars made a full circle of the heavens, so it was natural to break a circle into 360 pieces. This led naturally to 360 degrees and motivated their "base sixty" system of numbers.



Do you know historically why there are 360° in a full circle? Who first used the number 360? Why didn't they choose 300 or 400 or 500 degrees to make a circle?



Mathematicians use the notation m($\angle QPR$) to mean the **measure of angle** QPR. Notice that $\angle QPR$ and $\angle QPS$ divide the $\angle RPS$ into two parts. The measure of each angle is a number, and we can add these numbers together to get the equations below:

$$m(\angle QPR) + m(\angle QPS) = m(\angle RPS)$$

$\overleftrightarrow{QS}$ is a line, so m($\angle QPS$) = 180°. Then we have

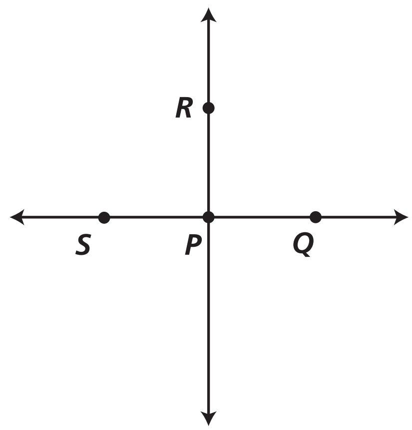
$$m(\angle QPR) + m(\angle QPS) = 180°.$$

DEFINITION 9.2: SUPPLEMENTARY
Two angles are **supplementary** if the sum of their measures is 180°.

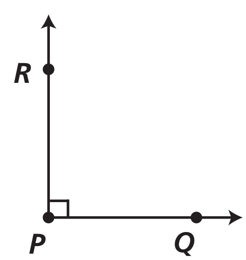
In the example above, $\angle QPR$ and $\angle QPS$ are supplementary angles. This means *$\angle QPS$ is the supplement of $\angle QPR$* and *$\angle QPR$ is the supplement of $\angle QPS$*.

Now divide a straight angle in half. Each angle formed is a right angle and measures 90° because $\frac{1}{2}$ of 180° is 90°. When two lines or line segments meet and form a right angle, they are **perpendicular** to each other.

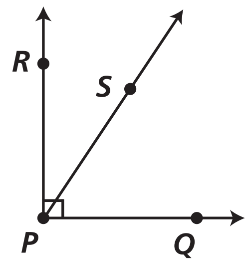
We often label the right angle with ∟ to indicate that it is 90°.



When two rays meet to form a right angle, they are perpendicular rays.



Next, divide the right angle $\angle RPQ$ into two parts using ray $\overrightarrow{PQ}$.



Because $\angle RPS$ and $\angle SPQ$ divide $\angle RPQ$,

$$m(\angle RPS) + m(\angle SPQ) = m(\angle RPQ) \text{ and}$$
$$m(\angle RPS) + m(\angle SPQ) = 90°.$$

DEFINITION 9.3: COMPLEMENTARY
Two angles are **complementary** if the sum of their measures totals 90°.

In the example above, $\angle RPS$ and $\angle SPQ$ are complementary angles. This means *$\angle SPQ$ is the complement of $\angle RPS$* and *$\angle RPS$ is the complement of $\angle SPQ$*.

EXAMPLE

1. One angle measures 34°. What is its complement? Check your work by creating and solving an equation.
2. One angle measures 34°. What is its supplement? Check your work by creating and solving an equation.
3. Two angles form an acute angle. If one angle is 34°, what are the possible measures of the second angle Check your work by creating and solving an inequality.
4. Two angles form an obtuse angle. If one angle is 34°, what are the possible measures of the second angle? Check your work by creating and solving an inequality.

EXERCISES

1. a. ∠ABC, ∠CBA, ∠B; acute; 55°

   b. ∠PRQ, ∠QRP,∠R; right; 90°

   c. ∠MNO,∠ONM, ∠N; obtuse; 172°

   d. ∠UVW, ∠WVU, ∠V; actue; 25°

   e. ∠QPS, ∠SPQ, ∠P; straight; 180°

   f. ∠QPR; ∠RPQ, ∠P; obtuse; 160°

SOLUTION

1. Call C the measure of the complement to the 34° angle. Two complementary angles equal 90°. In our case, we have the equation, C + 34° = 90°. Solving the equation we have C = 90° – 34° or C = 56°. The complement of the angle with measure 34° is an angle with measure 56°.

2. Call S the supplement of a 34° angle. Then S + 34° = 180°. Solving, S = 180° – 34° or S = 146°.

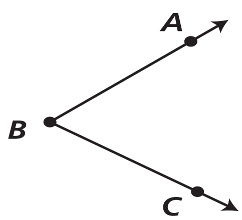
3. If two angles combine to form an acute angle, their sums must be less than 90°. Angle A which plus 34° is less than 90°. Write the inequality as A + 34° < 90°. Solve to get A < 90° – 34° so A < 56°.

4. If two angles form an obtuse angle, their two measures must be less than 180° but greater than 90°. Call the angle B. The inequality is 90° < B + 34° < 180°. Solve for B, which means isolate B algebraically, to get 90° – 34° < B < 180° – 34° or 56° < B < 146°. In other words, only angles greater than 56° but less than 146° can be combined with a 34° angle to form an obtuse angle.

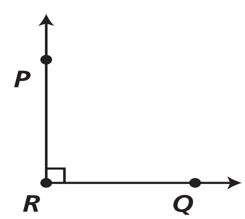
EXERCISES

1. Name each angle shown. Classify as acute, obtuse, right, or straight. Use a protractor to find the measure of each.

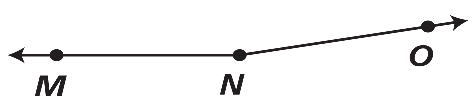
a.



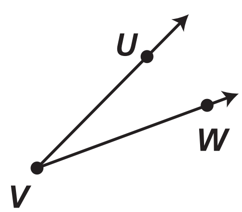
b.



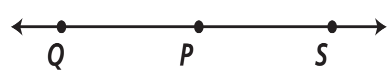
c.



d.



e.



2. obtuse, 115°

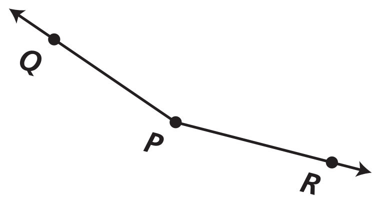
3. obtuse, 155°

4. Answers will vary.

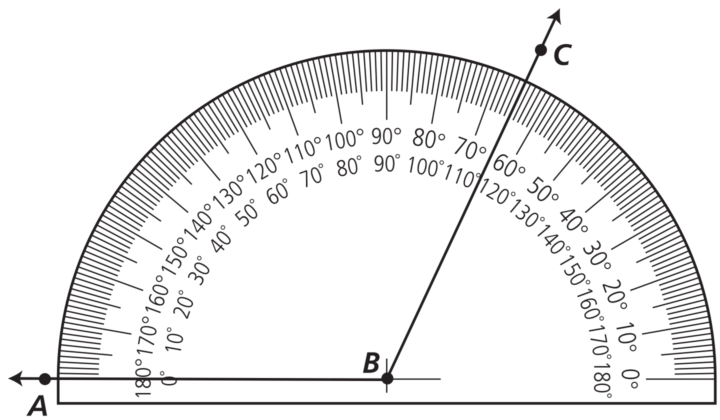
Acute angles: ∠ZYX, ∠TYZ, ∠ZYW, ∠WYX

Obtuse angles: ∠TYW and ∠ZYX

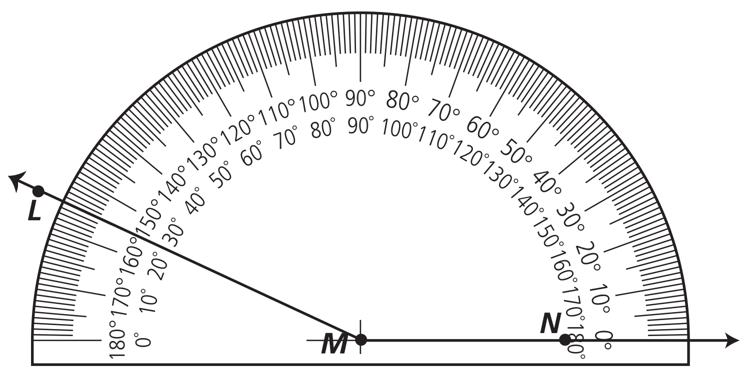
f.



2. Classify angle ABC in the figure below as acute, obtuse, right, or straight. Use the protractor in the figure to state its measure.



3. Classify ∠LMN in the figure below as acute, obtuse, right, or straight. Use the protractor in the figure to state its measure.



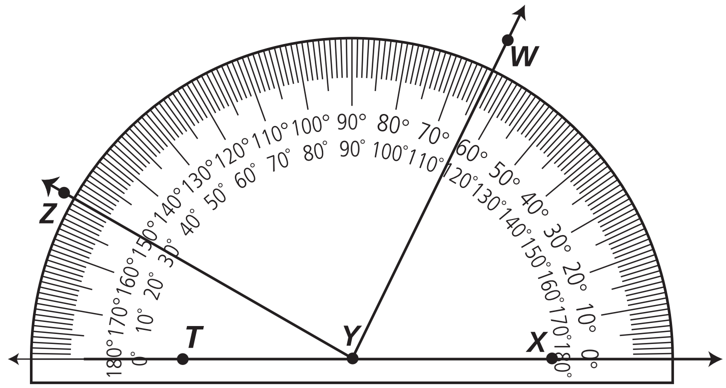
4. Name at least 2 acute and 2 obtuse angles. Give the measures of each of these angles. Are there any right or straight angles? If so, name those as well.

5. Answers will vary.

7. a. 38°, 52°, 90°

   b. The sum of the measures of the first two angles is equal to the measure of the third angle.

      In this case, the first two angles sum to 90°, so they are complementary.



5. Draw each of the following angles using a ruler and a protractor. Write the measure of each of your angles. Remember to label your measurements with degree signs.

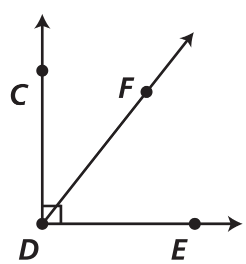
   a. Straight angle c. Acute angle
   b. Obtuse angle d. Right angle

6. Draw an angle with each of the following measures:

   a. 30° b. 120° c. 77° d. 45°

   e. 113°

7. a. Using a protractor, give the measures of the angles below.



   m(∠ CDF) = _______ m(∠ FDE) = _______ m(∠ CDE) = ______

   b. What relationship do you notice about the three angles?

8. a. 117°, 63°, 180°

b. The sum of the measures of the first two angles is equal to the measure of the third angle. In this case, the sum of the first two angles is 180°, so they are supplementary.

9. a. ∠ STR, ∠ RTV

b. Answers will vary, but the two angles need to add to 90°.

c. These two angles are complementary.

10. a. ∠ JKP, ∠ PKL

b. Anwers will vary, but the two angles need to add to 180°.

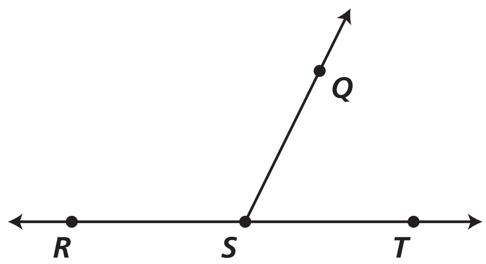
c. These two angles are supplementary.

11. x = measure of smallest angle, y = measure of second smallest angle, x + y = measure of largest angle.

$x + y + (x + y) = 180°$ and $x + y = 90°$

12. $80° < x < 85°$. If you have time, this discussion leads to the density of real numbers. Right now, if your students give you an answer like 4 or 5, you might ask if an angle like 81.234° exists. To show any doubting student this fact, ask your students to tell you two angles that are next to each other on the protractor scale. Then choose an angle in between them, probably exactly half-way, although that's not necessary. This could go on forever. You can also mention that a degree can be subdivided into minutes so that 1° = 60 minutes written 60′. Where have they seen that subdivision before? And students may extend their knowledge to find that an angle of 1′ = 60 seconds written 60″. You may wish to ask what 60 degrees and 30 minutes would look like all in degrees: 60.5°. This is another example of converting units.

8. a. Using a protractor, give the measures of the angles below.



m(∠ RSQ) = _______ m(∠ QST) = _______ m(∠ RST) = ______

b. What relationship do you notice about the three angles?

9. Draw right angle ∠STV using a protractor and a ruler. Draw ray $\overrightarrow{TR}$ within the right angle.

   a. Name the two angles you created within your right angle.
   b. Give the measure of each of the angles you created.
   c. How do you describe the relationship of the two angles?

10. Draw stright ∠JKL using a protractor and a ruler. Draw ray $\overrightarrow{KP}$ within the straight angle.

   a. Name the two angles you created within your straight angle.
   b. Give the measure of each of the angles you created.
   c. How do you describe the relationship of the two angles?

11. The sum of the measure of three angles is 180 degrees. The measure of the largest of the three angles is the sum of the measures of the two smaller angles. What is the measure of the largest angle?

12. Start with an angle A = 17°.

   a. What are the measures of angles B that can be combined with A to form an acute angle? Begin by writing an inequality that relates A and B with the condition.

   b. What are the measures of angles C that can be combined with angle A to form an obtuse angle? Begin by writing an inequality that relates A and C with the condition.

14. Point B: (-2 - 2, 3 - 3) = (-4, 0)
    Point C: (-2, -3)
    Point D: (2, 3)

15. 8 hours and 40 minutes

Ingenuity

16. We know that since the three angles make up a straight angle, the sum of the three angles is 180 degrees. So the two unknown angles together make 180 - 52 = 126 degrees. So since the two unknown angles are equal, each must have measure 126 ÷ 2 = 63 degrees.

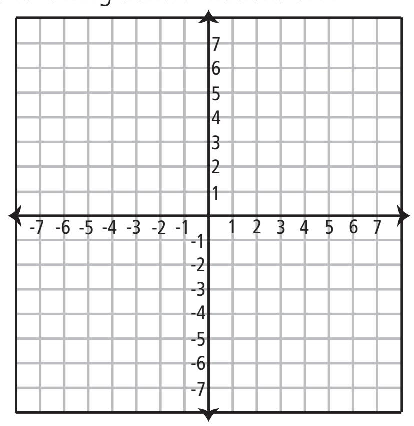
Investigation

17. a. We will make 12 turns, since 30 x 12 = 360. This connects the idea of angles as rotations.
    b. We will make 8 turns, since 45 x 8 = 360.
    c. Our first instinct is to try dividing 360 by 80, but this time we do not get a whole number as our answer. If we turn 4 times, we have turned a total of 320 degrees. If we turn 5 times, we have turned a total of 400 degrees, which puts us 40 degrees east of north. However, if we turn 4 more times, we will turn 320 degrees more, pointing us due north again. So this time we turn a total of 9 times. Note that 80 x 9 = 720, which is a multiple of 360.
    d. We need to keep turning until the total number of degrees we have turned is a multiple of 360. This means that we are looking for the LCM of 105 and 360. The LCM of 105 and 360 is 2520, so we have to turn a total of 2520 [divided by] 105 = 24 times.
    d. The LCM of 108 and 360 is 1080, so we need to turn a total of 1080 [divided by] 108 = 10 times.
    e. If we turn N degrees each time, then we can find the number of turns by finding the LCM of N and 360, and then dividing the LCM by N. Alternatively, since 360 x N = GCD(360, N) x LCM(360, N), we have LCM(360, N) / N = 360 / GCD(360, N). So we can also divide 360 by the GCD of 360 and N.

13. How many acute angles can you find with measures between 80° an 85°?

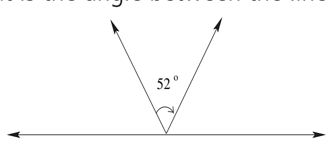
Spiral Review:

14. If point A is located at (-2, 3), find the following transformations of A.
    Point B: translated 2 left and 3 down
    Point C: reflected over x-axis
    Point D: reflected over y-axis



15. Krista left for school at 7:45 a.m. and returned home at 4:25 p.m. How much time elapsed between the time she left for school and returned home?

16. **Ingenuity:**
    In the figure below, two rays rest on a straight line. The two rays make an angle of 52° rest on the line so that the angles between the line and the rays on both sides are equal. What is the angle between the line and each of the rays?



17. **Investigation:**
    Suppose we turn due north, and then make a sequence of clockwise turns until we face due north again. For example, if we turn 90° each time, we will face north after exactly 4 turns.

    a. How many turns will we make if we turn 30° each time?

    b. How many turns will we make if we turn 45° each time?

    c. How many turns will we make if we turn 80° each time?

    d. How many turns will we make if we turn 108° each time?

e. Can you find a way to answer questions like (a) through (e) without actually going through the sequence of turns and keeping track of the angles? Explain.

Section 9.2 - Triangles

Big Ideas:
Discovering relationships between special angles; Classifying triangles according to their sides and angles.

Key Objectives:
- Characterize triangles as a specific polygon
- Classify and construct triangles in terms of lengths of sides and measure of vertex angles
- Discover relationship between angles in a triangle
- Use algebra to find missing information about the angles in triangles

Materials:
Grid paper, Ruler, Protractor, Uncooked spaghetti noodles.

Pedagogical/Orchestration:
- This section lays the groundwork for high school geometry, so inform your students that they will see this again in high school geometry.
- Encourage your students to develop strong habits in constructing accurate geometric models. Attention to detail and resistance to sloppy habits will reward students handsomely.
- Have students use their skills in measurement to make conjectures about angles in a triangles.

Activity:
"Tessellation Project", "Triangle Picture Book", "Triangle Angles Puzzle" and "Spaghetti Activity"

Vocabulary:
triangle, congruent, conjecture, tessellation, right triangle, polygon, acute triangle, obtuse triangle, scalene triangle, equilateral triangle, isosceles triangles

TEKS:

6.6(B); 6.8(C); 6.12(A); 6.13(A,B) New: 6.8(A, B); 6.10(A)

Launch for Section 9.2:

Triangles: This launch incorporates Exploration 1a as an activity. Ask students if any 3 lengths can form a triangle. Most students will say yes, and a few will disagree.

Ask students to characterize what makes a triangle. Characteristics should include: Three sides; sides are line segments; closed (beginning and ending point coincide); simple (no crossing over); three interior or vertex angles; and it is a polygon. They may talk about types of triangles in terms of angles (right, obtuse, acute) or sides (scalene, isosceles, right). We will formally define these terms in this section but the students may already know about the classifications.

Knowing that triangles have three sides, ask the students if the sides of a triangle can be of any length and allow the students to conjecture and explain their reasoning.

Tessellation Project

Objective: The students will learn about M.C. Escher and his marvelous mathematical and artistic work with tessellations. After previewing the works of Escher, the students will use their mathematical skills to make their own tessellated creation.

Materials:

Books from the library on M.C. Escher. If you are able to display images from your computer, you can find numerous websites with his works, as well as student designed tessellations.

http://www.mcescher.com/

http://www.mathacademy.com/pr/minitext/escher/

Cardstock (cut into 4”x4” squares)
Scissors
Tape
White, unlined paper for rough drafts
White or lightly colored poster paper Lightly colored construction paper works well, too.
Markers, crayons or colored pencils

Activity Instructions:

1. Spend some time sharing with your class some background information about the artist M.C. Escher. You can find books in your library or information on the web site that your students find interesting. If possible, try to find as many examples of tessellations, created by Escher and others, as possible to share with the class. After your students have seen enough examples to have some idea what a tessellation looks like, they are ready to begin the activity. Distribute a 4”x4” square of cardstock and a pair of scissors to each student.
2. Demonstrate for the class how to make two cuts, either curvy or jagged, on the 4x4” square. One cut is on a vertical edge, and the other cut is on a horizontal edge. After making the first cut, slide, or translate, that cut piece to the other parallel side of the 4”x4” square and tape it back on to make sure that the cut pieces will fit well. Follow the same instructions for the second cut.

3. When finished, the students will no longer have a square, but instead a new shape that resembles a puzzle piece. Hand each student a sheet of rough draft paper. Have them set their puzzle piece directly in the middle of this sheet of paper and trace their puzzle piece. They will then continue tracing their puzzle piece until the entire sheet of paper is covered with their tracings. There should be no gaps, spaces, or overlapping anywhere on the paper.
4. This is when things get fun! Now the students try to take this plain white piece of paper and turn it into something original and creative. Sometimes their tracings will resemble a car, duck, person, star… . The sky's the limit. Once their creative minds envision a picture in their design, they can start drawing and coloring.
5. When they have finished their rough drafts, they will bring them to you and you will check over their work. If you have any suggestions or comments, please share them at this time. Then, they are ready for their poster paper so they can redo the whole thing again as a final draft.

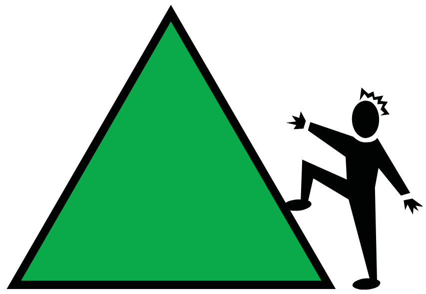
Just a couple of tips:

*I have done this project for several years, and the students always enjoy it. Some students will finish very quickly and want to do another poster. Instead, I suggest you try to pair them with a student who is struggling to complete his or her project.

*Some students need help turning their rough draft tracings into a design. Encourage them to turn their paper in several directions. Sometimes the new perspective will help inspire them to see a design. If not, put them on a computer where they can research examples of tessellations. Sometimes examples help stimulate their creativity. Another idea is to allow them to walk around the room with their design and ask for input from their peers. It's amazing how creative students can be.

*Please don't let your students be sloppy. If they work carefully and take pride in their work, they will create something wonderful. It can hang in your room or on the walls in the hallway. Some exceptional pieces might even be framed and hung in the office or library.

Triangle Picture Book



Objective: To have students recognize and classify different types of triangles in real world objects.

Materials:
Construction Paper
Markers, colored pencils, or crayons
Magazines
News Papers
Scissors
Glue
Access to computer with colored printer – optional

Activity Instructions:

1. Explain to your students that they will create a picture book organized as follows:
   Front – Cover Page
   Pg. 1 – Table of Contents
   Pg. 2 -- Equilateral Triangles
   Pg. 3 -- Scalene Triangles
   Pg. 4 – Isosceles Triangles
   Pg. 5 – Acute Triangles
   Pg. 6 – Obtuse Triangles
   Pg. 7 – Right Triangles
   Pg. 8 – Credits, where they name their sources
2. The students first need to find several pictures of triangles using magazines, newspapers, and/or computer printouts. Be very specific that all pictures must be REAL WORLD examples. It is not acceptable for a student to simply find a picture of a right triangle and put this in their book. The pictures that make it into their book must be objects that are examples of different types of triangles. It might be helpful to your students if you can provide some examples of pictures that you have previously found.
3. As your students cut out or print their pictures, remind them that they need to remember to name their sources. Have them keep their pictures in little envelopes with the name of their sources printed on the outside. For example, if a student finds 5 pictures in Architectural Digest, put the magazine title and issue month and year on the credits page. If a student finds a picture on the internet, name the website where he or she found the picture.
4. After your students have searched for their pictures, they begin creating their book in the format

listed above. Encourage the students to make their books neat and colorful. Have them lay their pages out before they start attaching the pictures. They should first title their pages and number their pages in pencil, then go back and rewrite in marker or crayon.

5. When finished, display the books in the classroom so that all students can have an opportunity to see their peers' books. If there are any that are particularly good, it might be fun to have the student share it with the whole class.

Triangle Angles Puzzle

Objective: Students will prove that the sum of the interior angles of a triangle is equal to 180°. This project can be thought of as putting a puzzle together.

Materials:
Multi-color sheets of construction paper
Glue sticks
Scissors
Protractors
Rulers

Activity Instructions:

Teacher provides each student with ½ of a sheet of construction paper in two different colors. Each student takes one of the pieces of construction paper, draws a medium-size triangle of any type (isosceles, equilateral, right, etc.), and labels each angle inside with numbers: 1,2,3. Students cut out their respective triangles.

Using their rulers, students draw a straight line in the middle of the remaining ½ of construction paper. Teacher shows students how to tear off each corner of the triangle. Students collect all three corners and glue them together on top of the straight line drawn on their construction paper. So they put the puzzle together!!

Students use their protractors to measure and prove that the sum of the three angles glued on the paper equals 180 degrees.

At the end, students will have a clear, concrete picture of how the sum of three angles of any triangle is equal to 180°.

Spaghetti Activity:

Materials:
uncooked spaghetti noodles

Activity Instructions:

- Give each student one uncooked spaghetti noodle.
- Have the student break the spaghetti into three pieces.
- Have the students measure each piece.
- Ask if the students can form a triangle using the three pieces.
- Have the students conjecture why a triangle could or could not be formed.
- Give the students another uncooked spaghetti noodle and have them break the noodle into three pieces so that a triangle can be formed.
- Have the students validate their conjecture.

The next question to ask is, "Why is it that some lengths cannot form a triangle, but others can? What has to be true about the lengths in order for them to form a triangle?" Let students have a few minutes to ponder this and maybe experiment with more spaghetti. If students are not getting anywhere, place a long piece of spaghetti on the overhead and 2 other pieces that sum to the length of the long piece positioned end to end right above and parallel to the long piece. Ask students if those pieces will be able to form a triangle. At this point, students should start seeing that the sum of the two shorter pieces must be greater than the longest piece. The triangle inequality theorem states that any side of a triangle is always shorter than the sum of the other two sides.

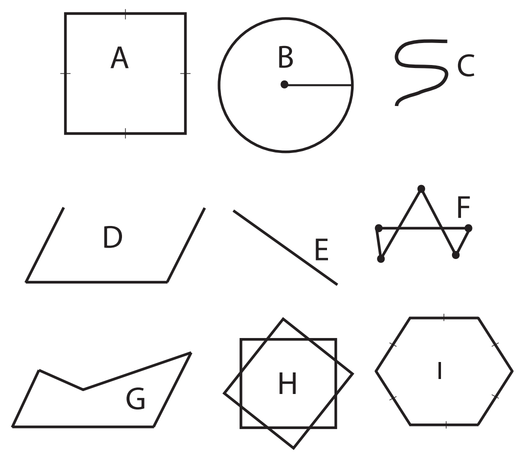
EXPLORATION 1

Some of the observations students may make in the discussion of exploration 1 should include: some are closed shapes (inside, outside) others are open; some use line segments to form the shape, others are not line segments; some shapes cross themselves and others do not;

SECTION 9.2 TRIANGLES

EXPLORATION 1

Consider the following figures and describe them by features that you observe. Identify common features as well as differing features among the shapes.



DEFINITION 9.4: POLYGON
A **polygon** is a closed, plane figure formed by 3 or more line segments that join with exactly two other line segments. The point where the two line segments meet is called a **vertex** of a polygon.

Classify the shapes in Exploration 1 as polygon or not a polygon. We often note the number of sides a polygon has. Include the number of sides each of the polygons has in Exploration 1.

A triangle is one type of a polygon. Triangles are closed two-dimensional figures whose three sides are line segments. Triangles can be classified by different properties - their size, shapes, and angles. In Exploration 2, you may discover different properties of triangles by making measurements of the triangles by lengths and angles.

EXPLORATION 2

a. Students should notice that the sum of any two sides is greater than the third side. You may wish to use the Spaghetti Activity here for part a.

b. Have a protractor handy.

c. The students should notice that the base angles in an isosceles triangle are equal. This foreshadows Problem 1 and 2.

EXPLORATION 2

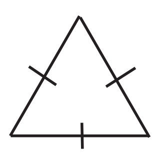
a. Draw a line segment that is five units long. Now draw two other line segments to complete a triangle. Repeat this process several times. What do you notice about the sum of the lengths of the other two sides relative to the length of the original side?

b. Make a triangle with two of the sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle having two sides of equal length.

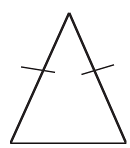
c. Make a triangle with all three sides of equal length. In groups, reflect on what you notice about each of these triangles. Measure and then make a rule about the angles of a triangle with all three sides of equal length.

DEFINITION 9.5: EQULATERAL TRIANGLE
A triangle in which all three sides have the same length is called an **equilateral triangle**. This term comes from the Greek *equi*, meaning "the same," and Latin *latus*, meaning "side." You have probably discovered in the exploration that all the angles of an equilateral triangle have the same measure.

A way to indicate that lengths are equal in a given measure is using tick marks as indicated in an equilateral triangle as shown below.



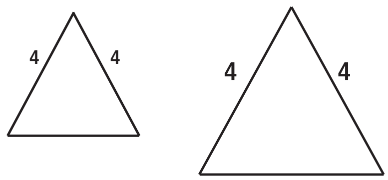
DEFINITION 9.6: ISOSCELES TRIANGLE
A triangle with two sides of equal length is called an **isosceles** triangle.



Let's explore isosceles triangles a little more.

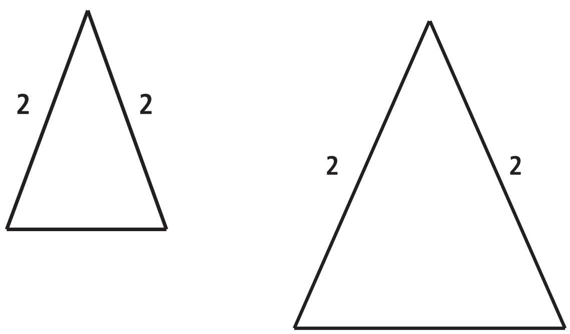
PROBLEM 1

The measures of the base angles should be congruent. The triangles may look like



PROBLEM 2

The measures of the base angles should be congruent.



PROBLEM 1

Draw two different isosceles triangles with the two equal sides of length 4 inches. Measure each of the angles. What do you notice about their measures?

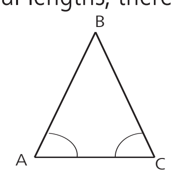
PROBLEM 2

Draw two isosceles triangles with the two equal sides of length 2 inches. Measure each of the angles. What do you notice about their measures?

In each of the problems, did you notice that the angles opposite the equal sides are of equal measure equal? This is actually one of the properties of all isosceles triangles:

- The angles opposite the equal sides are always of equal measure.

Just as we use a tick mark for equal lengths, there is a mark that is often used to indicate angles of equal measure.



When two lengths or two angles have equal measure, then we say that they are **congruent** and have a special notation, $\cong$, to indicate congruence. For example, $\angle A \cong \angle C$. The angle measures are equal and in that case we write $m(\angle A) = m(\angle C)$.

Conversely, if two of the angles in a triangle are equal, then the sides opposite these equal angles will be equal and the triangle will be isosceles. These are properties that you will learn when you study geometry. Do you see why they might be true?

It is also possible that all three sides of a triangle have different lengths. This leads to the following definition.

DEFINITION 9.7 SCALENE TRIANGLE
A triangle with all three sides of different lengths is called a **scalene** triangle.

PROBLEM 3

In a scalene triangle, there are no congruent angles.

The tessellation project is appropriate here.

PROBLEM 3

Draw two scalene triangles. Measure each of the angles. What do you notice about their measures?

In a scalene triangle, the three angles will also have different measures, since if two of the angles were equal, then the sides opposite these equal angles would also be equal.

EXPLORATION 3

Draw a large triangle on a sheet of paper, using a straight edge. Color or label the three angles of the triangle with different colors. Carefully cut out the triangle. Next, cut the triangle into 3 triangular pieces, each including one angle from the triangle. Put the angles together. What is the sum of the three angles of the triangle? Compare your result with others.

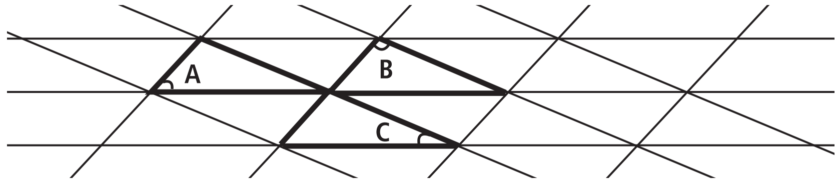
In each case, the sum of the measures of the angles in the triangles appears to be 180°. This leads to a conjecture: "The sum of the measures of the angles in any triangle is 180°." This is a **conjecture**, because it is a statement we think might be true based on our observations, but we have not yet proved it is always true. Is there a way to give a convincing argument or proof that the sum of the measures of the angles in *any* triangle is 180°? To answer that, investigate some triangles.

EXPLORATION 4

Divide the class into groups. In each group, draw a small triangle and several copies of it on lined paper using a straight edge. Be as exact in your work as possible. Use these copies to *tessellate* the paper or plane. A **tessellation** or tiling of the plane with some shape is a way of covering the plane with that shape with no gaps. This tessellation can be used to show that the sum of the measures of the angles of any triangle adds up to 180°.

To do this, begin by putting your triangles together to make a series of equal four-sided figures whose opposite sides are parallel. Use these to cover the plane. Label the angles of one of the triangles *A, B and C*. Place it in the middle of a sheet of paper and, using a straight edge, draw lines parallel to the three sides. Then draw three sets of equally-spaced parallel lines like the example.

Triangle Angles Puzzle before the Triangle Sum Theorem is a good confirmation.



This forms tessellated tiles in which each tiling piece is a triangle congruent to triangle *ABC*. **Congruent** means that all the tiling pieces, or triangles, have exactly the same size and shape. Label each of the angles in the picture *A*, *B*, or *C*. It is now easy to see something quite remarkable: the measures of angles *A*, *B*, and *C* sum to a straight angle. Explain why.

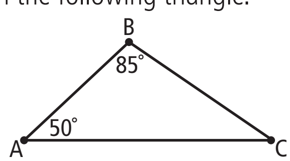
We can now state the **Triangle Sum Theorem**:

TRIANGLE SUM THEOREM
The sum of the measures of the angles in any triangle equals 180°.

The tessellation is a sketch of the proof that the sum of the measures of the angles in a triangle always adds up to 180°. How can knowing that the sum of 3 angles of a triangle is 180˚, help you figure out missing angles?

EXAMPLE

Find the measure of $\angle$C in the following triangle.



SOLUTION

m ($\angle$ A) + m ($\angle$ B) + m ($\angle$ C) = 180˚
50˚ + 85˚ + m ($\angle$ C) = 180˚
135˚ + m ($\angle$ C) = 180˚
m ($\angle$ C) = 45˚

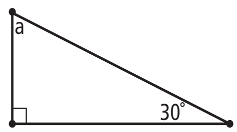
PROBLEM 3

a. 60° b. 75° c. 140°

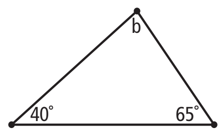
PROBLEM 3

Find the measure of the missing angles:

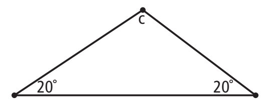
a.



b.



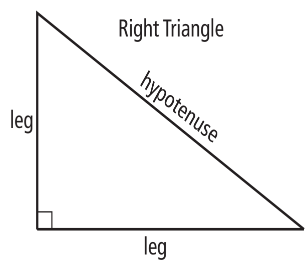
c.



In geometry, you will learn how to prove many interesting properties of geometric shapes using only the basic ideas above.

Much of the geometry that you study in middle school comes from the studies developed by Euclid several thousand years ago. He based the study of geometry on the foundations of axioms, postulates, and theorems. Part of the excitement of mathematics involves seeing new relationships based on simple ideas, like the sum of the angles in a triangle.

Triangles can also be categorized by their angles. One kind of triangle is a **right triangle**. A right triangle is a triangle with a right angle, which we know is an angle with measure 90°.



The longest side of a right triangle is called the **hypotenuse**. The right angle is **opposite** the hypotenuse. The two shorter sides are called the **legs** of the right triangle.

You will eventually learn a special theorem that relates the lengths of the legs of a right triangle to the the length of the hypotenuse. This theorem, the **Pythagorean Theorem**, enables you to find the length of any side of a right triangle if you are given the lengths of the other two sides.

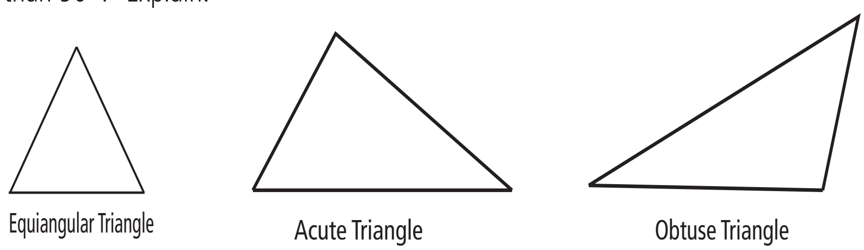
If you could cut both triangles out and place one on top of another, under the correct orientation, they would exactly overlap. This is intuitively what we mean when we say that two figures are congruent.

You may draw attention to two triangles with just congruent corresponding angles such as two equilateral triangles one with sides of length 2 and another with length 4. What can be said about the two triangles? This foreshadows the notion of similarity which is a "looser" relationship than congruence.

EXERCISES

1. a. actue, scalene b. acute, scalene c. obtuse, isosceles d. obtuse, isosceles
   e. right, scalene f. obtuse, scalene g. actue, scalene h. obtuse, scalene

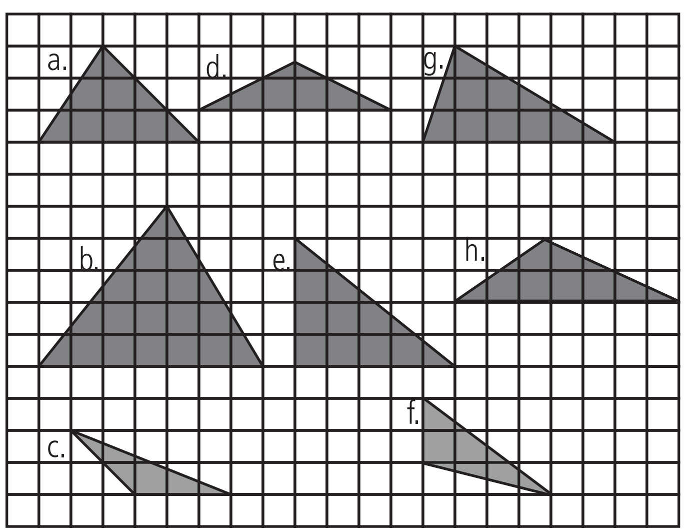
In addition to right triangles, there are other triangles that can be classified by their angles. A triangle with all three angles equal is an equilateral (equiangular) triangle. If all three angles of a triangle are acute, or less than 90°, the triangle is called an **acute** triangle. If one of the angles is larger than 90°, the triangle is called an **obtuse** triangle. Is it possible for a triangle to have two angles larger than 90°? Explain.



Two triangles are said to be congruent to each other if their corresponding sides and corresponding angles are **congruent** to each other. Experiment by drawing two different triangles on the same grid paper so that they each have the same lengths on all three sides. What would you say about their shapes? Measure their angles to verify that they also have the same angle measures. In some sense, they are exact copies of each other.

EXERCISES

1. For each triangle below, classify the triangles by their angles (right, acute obtuse) and then by their sides (equilateral, isosceles, or scalene). Explain your decision.



2. 30°
3. 45°
4. 60°
5. 60°

6. 80°

7. 45°

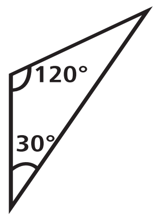
8. 60°

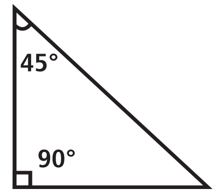
9. 40°

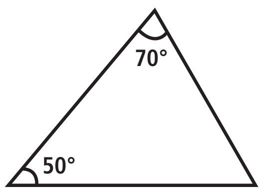
10. $X + 2X + 3X = 180°$; $X = 30°$; $2X = 60°$; $3X = 90°$

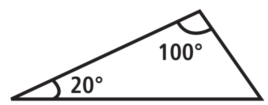
11. Measure of each angle should equal 60°

Find the measure of the angle missing in each of the triangles in Exercises 2–5.

2. 

3. 

4. 

5. 

6. A triangle has two angles whose measures are 35° and 65°. Find the measure of the third angle.
7. A right triangle has another angle whose measure is 45°. Find the measure of the third angle.
8. All of the angles in $\triangle ABC$ have the same measure. What is the measure of each of the angles?
9. A 120° angle is trisected, or divided into three equal angles. What is the measure of each of these angles? What would this look like?
10. A triangle has three angles with measures x, $2x$ and $3x$. Find the measure of each of these angles.
11. Use a protractor and ruler to draw an equilateral triangle with sides 5 inches long. What is the measure of each of the angles?
12. Consider each of the sets of three numbers. Determine which of the sets contain numbers that can be the lengths of sides of a triangle. If a triangle cannot be formed using the lengths, explain your reason. If a triangle can be formed, explain your reason and identify the kind of triangle formed such as scalene, isosceles, or equilateral.

13. Spiral Review (6.13 A) Set R {...,13, 15, 17, ...}; Set S {..., 21, 24, 27, ...}

14. Spiral Review (6.2 A) $8\frac{2}{5}$ or $42 \div 5$

Ingenuity

Since the sum of the angles of a triangle is 180 degrees, a triangle can have only one obtuse angle So since angle. A is larger than angle B, angle B cannot be the obtuse angle.

If angle A is the obtuse angle, then it can measure at most 92 degrees, since otherwise it would be at least 93 degrees, and angle B would measure at least 93 - 6 = 87 degrees, and 93 + 87 = 180, leaving no degrees for angle C. It is possible that angle A measures 92 degrees and angle B measures 86 degrees; this means that angle C measures 180 - 92 - 86 = 2 degrees. It is also possible that angle A measures 91 degrees and angle B measures 85 degrees; in this case, angle C measures 180 - 91 - 85 = 4 degrees. These are the only possibilities if angle A is the obtuse angle.

If angle C is the obtuse angle, then we know that the sum of angles A and B must be less than 90 degrees. The smallest possible value of the sum of the measures of angles A and B is 7 + 1 = 8 degrees; in this case, angle C measures 172 degrees. We can chart the possible measures of angles A, B, and C in a table:

$m(\angle A)$	$m(\angle B)$	$m(\angle C)$
7	1	172
8	2	170
9	3	168
10	4	166
...		
47	41	92

Notice that the measure of angle C can be any even number of degrees from 92 to 172. This is a total of 41 possibilities. Keeping in mind that angle C can also have measure 2 or 4 degrees (if angle A is the obtuse angle), this is a total of 43 possibilities.

Investigation

16. a. It takes 2 diagonals to divide a pentagon into triangles. We get 3 triangles by doing this.

b. It takes 3 diagonals to divide a hexagon into triangles. We get 4 triangles by doing this.

c. The pattern continues, and it takes n-3 diagonals to divide an n-gon into triangles. We get n-2 triangles by doing this. To see that the pattern will continue, observe that when we draw a diagonal of an n-gon, we get one triangle and an (n-1)-gon. If the pattern has continued up to this point, then we can draw n - 1 - 3 = n - 4 diagonals of the (n-1)-gon and get n - 1 - 2 = n - 3 triangles. Adding on the first diagonal we drew, and the first triangle we got, we have a total of n - 3 diagonals and n - 2 triangles. (This is an example of a proof by mathematical induction.)

a. 3, 3, 3

b. 1, 2, 3

c. 3, 4, 5

d. 5, 7, 5

e. 1, 3, 1

Spiral Review:

13. Look at set R and set S below. List three additional numbers that would belong to both sets R and S.

Set R = {1, 3, 5, 7, 9, 11...}

Set S = {3, 6, 9, 12, 15, 18...}

14. An isosceles triangle has legs with length $2\frac{3}{5}$ inches and a base of $3\frac{1}{5}$. What is the perimeter of the triangle? Represent the perimeter as both a mixed fraction and an improper fraction.

15. **Ingenuity:**

In triangle ABC, the measure of angle A is 6º more than the measure of angle B. If triangle ABC is an obtuse triangle, how many possible whole number measures are there for angle C?

16. **Investigation:**

A *diagonal* of a polygon is a line segment that connects one vertex of the polygon to another vertex, but is not a side of the polygon. We can divide a quadrilateral (a four-sided polygon) into two by drawing a single diagonal.

a. How many diagonals do we have to draw in order to divide a pentagon (a five-sided polygon) into triangles? How many triangles do we get?

b. How many diagonals do we have to draw in order to divide a hexagon (a six-sided polygon) into triangles? How many triangles do we get?

c. How many diagonals do we have to draw in order to divide an n-gon (an n-sided polygon) into triangles? How many triangles do we get?

Section 9.3 – Quadrilaterals and other polygons

Big Idea:
Classifying quadrilaterals and other polygons by their characteristics.

Key Objectives:
- Classify quadrilaterals as a specific polygon
- Classify and construct quadrilaterals based on sides and angles
- Identify regular polygons
- Characterize polygons by the number of sides
- Define parallel lines
- Discover relationship between angles in a quadrilateral
- Use algebra to find missing information about the angles in quadrilaterals

Materials:
Grid paper, Protractor, Ruler

Pedagogical/Orchestration:
- Have students use their skills in measurement to make conjectures about angles in a quadrilateral.

Activity:
"Polygon Riddles", "Quadrilateral Angles Puzzle" and "Polygon Practice"

Vocabulary:
quadrilateral, polygon, regular, parallelogram, trapezoid, parallel, rectangle, square, rhombus

TEKS:
6.6(B); 6.8(C); 6.12(A); 6.13(A,B) New: 6.8(A, B); 6.10(A)

Launch for Section 9.3:
Use Two-dimensional shapes lesson and revisit the Spaghetti Activity.

Spaghetti Activity - Quadrilateral: Break the spaghetti into four lengths. How can this be done? Discuss different quadrilaterals that can be formed in terms of their lengths. Some of the results should include: all four lengths are equal, leading to a square or a rhombus; all four lengths are unequal; two pairs of sides are equal; ideas about parallel sides may also surface. Students will most likely recognize from past experiences the terms rectangles, squares, trapezoids, paralleogram.

Two-Dimensional Shapes

(Suggestion: Make "Magic Strip" foldable for vocabulary)

POLYGONS and REGULAR POLYGONS

The word POLYGON comes from the Greek words poly, meaning "many," and gon meaning "angles." A polygon is made by joining a number of line segments to make a closed shape whose sides do not intersect each other. So each polygon can have many sides. The simplest polygon is a triangle. A quadrilateral is a polygon with "quad" or 4 sides. Polygons are named by the number of sides they have. They have the same number of angles as sides.

Regular Polygons have equal sides and equal angles.

Here are some common polygons:

Name of Polygon:	Number of Sides:	Sum of the Measure of All the Angles:
Triangle	3	180°
Quadrilateral	4	360°
Pentagon	5	540°
Hexagon	6	720°
Octagon	8	1440°

Let's explore the characteristics of the following polygons:

1. Quadrilaterals:
   a. Rectangle
   b. Square
   c. Trapezoid
   d. Parallelogram
   e. Rhombus

2. Triangles:
   a. Characterized by their angles:
      Right Triangle
      Obtuse Triangle
      Acute Triangle
   b. Characterized by their sides:
      Scalene Triangle
      Isosceles Triangle
      Equilateral Triangle

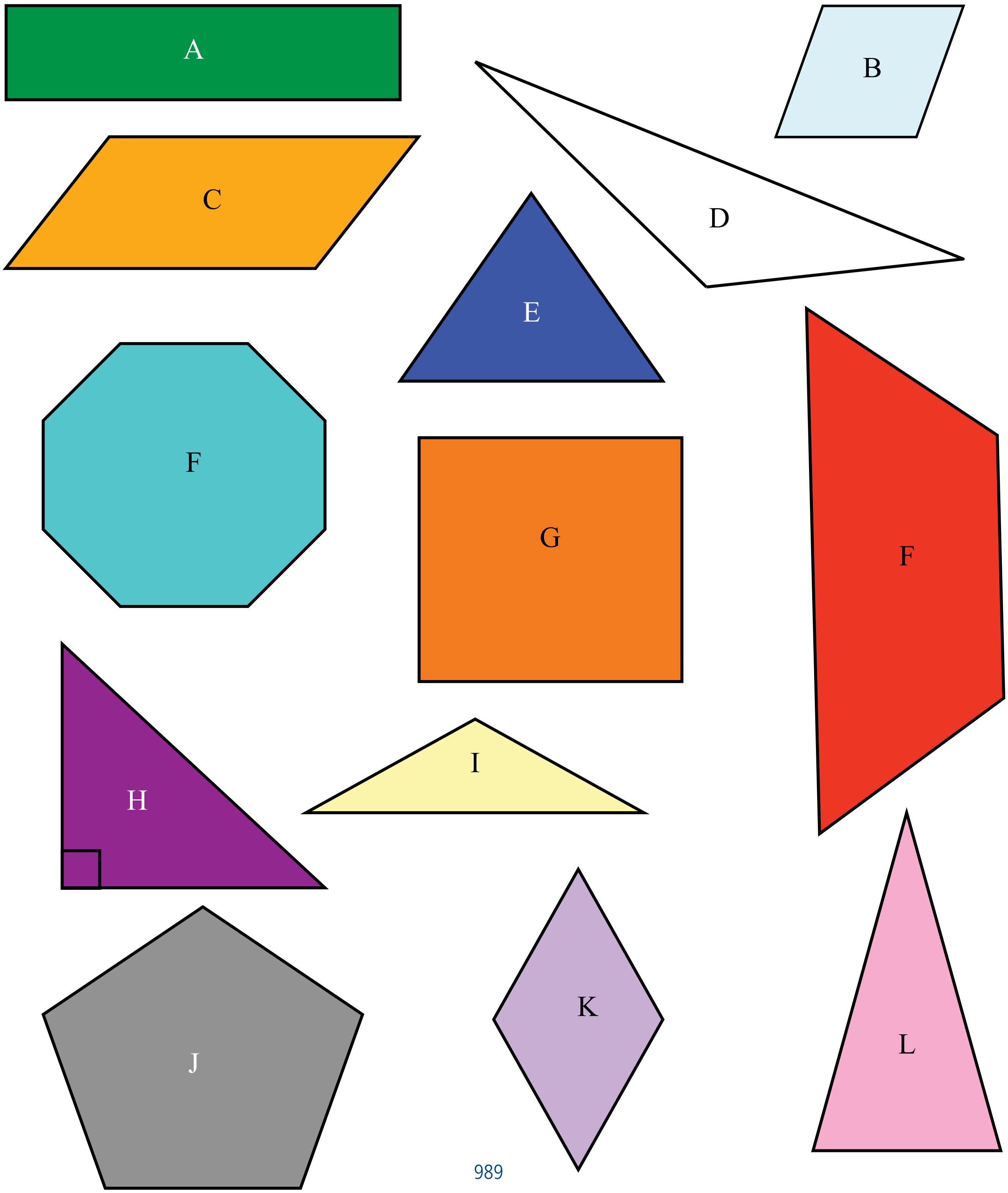
3. Other Polygons:
   Hexagon
   Pentagon
   Octagon

Polygon Riddles

Use the GEO-pieces (found on the next page) and your notes to answer the riddles.

1. I am quadrilateral with one pair of parallel sides.
2. I am a triangle with no equal sides.
3. I am a quadrilateral with 2 pairs of parallel sides.
4. I am a three-sided polygon with one right angle.
5. I am a parallelogram with four equal sides and my opposite angles are congruent.
6. I am a polygon with four sides, four right angles and the sum of my four angles is 360 degrees.
7. I am a polygon with three sides and the sum of my angles is 180 degrees.
8. I am a polygon with one obtuse angle.
9. I am a 5-sided polygon.
10. I am an 8-sided polygon.
11. I am a three-sided polygon with two congruent sides.
12. I am a triangle with equal sides and equal angles.
13. I am a 3-sided polygon with 3 acute angles.
14. I am a rectangle but not all rectangles are the same as me.
15. I am a parallelogram with only equal sides.

▲ ▶ ■ GEO-pieces ▲ ▶ ■



Quadrilateral Angles Puzzle

Objective: Students will prove that the sum of the interior angles of a quadrilateral is equal to 360 degrees.

Materials:
Multi-color sheets of construction paper
Glue sticks
Scissors
Protractors
Rulers

Activity Instructions:
Teacher provides each student with ½ of sheet of construction paper in two different colors. Each student starts by taking one of the pieces of construction paper and draws a medium-size quadrilateral of any type (square, rectangle, parallelogram, trapezoid, etc.) and labels each angle or corner with numbers: 1,2,3, 4. Students cut out their respective quadrilateral.

Using their rulers students draw a straight line in the middle of the remaining ½ of construction paper. Then, the teacher shows students how to tear off each corner of the quadrilateral. Students collect all four corners and glue them together on top of the straight line drawn on their construction paper. This is like putting the quadrilateral puzzle together!! The piece of the quadrilateral that is left over once the four corners are cut off may be glued next to the number line and as a separate piece of the puzzle.

Students take time to use their protractors to measure the sum of the four angles glued on the paper to prove that the sum is indeed equal to 360 degrees.

At the end, students will have a clear, concrete picture of how the sum of the measures of the four angles equals 360 degrees.

Polygon Practice

Practice finding the angle measures of the following polygons.

1. What is the sum of the angle measures of a triangle? ______	2. What is the sum of the angle measures of a quadrilateral? ______	3. A quadrilateral has angles that measure 90°, 100° and 120°. What is the measurement of the fourth angle? ______
4. A parallelogram has opposite angles that each measures 100°. What is the measurement of each of the other two angles? ______	5. A triangle has angles that measure 40° and 90°. What is the measurement of the third angle? ______	6. An isosceles trapezoid has an angle that measures 45°. What are the measurements of the other three angles? ______
7. What are the angle measures of a rhombus if one of its angles is 25°? ______	8. The sum of the angle measures of an equilateral triangle is 180°. What is the measure of each angle? ______	9. The sum of the angle measures of a regular hexagon is 720°. What is the measure of each angle? ______

10a. Suppose four angle measures are 25°, 75°, 50° and 30°. Can the measures of these three angles be used to form a quadrilateral? ______ Justify your answer. ______

______

10b. Write an equation to prove that the sum of the angle measures is 180°. Use a variable to represent the unknown angle measure.

Equation: ______

Solve and find the missing number in the equation:

Have students draw an example of each. 12-gon gets to be tricky.

SECTION 9.3 QUADRILATERALS AND OTHER POLYGONS

"Tri" in triangle gives us a clue about the number of sides in a polygon. Similarly, "Quad" in quadrilateral refers to its four sides. Other prefixes provide clues to the number of sides a polygon has. Here are some of the prefixes that are used with polygons:

Number of sides of a polygon	Prefix	Name of polygon
3	Tri	Triangle
4	Quad	Quadrilateral
5	Penta	Pentagon
6	Hexa	Hexagon
8	Octa	Octagon
10	Deca	Decagon
12	Dodeca	Dodecagon
n (in general)		n-gon

All **quadrilaterals** have the common characteristic that they are polygons with four sides. Just as with triangles, quadrilaterals can be classified by properties of sides and angles. We first make an important observation about lines and line segments.

Let's think carefully about what seems like a simple concept, the idea of "parallel" lines. The question is how you decide whether two lines are actually parallel. In fact, what does it mean to say that they are parallel in the first place?

One way to describe parallel lines is to say that two lines in a plane are **parallel** if they never intersect, even if they are extended forever in both directions.

As we examine different types of quadrilaterals, we consider the lengths of sides as well as whether opposite sides are parallel or not.

Common quadrilaterals include:

- **Parallelogram**: A quadrilateral with opposite sides parallel. The lengths of the opposite sides will be congruent.
- **Rectangle**: A parallelogram with a right angle.

PROBLEM:

A. parallelogram, rhombus
B. parallelogram, rectangle
C. parallelogram, rectangle, rhombus, square
D. parallelogram, rectangle, rhombus, square
E. trapezoid

EXPLORATION

Students should discover that the adjacent angles are supplementary and the opposite angles are congruent.

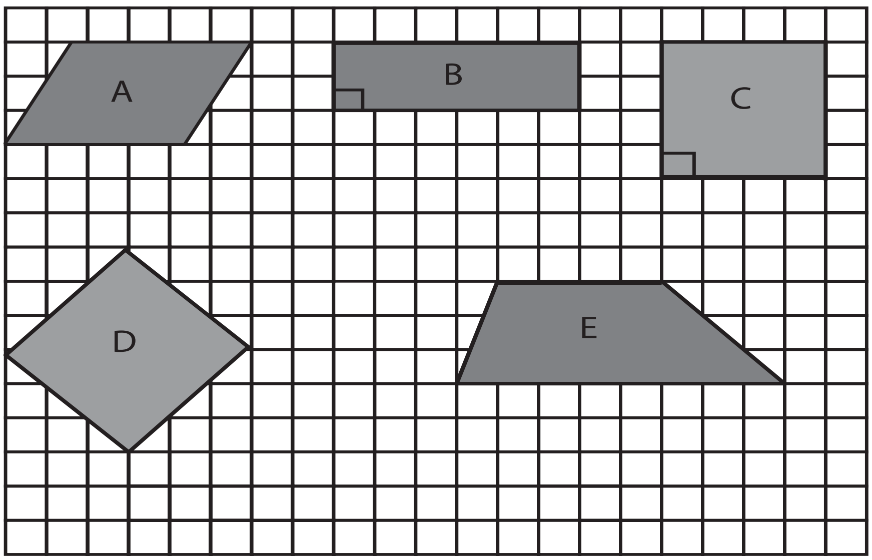
The quadrilateral can be divided into two triangles by drawing in one diagonal.

Polygon Riddle Activity would be a good activity to do at this point.
Polygon Practice is appropriate as an assessment.

- **Square**: A rectangle with all four sides of equal length.
- **Trapezoid**: A quadrilateral with exactly one pair of opposite sides parallel.
- **Rhombus**: A parallelogram with all four sides of equal length.

PROBLEM

Classify the following quadrilaterals with the appropriate term or terms that apply..



EXPLORATION

Draw three different parallelograms on a grid paper. Measure the angles in each parallelogram. Make two conjecture about the relationship among the angles that you think must be true for all parallelograms. Explain your reasoning.

In fact, opposite angles in a parallelogram are congruent to each other. Consecutive angles in a parallelogram are supplementary.

Rectangles and squares give us a clue that the sum of the angles is 360 degrees. It is not so clear when you examine general quadrilaterals. You could consider breaking a quadrilateral into two triangles. Can you see a way to use what you know about the sum of the angles in a triangle to make a conjecture about the sum of the angles in a quadrilateral?

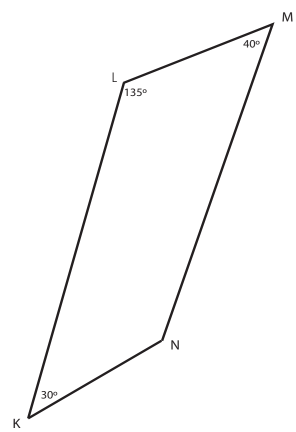
Quadrilateral Angles Puzzle Activity

Draw a trapezoid and a parallelogram on grid paper. Label the four angles in each of your quadrilaterals. Use your protractor to carefully measure the four angles in each shape and record the measures. What do you notice in your findings? Make two conjectures about the relationships among the angles that you think must be true for all parallelograms.

We summarize the observation you made about the angles in a quadrilateral: the sum of the angles in a quadrilateral is 360°.

The Polygon Riddles Activity will give you a chance to check your understanding of polygon classification.

Sometimes one or more measures of angles are missing in a quadrilateral. How can knowing that the sum of the four angles is 360° help us to find the missing measures?



EXAMPLE 1

Find the measure of $\angle$ N.

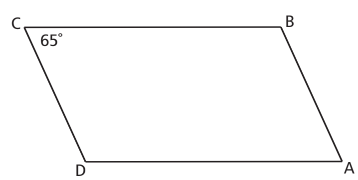
SOLUTION

$m(\angle K) + m(\angle L) + m(\angle M) + m(\angle N) = 360^\circ$
$30^\circ + 135^\circ + 40^\circ + m(\angle N) = 360^\circ$
$205^\circ + m(\angle N) = 360^\circ$
$m(\angle N) = 155^\circ$

EXAMPLE 2

Find the measure of the missing angles

in the parallelogram ABCD to the right.



A regular triangle is referred to as an equilateral or equiangular triangle. A regular quadrilateral we call a square.

EXERCISES

1. Drawings will vary.

SOLUTION

$m(\angle A) = m(\angle C)$ so $m(\angle A) = 65°$. $m(\angle A) + m(\angle B) + m(\angle C) + m(\angle D) = 360°$. So $m(\angle B) + m(\angle D) = 360° - 130° = 230°$. $m(\angle B) = m(\angle D)$ so $m(\angle B) = 230 \div 2 = 115° = m(\angle D)$.

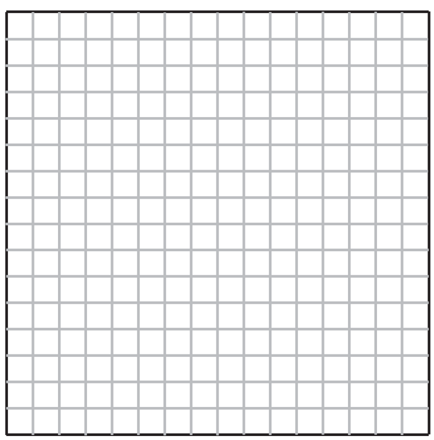
$m(\angle A) = 65°$, $m(\angle B) = 115°$, $m(\angle C) = 65°$, $m(\angle D) = 115°$

The sides of polygons are in general of varying lengths. If a polygon has all sides of equal length and the angles also of equal measure then we give it a special name. A polygon is said to be **regular** if the sides and the angles are all of equal measure.

Can you think of any polygon with 3-sides that are regular? With 4-sides?

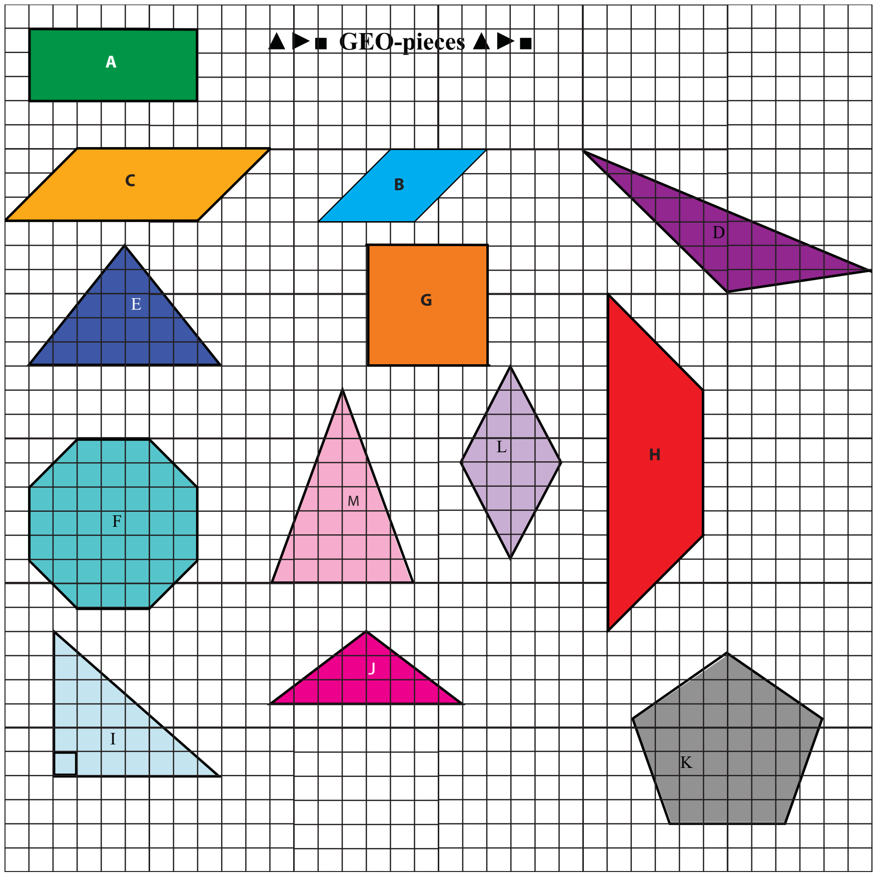
EXERCISES

1. Draw the five shapes specified on graph paper. Be sure all the requirements are met.
   a. A parallelogram that is not a rectangle.
   b. A rectangle that is not regular.
   c. A trapezoid with a right angle.
   d. A parallelogram whose four sides are congruent but is not regular
   e. A regular quadrilateral.



2. A. Quadrilateral, Parallelogram, Rectangle
   B. Quadrilateral, Parallelogram
   C. Quadrilateral, Parallelogram
   D. Triangle, obtuse scalene triangle
   E. Triangle, equilateral triangle
   F. Octagon
   G. Quadrilateral, Parallelogram, Rectangle
   H. Quadrilateral, Trapezoid
   I. Triangle, Right Triangle
   J. Triangle, Obtuse Triangle, Isosceles Triangle
   K. Pentagon
   L. Quadrilateral, Parallelogram, Rhombus
   M. Triangle, Acute Triangle, Isosceles Triangle

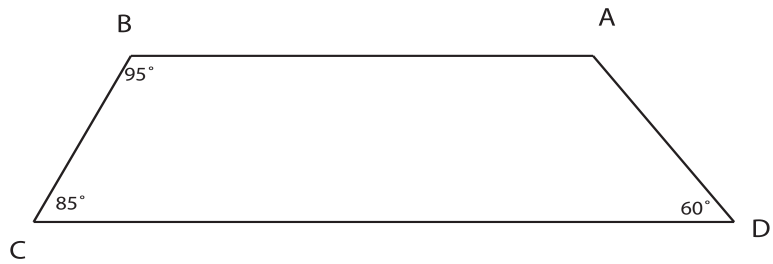
2. Classify the following shapes, writing all the names that apply to each figure.



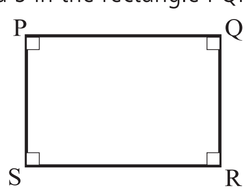
3. a. $m(\angle A) = 120^\circ$
   b. 180°
   c. $m(\angle C) = 110^\circ$, $m(\angle D) = 70^\circ$
   d. $X = 10$, $Y = 8$
   e. $m(\angle G) = 45^\circ$, $m(\angle H) = 135^\circ$

3. Find the measure of the specified part.

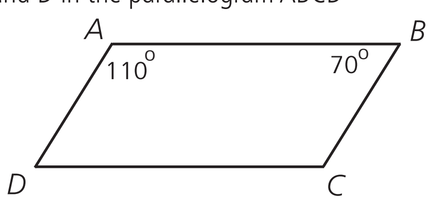
   a. Angle A in the trapezoid ABCD



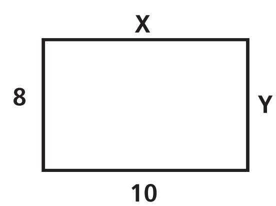
   b. Sum of Angles P and S in the rectangle PQRS



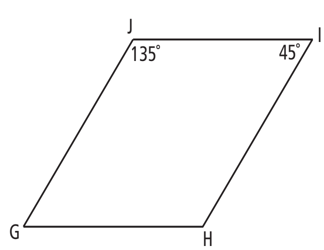
   c. Angles C and D in the parallelogram ABCD



   d. Lengths of sides X and Y in the rectangle PQRS

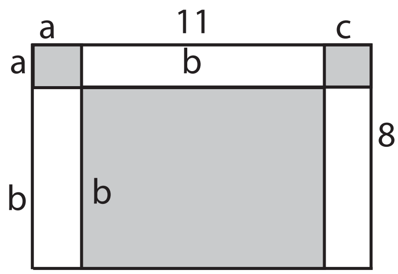


   e. Angle G and Angle H in rhombus GHIJ



3. f. m ($\angle$ M) =70 °, m ($\angle$ N) = 70 °

4. Lable the figure as below:



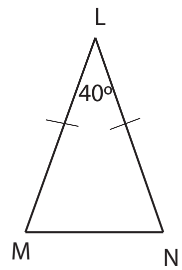
a + b + c = 11
a + b = 8 so 8 + c = 11 and c = 3. Similarly, c + b = 8. Since a + b = 8 and a = c, we find that b = 5. It follows that a = 3.

5. 120°

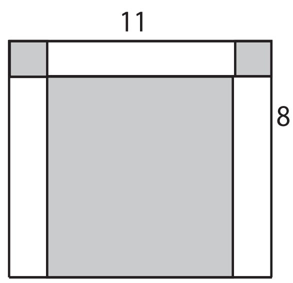
6. \$672 ÷ \$8 = 84 cars

7. 9″
The length of the right side of the big square is 75 inches, so the distance from the top of the bottom square to the top of the big square is 75 - 33 = 42 inches. But the height of the top square is also 33 inches, so the height of the small square in the middle is 42 - 33 = 9 inches. By the same reasoning, we can show that the width of the small square is also 9 inches.

f. $\angle M$ and $\angle N$ in isosceles triangle LMN.



4. In the 11 by 8 rectangle below the **shaded regions represent squares**. Find the side lengths of each of the squares.

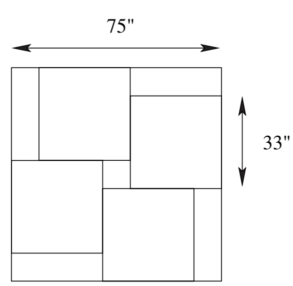


5. Determine the sum of the interior angles of a hexagon. Explain how you arrived at this measure. If the hexagon is regular, what is the measure of each interior angle?

Spiral Problems

6. The student council raised $672 by washing cars on Saturday. If they charged $8 to wash each car, how many cars did they wash?

7. In the figure below, four squares, each having sides of length 33 inches, are nested inside a larger square with sides of length 75 inches. What is the length of the sides of the small square in the middle?



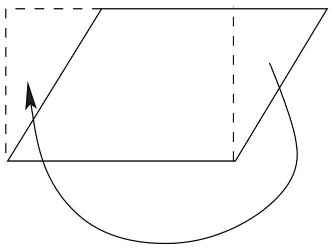
INGENUITY

8. a. All containers hold more than a quart
   b. large and super
   c. 20 cups

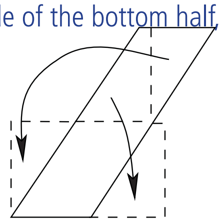
9. INVESTIGATION

The purpose of this Investigation is to explore the idea of cutting up various quadrilaterals into parts which can be rearranged to form rectangles. This technique is an important tool when we derive the areas of parallelograms and trapezoids in the next section. This Investigation may be an excellent problem to explore in class; cut out some parallelograms and trapezoids, and let students try cutting them up and rearranging the parts.

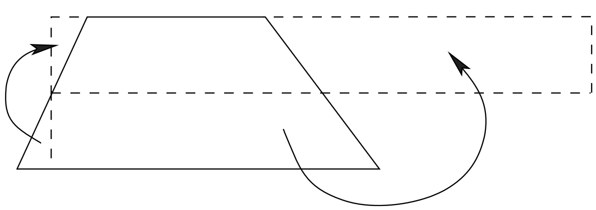
a. Yes, we can make a rectangle by cutting a triangle off of the original parallelogram and moving it to the opposite side as shown below:



b. This case is a bit more difficult. We cannot use the same method we used in part (a), since we cannot draw a vertical line through the parallelogram from the top side to the bottom. However, we can cut the parallelogram in half using a horizontal line, and then cut a triangle off of the right side of the top half. We move the triangle to the left side of the bottom half, and the rest of the top half to the right side of the bottom half.



c. Again, we begin by cutting horizontally along the midline of the figure. We then cut a triangle off the left side of the bottom half, and align it with the left side of the top half. We align the rest of the bottom half with the right side of the top half, as shown below.



Ingenuity:

8. The table below shows the liquid capacity of 4 sizes of floor cleaners.

Size	Name of polygon
Mini	45 fluid ounces
Regular	100 fluid ounces
Large	130 fluid ounces
Super	160 fluid ounces

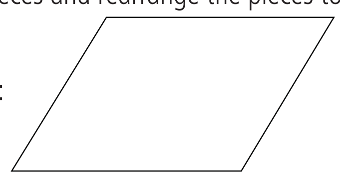
a. Which containers hold more than a quart?

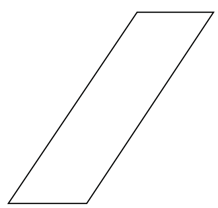
b. Which containers hold more than a gallon?

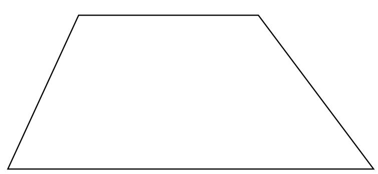
c. How many cups does the super size contain?

Investigation:

9. For each of the following figures, determine whether it is possible to cut a figure into several pieces and rearrange the pieces to form a rectangle.

a. A parallelogram: 

b. Another parallelogram: 

c. A trapezoid: 

Section 9.4 - Perimeter and Area

Big Idea:

Discovering properties of certain two-dimensional figures which lead to area formulas.

Key Objectives:

- Apply the definition of perimeter
- Derive the formulas for the areas of triangles and quadrilaterals
- Apply the definition of perimeter and area on composite and other 2-dimensional shapes.

Materials:

ruler, grid paper, color tiles

Pedagogical/Orchestration:

- Students will make conjectures about rules leading to the formulas for the area of rectangles, parallelograms, triangles and trapezoids

Exercises:

Break up exercises as follows:

- Exercises 1, 2, 7, 8, 10, 11, 13 and 15 after problems following the parallelogram section.
- Exercises 3, 4, 5, 6 and 12 after the problems following the triangle section.
- Exercises 14 after the problems following the trapezoid section.

Vocabulary:

Perimeter, area, altitude, height, base, length, width

TEKS:

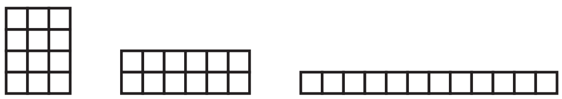
6.8(A,B); 6.13(A,B)

New: 6.8(B, C, D)

Launch for Section 9.4:

a. Teacher prepares color tiles so that each student has 12 tiles. Ask the students to form different rectangles using all the tiles. At the same time, students fill a table that will help them kept track of the rectangles. For example:

Number of Tiles	Length	Width	Perimeter
12	4	3	14
12	6	2	16
12	1	12	26



Students will share what they noticed about the relationship between area and perimeter. Students should note that area is the inside of the rectangles. Perimeter is the length around each rectangle.

b. Now have students use the same 12 tiles to form irregular shapes/polygons so that their perimeter and area are not the same as the rectangles done before. Again, students fill in a table and use it to discuss how the area and perimeter change. For example:

Area	Perimeter
12	24
12	
12	
12	
12	
12	

EXPLORATION 1

Answers may vary, depending on criteria. Encourage discussion and debate.

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The idea of area and perimeter may come up naturally. If they do not, listen to what the students do observe.

Have students decide what we mean by biggest. The students will see that the criteria will produce different answers to the question.

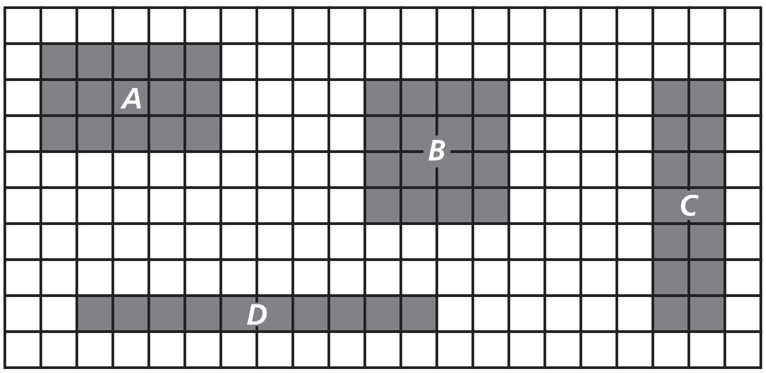
1. 15, 16, 14 and 10 sq. units
2. B
3. Answers will vary.
4. A: 16; B: 16; C: 18; D: 22
5. D
6. Answers will vary.

SECTION 9.4 PERIMETER AND AREA

How do you find the area of a 3 cm x 4 cm rectangle? How is this different from the distance around a 3 cm x 4 cm rectangle?

EXPLORATION 1

Of rectangles *A*, *B*, *C* and *D*, which is the biggest? Explain your answer.



DISCUSSION

There are several ways to think about what "biggest" means. One way to measure "biggest" is to find the area by counting the number of unit squares that are needed to cover each figure.

1. What are the areas of rectangles *A*, *B*, *C* and *D*?
2. Which one has the largest area?
3. Does this agree with the rectangle you chose?

Another way to measure the size of a rectangle is to add the lengths of all the sides. This sum is called the **perimeter**. Its name comes from the Greek words *peri*, meaning "around," and *metron*, meaning "measure."

4. What are the perimeters of the 4 rectangles?
5. Which one has the largest perimeter?
6. Does this agree with the rectangle you chose?

Have students discuss and present different ways to compute the perimeter: L + W + L + W or 2L + 2W or 2(L + W).

Notice the area can be obtained by counting the squares. Another way to find the area is to break the shape into smaller squares and rectangles. The area is equal to 24 square units. The perimeter is 22 units.

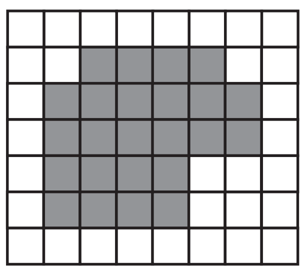
What is the definition of perimeter? What is the perimeter of a 3×4 rectangle? In general, if a rectangle has length L and width W, what is the perimeter of the rectangle? You saw that the perimeter of a rectangle is the sum of the lengths of the four sides. You can write this as perimeter of the rectangle $P = L + L + W + W = 2L + 2W = 2(L + W)$.

Notice that squares are special rectangles whose sides are all the same length. If we call the side length as s, then the perimeter of the square, $P = s + s + s + s = 4s$

Just as you found the perimeter of a rectangle by adding the lengths of the four sides, you can find the perimeter of any polygon.

DEFINITION 9.4: PERIMETER
The **perimeter** of a polygon is the sum of the lengths of all of its sides.

If the units of the length and width are inches, then the perimeter is measured in inches. If the unit of measure is not given, then the perimeter is measured in units.



DEFINITION 9.4: AREA
Area is the measure that it takes to cover the space inside the shape. When you formed shapes using tiles, the number of tiles used was the area of the shape.

Area is measured in square units. It can be labeled as square units or $units^2$ or u^2. What is the area of the shaded region above? Also, find the perimeter around the shaded region.

EXPLORATION 3

Draw the 2 x 6 rectangles that you formed with tiles in the Launch activity. What is its length? What is its width?

Now remember that 2 and 6 are dimensions that multiplied together gives the area of this rectangle. There is a formula for this that says: Area = length · width or A = L · W. Can you use this formula to find the area of a square? Draw a 6 x 6 square.

FORMULA 9.1 AREA OF A RECTANGLE
$A = L \cdot W$
where L is the length and W is the width of the rectangle

If you use the formula A = l · w you would see that 6 x 6 = 36 square units. Because the square is a special type of rectangle, it has its own formula.

$$\text{Area} = s \cdot s$$

$$A = s^2$$

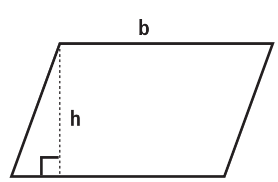
FORMULA 9.2 AREA OF A SQUARE
$A = S^2$
where L is the length of the side of the square

EXPLORATION 4

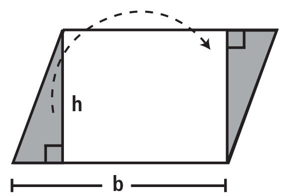
Draw four parallelograms using grid paper. For this exploration, make sure the longest side is on one of the grid lines.

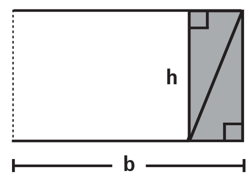
a. Measure the length of each of the sides and the measure of each angle. What do you observe?

b. Find the area of one of the parallelograms by cutting the parallelogram apart as illustrated below, and reassembling it to make a rectangle.



Label one of the horizontal parallel sides of the parallelogram the **base**, with length b. To find the height, draw a line segment between the two bases perpendicular to each base. The **height**, h, is the length of the perpendicular distance between the two bases. Notice that the height in parallelograms, unless they are rectangles, is not the same as the length of either of the two non-horizontal sides.



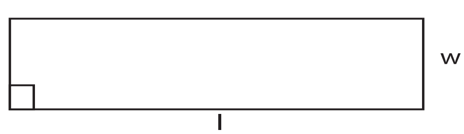


When reassembled, the parallelogram creates a rectangle with length, or base, b, and width, or height, h. That means the formula for the area A of a parallelogram is:

FORMULA 9.3 AREA OF A PARALLELOGRAM
$A = b \cdot h$ or $A = bh$.

The formula can be used for parallelograms with any positive rational number. For example, a a parallelogram with base = 3.6 and height 2.4 would have area = (3.6)(2.4) = 8.64 square units.

Rectangles are special parallelograms with four right angles. In this case, the height of the parallelogram can be thought of as the width of the rectangle and the base of the parallelogram the length. he area formula for a rectangle as we saw earlier can be written as A = L · W.



Ask students to complete the following exercise: 1, 4, 5, 10. 11. 12. 13. 14 and 16.

EXPLORATION 4

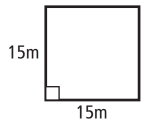
These areas will be approximate:

A = $8u^2$;	B = $12u^2$;	C = $4u^2$;	D = $8u^2$;	E = $10u^2$;	F = $8u^2$
G = $4u^2$;	H = $9u^2$;	I = $7.5u^2$;	J = $14u^2$		

PROBLEM 1

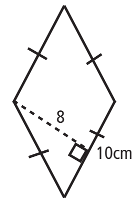
Find the area and perimeter of each figure. Draw an accurate parallelogram that is not a rectangle, then decompose and rearrange the parts of the parallelogram to create a rectangle and then use the rectangle formula to find its area.

a.

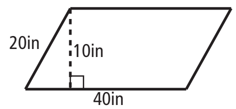


b.

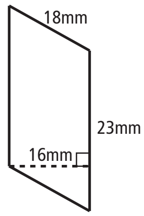
c.



d.

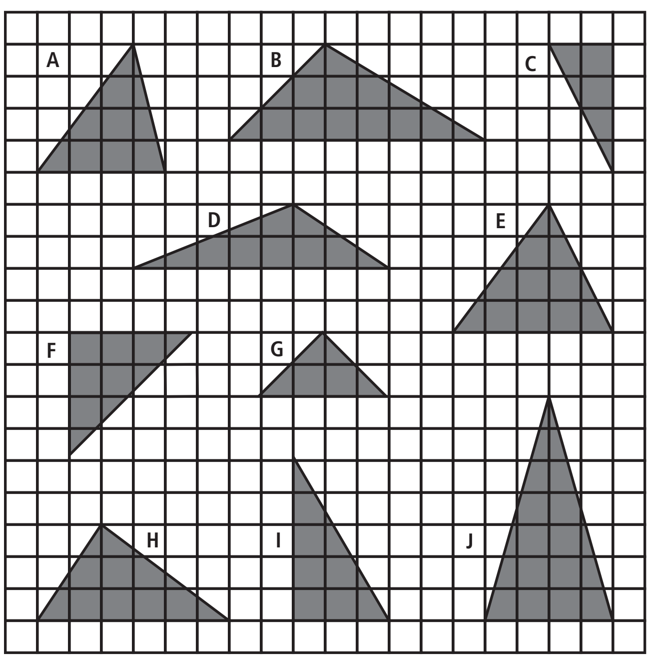


e.



EXPLORATION 4

Using the grid below, estimate the area of each triangle in square units. The bottom is usually called the **base** of the triangle. Sometimes you may have to rotate the triangle to find its base.



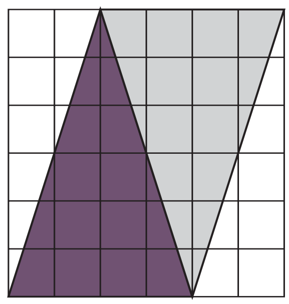
a. Expect students to cut and piece together unit squares to find the total area. The ways in which they do this vary and will indicate answers for part (b).
b. They might see that each triangle is half of a rectangle with the length of the rectangle being the base of the triangle and the width of the rectangle being the height of the triangle. If they don't discover this, move on to part (c).
c. The students might discover and need to name the "height" of the triangle. Let them discover that this measurement is useful. Help them discover the connection between the rectangle with area $L \cdot W$ = base · height $= b \cdot h$.

a. How did you compute the areas of each triangle?

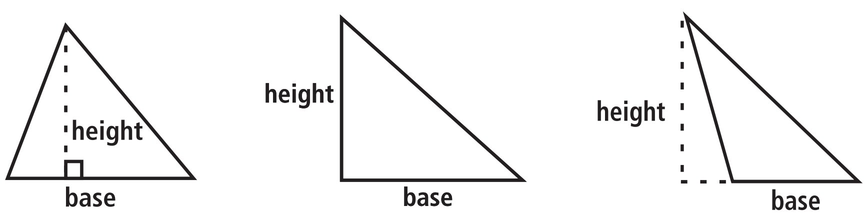
b. What patterns did you notice? Explain.

c. When you found the sum of the angles in a triangle in Section 10.2, you pasted two triangles together to form a four-sided figure. Using the triangles above, make a copy of each triangle and paste it together with the original triangle. What shape do you get? Use this process to find a rule for the area of these triangles.

What is the area of a triangle and is there a formula to compute this area? You have seen that taking any triangle, copying it exactly and putting the two triangles together creates a parallelogram.



You can use the formula for the area of the parallelogram and take one-half of it to compute the area of the triangle. Be careful in identifying the base and the height of the triangle. The base must be a side of the triangle, and the height, or **altitude**, must be perpendicular to the base, or an extension of the base, and be drawn from the vertex opposite the base.



Notice that a height of a triangle may be a side of a triangle as in a right triangle, In general, a height of a triangle is not a side of a triangle but an additional line segment that must be drawn in.

With those restrictions, the formula for the area A of a triangle with height h and base b is:

You may wish to have a big triangle on which the students can point or drawn in what the height would look like. What they should note is the perpendicularity and for an obtuse triangle the fact that a height could go "outside" the triangle and onto the extension of a base. These are introductory ideas but one that expands the idea that there is more than one height and base pair that can be used in a triangle but that the product of the base and height divided by 2 will all be equal as the area of the same triangle should be. You need not quantify the sides and the heights but provide some foreshadowing for students to see the base/height pairs and for students to better understand what perpendicular means.

PROBLEM 2

a. $7 + 8 + 9 = 24$ inches b. $12 + 9 + 15 = 36$ feet

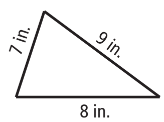
FORMULA 9.4: AREA OF A TRIANGLE

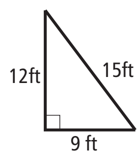
$$A = \frac{1}{2}b \cdot h \text{ or } A = \frac{1}{2}bh \text{ or } A = \frac{bh}{2}.$$

In the three triangles given above, a base and a height are specified. Can you see another possible base and height pair? Remember there are three sides to a triangle so there are three possible choices for the base. If you use the area formula, do you suppose the area of the triangle would be the same for any base and height combination?

PROBLEM 2

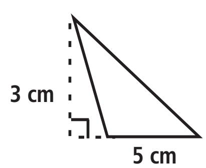
Find the perimeter for each triangle.

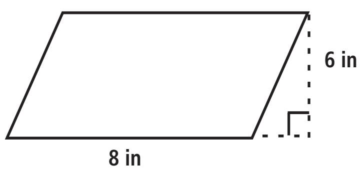
a.


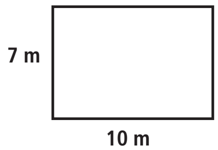
b.


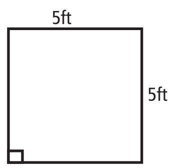
EXAMPLE 1

Find the area of the following polygons.

a.


b.


c.


d.


Students should explain that based on the area of a parallelogram formula, $A = bh$, the area of this parallelogram of two trapezoids should be $(b_1 + b_2)h$. The area of one trapezoid is then $\frac{1}{2}(b_1 + b_2)h$

SOLUTION

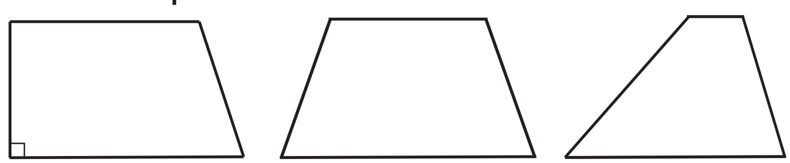
a. $A = \frac{bh}{2} = \frac{5 \bullet 3}{2} = \frac{15}{2}$ sq. cm.

b. $A = bh = 6 \cdot 8 = 48$ sq. in.

c. $A = bh = LW = 10 \cdot 7 = 70$ sq. m.

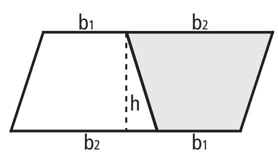
d. $A = LW = s \cdot s = s^2 = 5^2 = 25$ sq. ft.

A quadrilateral with two pairs of opposite sides parallel is called a parallelogram. If a quadrilateral has exactly one pair of opposite sides parallel, this 2-dimensional figure is called a **trapezoid**.



Draw a trapezoid on a grid paper. Draw an upside-down congruent trapezoid next to the first. What is the resulting shape?

The two trapezoids should form a large parallelogram similar to the picture below.



Using what you know about parallelograms, what is the area of this parallelogram consisting of two trapezoids? Because the parallelogram consists of the same trapezoid twice, what should the area of one trapezoid equal?

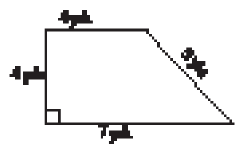
FORMULA 9.5: AREA OF A TRAPEZOID

$$A = \frac{1}{2} \bullet (b_1 + b_2) \bullet h = \frac{(b_1 + b_2)h}{2}$$

The formula can be used for trapezoids with any positive rational number, for example a trapezoid with base1 = 5.5, base 2 = 7.3 and height = 3.1 would have area = ½(5.5 + 7.3)3.1 = 19.84 square units.

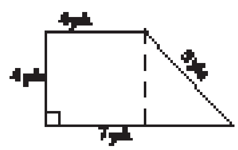
EXAMPLE 2

Find the area and perimeter of the following trapezoid. For the area, consider decomposing the figure into two recognizable shapes and finding these two smaller areas to attain the total area of the trapezoid:



SOLUTION

The trapezoid in Example 2 can also be thought of as a composite figure that is made up of other figures put together. In our case, the trapezoid is a composite of a square and a triangle as you can see below:



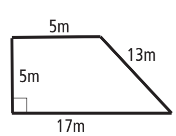
The area of the square is 4 x 4 = 16.

The area of the triangle is ½(4 x 3) = 6.

The area of the trapezoid is 16 + 6 = 22

PROBLEM 3

Find the area and perimeter of the following trapezoid. For the area, try decomposing the figure into two recognizable shapes, then finding the two smaller areas to find the total area of the trapezoid.



EXERCISES

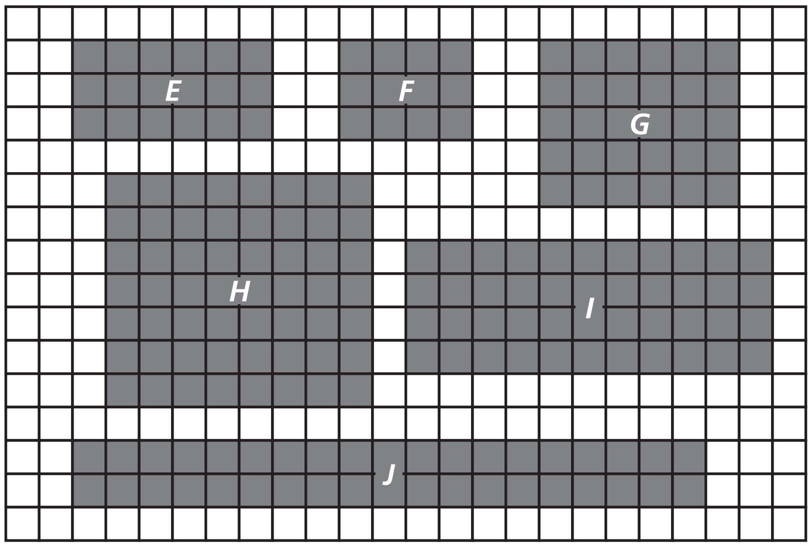
1.

Rectangle	Length	Width	Area	Perimeter
E	6	3	18	18
F	4	3	12	14
G	6	5	30	22
H	8	7	56	30
I	11	4	44	30
J	19	2	38	42

Answers will vary. Possible relationships that hold true are that area = length times width or that perimeter equals twice the length plus twice the width. Or it appears at times that as the length and width go up, the area and perimeter go up.

EXERCISES

1. Calculate the area and perimeter of each rectangle on the following grid. Make a table like the one that follows, showing the length, width, area and perimeter of each rectangle. Assume the length is the horizontal distance.



Rectangle	Length	Width	Area	Perimeter
E				
F				
G				
H				
I				
J				

What patterns do you notice in the values in the table? Do you see any relationships between the four categories that you can state as a rule?

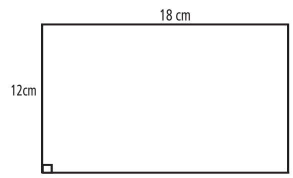
2. a. Area = 216 square centimeters; Perimeter = 60 cm.
   b. Area = 196 square inches; Perimeter = 56 inches
   c. Area = 1064 square millimeters; Perimeter = 132 mm
   d. Area = 100 square centimeters; Perimeter = 40 cm

3. a. Area = 96 square inches; Perimeter = 48 inches.
   b. Area = 322 square inches; Perimeter = 78 inches
   c. Area = 42 square inches; Perimeter = 28 inches

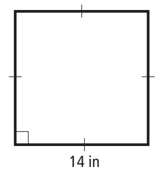
4. Row 1: 12, 6, 9 sq units
   Row 2: 7.5, 17.5, 17.5 sq units
   The areas of the triangles are half the areas of the rectangles.

2. Find the area and perimeter of the following rectangles and squares. Refer to the area formulas for rectangles and squares.

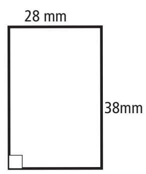
a.



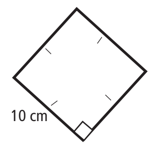
b.



c.

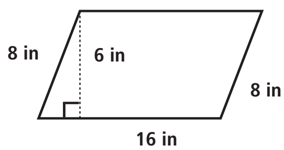


d.



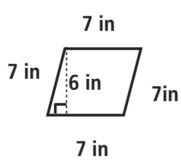
3. Find the area and perimeter of the following parallelograms.

a.

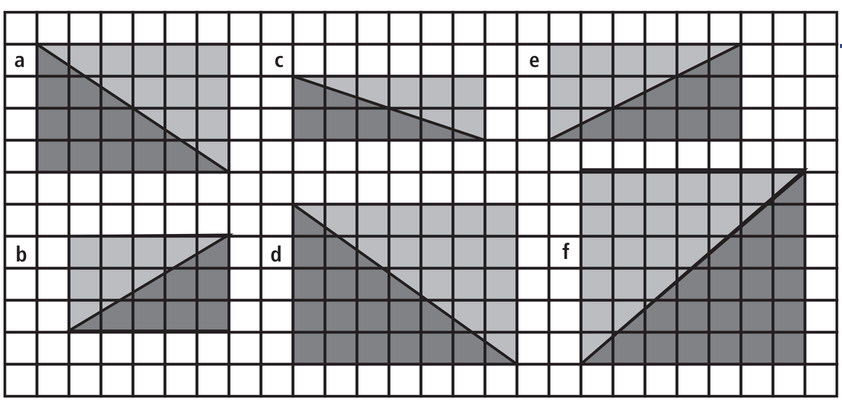


b.

c.



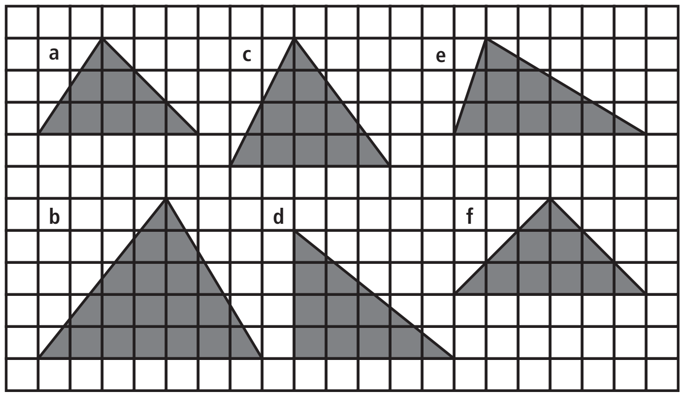
4. Calculate the area of the triangles below. How does the area of each rectangle relate to the area of the triangle inside it?



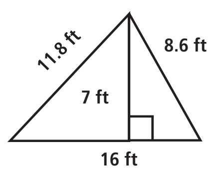
5. Row 1: 7.5, 10, 9 sq units
   Row 2: 17.5, 10, 9 sq units

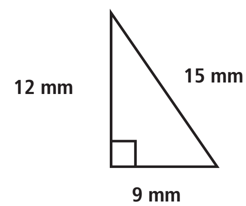
6. a. Area = 1/2(*bh*) = *1/2* · (16) · (7) = 56 square feet; Perimeter = 36.4 feet.
   b. Area =1/2(*bh*) = *1/2* · (9) · (12) = 54 square millimeters; Perimeter = 36 millimeters
   c. Area = 1/2(*bh*) = *1/2* · (10) · (12) = 60 square meters; Perimeter = 36 meters

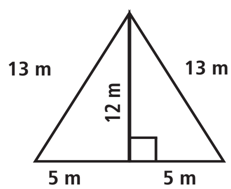
5. Calculate the areas of the triangles below using the area formula for a triangle.



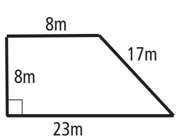
6. Find the area and perimeter of the following triangles.

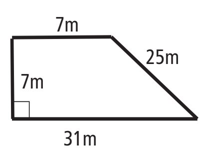
a. 

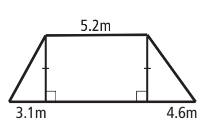
b. 

c. 

7. Find the area of the following trapezoids by decomposing the trapezoid into squares and triangles.

a. 

b. 

c. 

11. Area = 25 sq ft + 1040 sq ft + 75 sq ft = 1140 sq ft.

9. Assuming the field is a rectangle, you will need to know the length and width of the area to be covered.

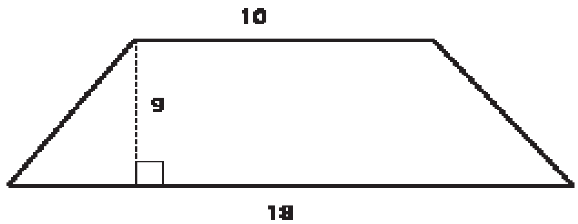
10. Let l = length. $l = 2w = 2(28) = 56$ in. $P = 2(l + w) = 2(56 + 28) = 168$ in.

11. 24 inches

12. Generally, if the two dimensions are closer, the area increases and the perimeter decreases.

14. Although not drawn to scale, students may decompose the figure into shapes that they can find the areas of and add together these areas. $(45)(82) + \frac{1}{2}(3)(7) + (8)(7) = 3690 + 10.5 + 56 = 3756.5$ sq ft

8. Find the area of the following trapezoid by arranging a copy of the given trapezoid with the original to form a parallelogram.



For problems 9 - 11, draw a picture to match the information provided. Solve the problems and write your answer in a complete sentence.

9. The school district is going to buy a cover for the artificial turf on the district football field. What must they know about the field so they can order the correct size to cover the entire field?

10. Mrs. Moreno wants to put a border around the rectangular bulletin board in her classroom. The board is twice as long as it is wide. If the width of the board is 28 inches, how much border will she need?

11. A regular hexagon has sides of length 4 inches. Determine the perimeter of the hexagon.

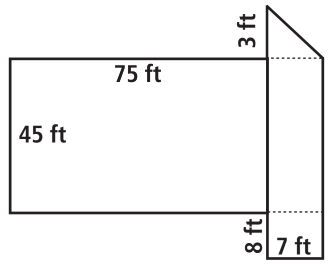
12. Using another sheet of grid paper, make and label as many different rectangles of area 36 square units as possible. Make another chart to organize the information about these rectangles. How is the perimeter of a rectangle related to the area of the rectangle?

13. A rectangular house has a square porch on the rear of the house as shown. Find the area of the house and the porch.

14. A rectangular house has a porch on the rear of the house as shown. What is the area of the house and porch combined? Note, this picture is not drawn to scale.

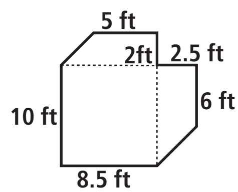
15. It will be easier if students use grid paper to do this. The rectangles with integer sides are 1 x 11, 2 x 10, 3 x 9, 4 x 8, 5 x 7, and 6 x 6. Organizing the data might avoid missed possibilities and is a good mathematical habit. The students may also report some non-integer dimensions.

16. a. The width of the figure is 11 feet and the height is 12 feet. Then the area of the rectangle the room fits into is 132 square feet. Now we need to subtract the triangle in the upper left corner (3.5 sq ft), the rectangle in the upper right corner (5 sq ft), and the triangle in the lower right corner (5 sq ft). So the total area is 132 – 13.5 = 118.5 sq ft. Alternatively: 85 + 10 + 15 + 5 + 3.5 = 118.5 sq ft if you subdivide the figure into three rectangles and two triangles.
    b. A = 344 sq ft
    c. (344 sq ft)(1 gallon / 200 sq ft) = 1.72 gallons



15. Draw as many different rectangles as you can that have perimeter 24 units.

16. A room has the following floor plan and dimensions.



a. Find the area of the room.

b. If the approximate perimeter is 43 feet and the walls in the room are 8 feet high, what is the total area of the walls?

c. It takes one gallon of paint to cover 200 square feet of wall. How many gallons will it take to paint the room?

17. 20 %, 25 %, 50 %, 75 %

18. a. 18 b. 24 c. 21 d. 24 Note: b and d have the same value.

Ingenuity

19. The sidewalk can be divided into eight parts: four 3′ x 3′ square corners, and four 3′-wide rectangles on the sides. Each corner square has area 9 square feet, so the total area of the remaining rectangles is 1743 - 4 x 9 = 1707 square feet. The sum of the lengths of the four rectangular pieces is equal to twice the length plus twice the width of the field - that is, the perimeter P of the field. So we know that 3′ x P = 1707 square feet. This means that P = 569 feet. However, we were asked for the outside perimeter of the sidewalk, not the inside perimeter. The outside perimeter of the sidewalk is equal to the inside perimeter, plus six additional feet for each corner. So the outside perimeter is 569 + 24 = 593 feet.

Investigation

20.

The perimeter of each of these figures is 28. This is because each figure has 14 horizontal unit segments - 7 on the top side of the figure and 7 on the bottom - and 14 vertical unit segments - 7 on the left side of the figure and 7 on the right side. The fact that these edges are staggered in some cases does not matter; in each case, the total number of unit segments is the same.

(e) Create a figure that has perimeter 28 and area 19.

We know that a 7 x 7 square has area 49, so if we can subtract an area of 30 without changing the perimeter of the figure, then we will have a figure with the desired properties. One way to do this is to remove a 5 x 6 rectangle from the upper right corner of the figure, leaving a skinny L-shaped figure.

Spiral Review:

17. List the fractions in order from least to greatest, representing the fractions as percents $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{3}{4}$
18. Which two expressions have the same value?
    a. $3(2)^2 + 2(7 - 4)$
    b. $2(6 - 2) + 4^2$
    c. $5(4\text{-}1) + 3 + 6 \div 2$
    d. $6 + 2 \cdot 3^2$

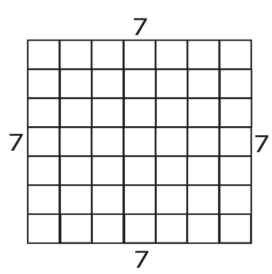
19. **Ingenuity:**

    A three-foot-wide sidewalk (shaded part) has been built around a rectangular field, as shown in the figure below. (note: the figure is not drawn to scale) If the total area of the sidewalk is 1743 square feet, what is the outside perimeter of the sidewalk?

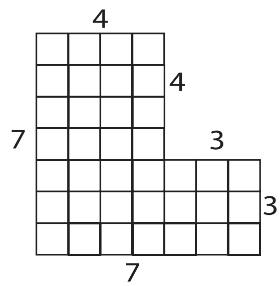
20. **Investigation:**

    Find the perimeter of each of the following figures:

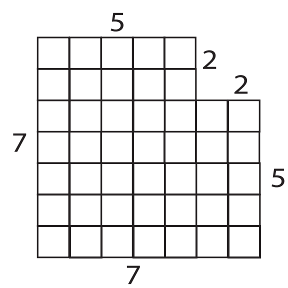
a.



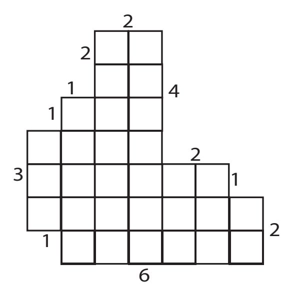
b.



c.



d.



e. Create a figure that has perimeter 28 and area 19.

Section 9.5 - Circles

Big Idea:
Discovering properties of circles

Key Objectives:
- Learn the vocabulary of circles.
- Understand the constant π.
- Develop the formula for the circumference and area of a circle.
- Use properties of circles to solve everyday problems.

Materials:
Compass, tape, ruler, grid paper, circles of various sizes, circles handout for Exploration 2.

Pedagogical/Orchestration:
- This lesson visually develops how to find the circumference and area of a circle.
- Circle measurement, both one- and two-dimensionally, is carefully and actively developed in this section. Encourage your students to cut and measure to prove to themselves that the formulas work. Take time to make sure your students know the difference between a symbol that is a variable and a symbol that is a constant.
- If, as many of us believe, discovering a fact for yourself makes it yours, this section will be a benefit to your students in remembering the formulas associated with circles and working with those formulas.
- It will be important for students to keep in mind that whenever the radii and diameters are drawn in, the center must also be noted. Encourage your students to mark the center whenever drawing a circle.

Activities:
"Discoverying Pi is Easy as π", "Circles, Circles, Circles" and "Radius Square Activity"

Exercises:
The beautiful designs in these exercises should draw students into working them.

Vocabulary:
Center, radius, circle, diameter, circumference, chord, semi circles

TEKS:
6.6(C); 6.8(A,B); 6.10(C); 6.12(A); 6.13(A,B) New: 7.5(B); 7.11(B, C)

Launch for Section 9.5:

Lead students through the questions in the first paragraph of Section 9.5. "What is the definition of a circle? How do you draw a circle? Try to describe a circle to someone without using the word 'circle.'" Let students struggle with this challenge and let students share good attempts. Go through Exploration 1 examining the definitions given. (For more on the discovery of the relationship of the diameter to the circumference see the Circles, Circles, Circles Activity on the CD.) Tell your students, "Today we will learn how pi is related to circles, and make many important discoveries including how to find the area and circumference of any circle."

Discovering Pi is Easy as π

Objective: The students will approximate the value of π by measuring the diameters and circumferences of various size circular lids. Students will discover that there are about 3 diameters in every circumference, regardless of the size of the circular lids.

Materials:

Lids of various sizes (at least 3 different size lids, per group)
Yarn
Ruler
Large piece of poster paper (one per group)

Activity Instructions:

1. Divide your class up into groups of 3 to 4. Make sure that each group has at least three lids of different sizes, poster paper, yarn and a ruler.
2. Students will trace each of their lids onto their poster paper. Using the yarn and the ruler, students will explore the size of the circles to find relationships between the radius, diameter, and circumference of each circle. This information should be organized and recorded on a piece of paper, to be shared and compared with the rest of the class when all groups are finished measuring.
3. During the sharing and comparing stage of this activity, the students should discover that regardless of the size of the circles, each radius is ½ of each diameter, each diameter is twice the size of each radius, each circumference is equal to approximately 3 diameters, and each circumference is also equal to approximately 6 radii.
4. Once your class finds these discoveries, ask the class to explain to you and to each other how this relates to the two formulas on their formula chart for circumference and the number pi.

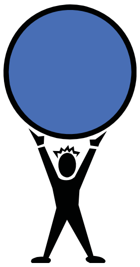
Extension:

Students will find 3 objects from home in which understanding the circumference, diameter, and radius of a circle was useful.

Literary Extension:

Read Sir Cumference and the First Round Table by Cindy Neuschwander and Wayne Geehan.

Circles, Circles, Circles



Objective: The class will approximate the value of π by measuring the diameter, d, and circumference, c, of various circles and finding the ratio of C to d by graphing the class data.

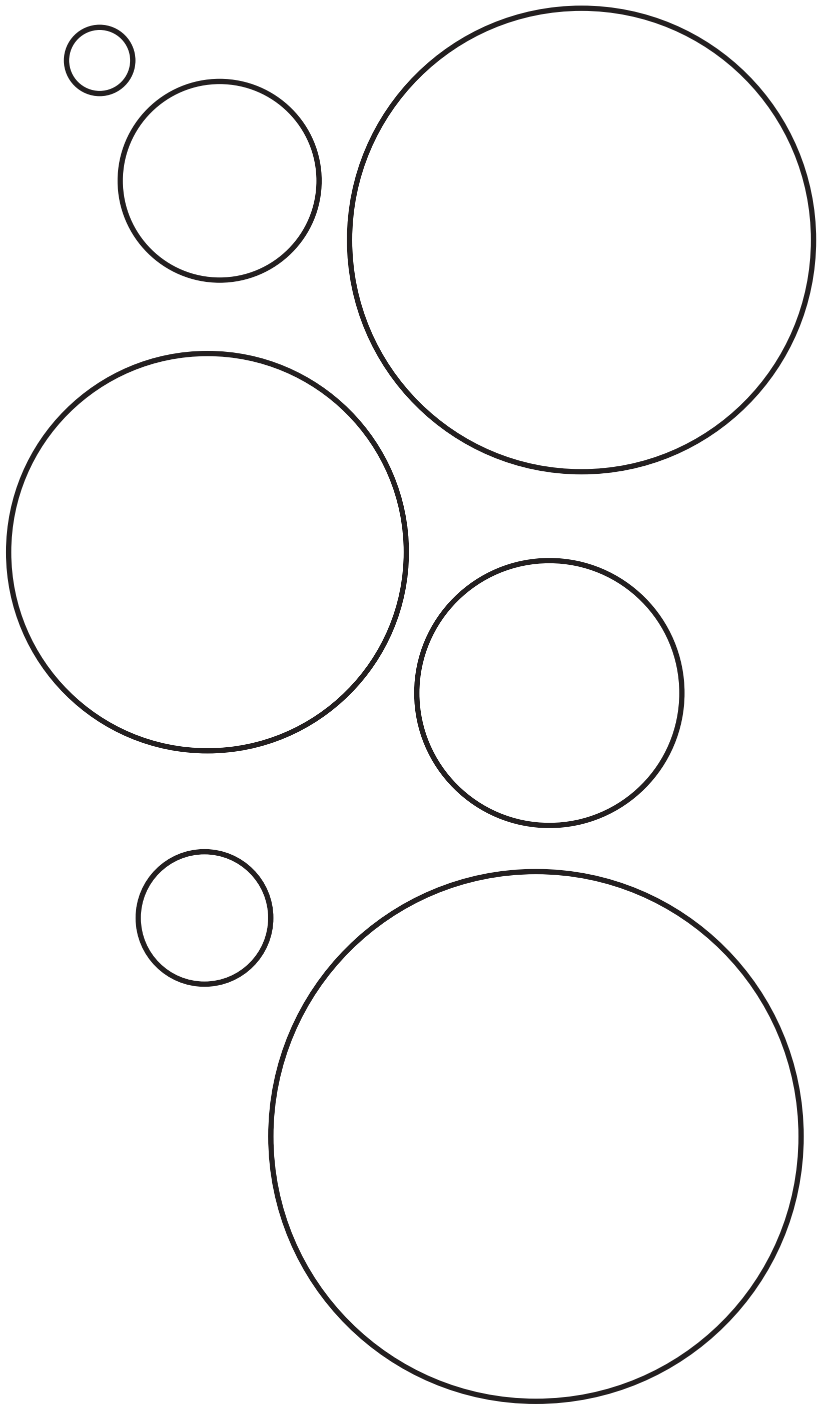
Materials:
Lots of objects, of various sizes, with circular bases
Measuring tape
String is not necessary, but some students may find it helpful for measuring
Graph Paper
Notebook paper

Activity Instructions:
This activity is best done in groups of 3 to 4. Each group is given 3 to 5 objects to measure. After each group has completed the measurements and graphs, the data should be compiled on a large class coordinate plane to discuss the results. Ask probing questions to lead the students to find relevant patterns in the data and ultimately find the relationship between the diameter and circumference of all circles.

Mathematicians have found a relationship between the diameter and circumference of a circle. Let's see if you can discover this relationship too. We will begin by measuring the circular bases of the objects on our tables and look for patterns.

1. Use a tape measure to find the diameter and circumference of each object. Record your results in a table with these column headings.
   Object Diameter Circumference
2. Make a coordinate graph of your data. Use the horizontal axis for diameter and the vertical axis for circumference.
3. Add your results to the class graph. Draw a line through the origin and as close to as many points as possible. Try to find the ratio of the vertical values and the horizontal values.

Circles Handout



Radius Square Activity

Objective: Students will develop that relationship between the radius and the area of a cirlce.

Materials:
Compass
Construction paper squares the size of the radius
Glue
Scissors
Card stock

Activity Instructions:

1. Students will draw a circle with the compass on the cardstock with a radius of 2 inches. (You can choose a different measurement if desired). Remember to mark the center of the circle.
2. Pass out 4 precut squares the size of the radius. Each square should be a different color.
3. Ask students,"Is the area of 1 radius square more than, less than, or equal to the area of the cirlce?" Have students place the radius square on the circle forming a right angle in the center. They should cut out the extra part outside the circle and glue it inside the circle. Students should see and conclude that the area of 1 radius square is less than the area of the circle.
4. Continue as with #3 with the second radius square. After gluing the entire radius square in the circle, students should conclude that the area of 2 radius squares is less than the area of the circle.
5. Repeat the process with the third radius square. Students should glue the third radius square inside the circle. Students will notice that almost the entire circle is now covered with the area of the 3 radius squares. However, the circle still needs "a little bit more" to be completely covered.
6. Since only a little of the 4th radius square is needed to completely fill the circle's area, students should cut the needed amount and completely fill the circle. They will glue the remaining part of the 4th radius square outside the circle. They should conclude that the area of 4 radius squares is greater than the area of the circle.
7. The final product will show the circle covered by 3 complete radii and a little bit more of the fourth. Students will recognize this a "pi." They can then be guided to the formula for area of a circle: $A = \pi r^2$.

Diameters are considered chords but not all chords are considered diameters.

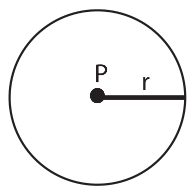
SECTION 9.5 CIRCLES

Everyone has seen circles of various sizes, but what is the definition of a circle? How do you draw a circle? Try to describe a circle to someone without using the word "circle."

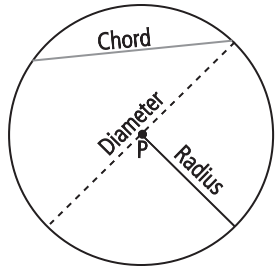
EXPLORATION 1

How do you draw a circle? Once you have drawn a circle, write directions that someone could use to draw a circle. Then state your definition for a circle.

In general, one way to draw a circle is by marking a point *P*, called the **center** of the circle. Then take a length of string, *r* units long, place one end of the string at point *P* and attach a pencil to the other end. Stretch the string to its full length and draw the circle with the pencil. Each point on the circle is *r* units from *P*. A circle is often named by its center. In this case, we have circle P.



Circle P is made up of all points that are distance r from P. The fixed distance r from the center P to a point on the circle is called the **radius** of the circle. All line segments that can be drawn from one point of the circle to another are called **chords**. A circle can have many chords. A line segment connecting two points on the circle and passing through the center P is called a **diameter**.The diameter is a special chord because it passes through the circle's center. The length of the diameter is equal to the length of 2 radii. Radii is the plural of radius. Notice that the diameter cuts the circle in half, forming two **semicircles**. Also notice that it is the longest line segment that can be drawn from one point on the circle to another. Are all diameters considered chords? Are all chords considered diameters?



The distance around the circle is called the **circumference** and is like the perimeter of a polygon.

Use the circles on the Circle Handout. Using a piece of string, carefully measure the radius and circumference. Place the circle on grid paper, measure the radius, diameter and circuference, and estimate the area, then complete the table below:

Circle	Radius	Diameter	Circumference
A			
B			
C			
D			
E			

Do you notice a relationship between the radius and the diameter? Using the variable *d* to denote the length of the diameter, express the diameter in terms of the radius *r*.

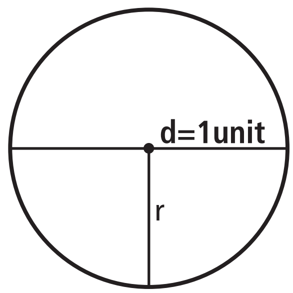
What is the relationship between the circumference of a circle, C and its diameter, d? Add a column to your table and compute the ratio of the circumference to its diameter for the five circles. What do you notice about the ratios? The ratio you computed approximates the exact ratio of the circle's circumference to its diameter, the number pi, written as the Greek letter π. This ratio, $\frac{C}{d} = \pi$, of the circumference to its diameter is the same regardless of the size of the circle.

Students need to get used to the multiple approximations for pi, the use of the pi key on the calculator, and having answers in terms of pi. If students do not have access to calculators, then at least make them knowledgeable of the fact that it exists. If students don't have calculators, then the $\frac{22}{7}$ approximation is possible at this point since they have been taught fractions. Only after teaching decimals should they use the decimal approximation.

FORMULA 9.4: Circumference of a Circle
The formula for the circumference of a circle with radius r and diameter d is $C = 2\pi r$ or $C = \pi d$.

In either version of the circumference formula, π can be thought of as a rate that is multiplied by the diameter of the circle.

Look at a circle with diameter 1 unit.



Remember, a unit can be any length you choose. Take out a string or tape ruler and measure the circumference. The circumference has a length of π units. The number π is approximately equal to the fraction $\frac{22}{7}$ or the decimal 3.14. These two approximations are not exactly equal to π. However, the two approximate values are very close to the actual value of π, which begins 3.1415926....

What happens when the circle is scaled by a factor of 2, making each dimension twice as large? When the radius doubles, what happens to the circumference? What pattern do you notice when you measure the scaled circumference and compare it to the original circumference? Just as with a square, when scaled by a factor of 2, the perimeter, or circumference, doubles. The ratio $\frac{C}{d}$ of the circumference to the diameter remains the same, π.

In summary, call C the circumference of a circle, d the diameter, and r the radius. Then,

$$d = 2 \cdot r \quad \text{and} \quad C = \pi \cdot d \quad \text{or} \quad C = \pi \cdot 2 \cdot r$$
$$= 2r \qquad\qquad = \pi d \qquad\qquad = 2 \cdot \pi \cdot r$$
$$= 2\pi r.$$

PROBLEM 1

a. 26π in or 81 .71 in

b. 52π in or 163.43 in

c. 42π cm or 132 cm

d. 33π m or 103.71 m

e. 30π ft or 94.29 ft

f. 36π mm or 113.14 mm

PROBLEM 1

12π cm or 37.71 cm

EXAMPLE 1

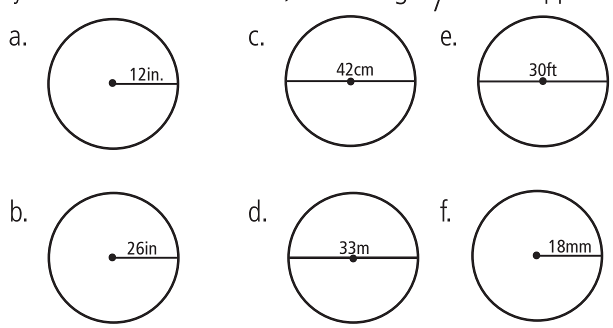
The diameter of a circle is 8 ft. Find the circumference. Use $\frac{22}{7}$ as an approximation for π. Round your answer to the nearest foot.

Solution:

Circumference of a circle is $C = \pi d$ and $d = 8$, so $C = 8\pi$. Using $\frac{22}{7}$ for π, we have
$C = \frac{22}{7} \cdot 8 = \frac{176}{7} = 25\frac{1}{7}$ ft. Rounding to the nearest foot, we have C = 25 ft.

PROBLEM 1

Find the circumference of the following circles with the given radius or diameter. Find your answer in terms of π, and using $\frac{22}{7}$ as an approximation for π.



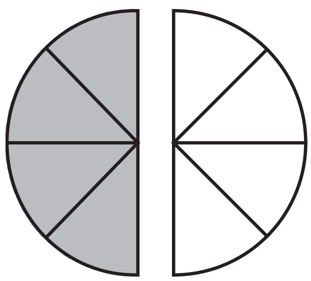
PROBLEM 2

The radius of a circle is 6 cm. Find the circumference of this circle.

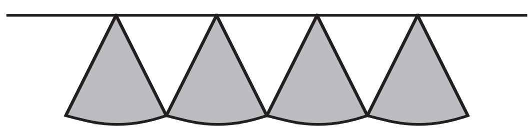
While circumference is the special term for the perimeter of a circle, its area has no special name other than area. The following exploration gives us a way to formulate the area of a circle.

EXPLORATION 3

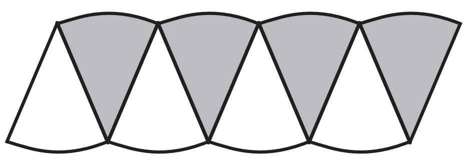
What is the area A of a circle whose radius is 1? Draw a circle with radius 1 and circumference 2π and cut it in half. Then cut each half into many small pie slices of equal size:



Take the slices from one half of the circle and lay the points of the slices along a line:



Do the same with the other half of the circle, filling in the spaces:

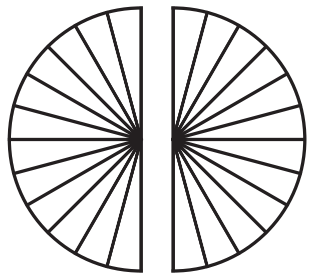


The shape looks a little like a parallelogram. The more slices, the closer the shape is to a rectangle. If this cutting process continued infinitely, the area of the circle with radius 1 would approximate the area of the rectangle with length π and width 1. One way to visualize this is to create slices in the circle with radius r, like the previous process when the radius was equal to 1.

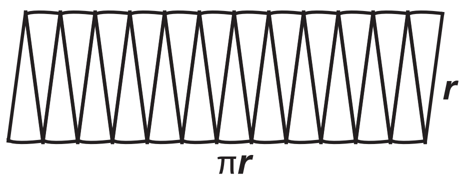
PROBLEM 3

$A = \pi r^2 = \pi(6 \div 2)^2 = \pi(3)^2 = 9\pi$ cm or $9(\frac{22}{7}) = 28.29$ cm

What happens to the area of the circle when its radius is a number r? One way to visualize this is to create slices in the circle with radius r, like the previous process when the radius was equal to 1.



Cut the circle into two equal semicircles as you did in the unit circle and fit one semicircle into the other semicircle.



What is the length of this rectangular shape? What is its width? What is the area of the rectangle?

In this rectangle the length is π r, which is half the circumference 2 π r and the width is r. The area of the rectangle is length times width so we have $\pi r \cdot r$ or πr^2. Any area is measured in square units, so if r is measured in inches, then πr^2 is measured in square inches. The formula for the area of a circle A with radius r can be summarized as:

FORMULA 9.5: AREA OF A CIRCLE
The area of a circle with radius r is $A = \pi r^2$ square units.

PROBLEM 3

Write an expression for the area of a circle with diameter 6 cm. Give answer in terms of π and then use $\frac{22}{7}$ as an approximate value for π and determine the area to the nearest whole number.

EXAMPLE 2

In general, exact answers here will have π in its answer while an approximation uses an approximate value of π such as 3.14 or $\frac{22}{7}$.

a. 8π square inches
b. 25.1 square inches
c. 16π square inches
d. 50.27 square inches

PROBLEM 4

a) $C = \pi d$, so $d = 12/3 = 4$

b) $C = 2\pi r$, so $r = C / 2\pi = 12 / 6 = 2$

EXAMPLE 2

A circle has radius 4 inches.

a. Find the exact circumference of the circle.
b. Approximate the circumference to the nearest tenth of an inch.
c. Find the exact area of the circle.
d. Approximate the area to the nearest hundredth of an inch.

SOLUTION

a. Apply the above formulas. The circumference is $C = 2\pi \cdot 4$ inches $= 8\pi$ inches.
b. If we approximate π by $\frac{22}{7}$, then the circumference $= 2 \cdot \frac{22}{7} \cdot 4 = \frac{176}{7} = 25\frac{1}{7}$, which is approximately 25.1 square inches.
c. Apply the area formula for a circle, $A = \pi r^2$. $A = \pi \cdot 4^2 = 16\,\pi$ in^2.
d. If we approximate π by $\frac{22}{7}$, then the area $= \frac{22}{7} \cdot 4^2 = \frac{22 \cdot 16}{7} = \frac{352}{7} = 50\frac{2}{7}$, which is approximately 50.29 square inches.

In mathematics, a product that includes a **constant**, or number, times a variable is written with the constant first, like $2x$. In the product of a constant and a variable, the constant is called the **coefficient** of the product $2x$. Even though π is a constant, not a variable, the product of π and a constant like 16 is usually written 16π.

PROBLEM 4

A circle has circumference 12 inches. Use 3 for π to approximate:

a) the diameter of this circle.
b) the radius of this circle.

EXERCISES

Explain to students that even when they use the π key on the graphing utility, though it looks like it's really precise, they are still just getting an approximation. The only way to get an exact answer is to leave the answer in terms of π.

1. center: S
   diameter: VT
   radii: VS, SP, ST
   chords: RV, RT, VT

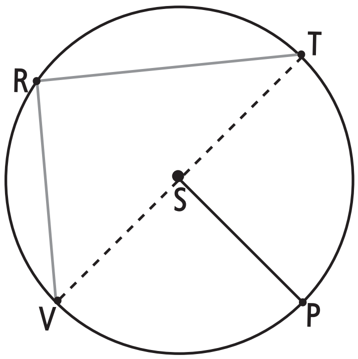
2. a. 20π in, 62.8 in
   b. 16π cm, 50.24 cm
   c. 22π mm, 69.08 mm
   d. 18π in, 56.52 in
   e. 28π ft, 87.92 ft
   f. 30π ft, 94.2 ft

3. a. 100π in^2, 312in^2
   b. 64π cm^2, 200.96 cm^2
   c. 121π mm^2, 379. 94 mm^2
   d. 81π in^2, 254.34 in^2
   e. 196π ft^2, 615.44 ft^2
   f. 225π cm^2, 706.5 cm^2

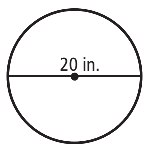
4. a. $C = 2\pi r = 2(30)\pi$
   b. 60πmm or 188.57mm

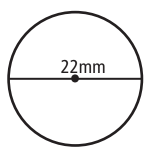
EXERCISES

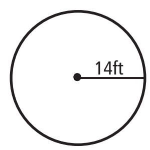
1. Use the diagram below to label the parts of the circle. Remember to use the correct labeling.

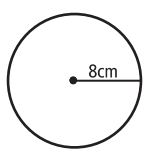


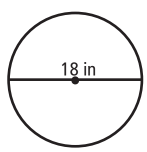
2. Find the circumference of each circle below. Use the circumference formula that corresponds to the circle part shown. Find the exact circumference and then approximate using $\pi \approx 3.14$. Remember to include your units.

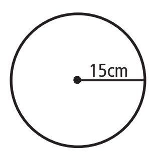
a. 

c. 

e. 

b. 

d. 

f. 

3. Use the circles from exercise 2 to find the area of each circle. Find the exact area and then approximate using $\pi \approx 3.14$. (Think carefully about the circle parts given and the part needed for the formula.) Remember to include your units.

Use the formulas for area and circumference of a circle to answer the following questions. For approximations, use $\pi \approx \frac{22}{7}$. Label your answers appropriately.

4. The diameter of a circle is 30 mm.

   a. Write an equation you would need to find the circle's circumference.

   b. What is the circumference of the circle?

5. 42π in or 132

6. 25π in^2 or 78.57 in^2

7. $A = \pi r^2$

8. a. 12π ft or 37.71 ft
   b. 36π ft^2 or 113.14 ft^2

9. a. $A = 4\pi ft^2$ or 12.57 ft^2
   b. 16ft^2
   c. $16 - 12.57 = 3.43$ ft^2

10. $\pi 6^2 - \pi 3^2 = 36\pi - 9\pi = 27\pi$

5. The spokes of a bike tire are 21 inches. What is the circumference of the bike tire?

6. A medium pizza has a diameter of 10 inches. What is the area of the pizza?

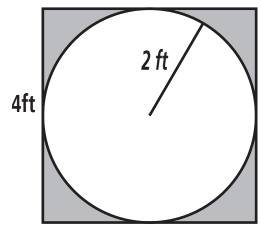
7. The large circular clock at the top of the church tower has radius of 4 feet. What formula would you need to find the area of the clock?

8. Local artists were asked to create a circular mural on the wall of the new library. The circle has diameter of 12 feet.

    a. The arc of the circle will have a rubber edging. How much rubber edging will they need?

    b. The circle will be covered with crystallized paint. How much space will they need to paint with this special paint?

9. A circle with radius 2 ft lies inside a square with each side 4 ft long.

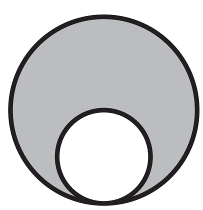


    a. What is the area of the circle?

    b. What is the area of the square?

    c. Find the area inside the box but outside the circle.

10. A circle with radius 3 in. is contained in a larger circle with radius 6 in. touching at the bottom.



What is the area outside the smaller circle and inside the larger circle?

11. Spiral Review (6.4A)
    $m = 2c = 2(10) = 20$

12. Spiral Review (6.3A)
    36 cookies = 3 dozen cookies
    1 egg= 3 dozen
    She needs 2 eggs for 6 dozen cookies.

13. **Ingenuity:**
    The area of the larger circle = $3^2 \cdot \pi = 9\pi$.
    The area of the two smaller circles $2(1.5^2 \pi) = 4.5\pi$.
    Area of twice the shaded region = Area of the larger circle – area of the two smaller circles
    $= 9\pi - 4.5\pi = 4.5\pi$.
    The area of the shaded region = $\frac{4.5}{2}\pi = 2.25\pi$ or approximately 7 cm.

14. **Investigation:**
    a. The length of the belt will be 3960 miles x 2π, which is approximately 24880 miles.
    b. and c.
    Let C be the length of the original belt. The length of the new belt is C + (6 feet). If we think of this as the circumference of a circle that encompasses the earth's equator, the radius of this circle is

    $\frac{C + 6\text{ feet}}{2\pi} = \frac{C}{2\pi} + \frac{6\text{ feet}}{2\pi}$ which approximately equals R + (1 foot), where R is the radius of the earth. So the radius of the circle formed by the new belt is one foot greater than the radius of the earth, which means that the new belt will have almost one whole foot of slack! This is a surprising result for most people who work this out, because we added only a tiny fraction (less than a millionth) of the length of the belt to make the new belt.

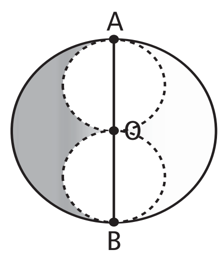
Spiral Review:

11. Casey is C years old. Melissa's age, m, is two times Casey's age. Write an expression representing Melissa's age in terms of Casey's age. When Casey is 10 years old, how old is Melissa?

12. Susan's sugar cookie recipe requires 1 egg for 36 cookies. How many eggs does Susan need for 6 dozen cookies?

13. **Ingenuity:**

    In the figure below, $\overline{AB}$ is the diameter of the larger circle and O is the center of the larger circle. $\overline{AO}$ and $\overline{OB}$ are diameters of the two smaller circles. The length of the diameter $\overline{AB}$ of the larger circle is 6 cm. What is the shaded area?



14. **Investigation:**

    The earth's radius is approximately 3960 miles. For the purposes of this problem, we will assume that the earth is a perfect sphere without any irregularities. Suppose we make a belt that is just large enough to encircle the earth's equator.

    a. What is the approximate length of this belt?

    b. Guess the answer to the following question: suppose we now make a belt that is 6 feet longer than the original belt. We then encircle the equator with this belt, and hold it over the original belt so that the amount of room under the belt is constant all the way around the equator. How far above the ground will the new belt be?

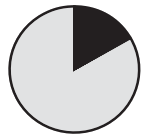
    c. Once you have guessed the answer to this question, use your knowledge of geometry to get a definitive answer.

15. **Challenge:**

The area of the circle is given by $A = \pi \cdot r^2 = \pi \cdot 4^2 = \pi \cdot 16$. Since we want only a fraction of the circle, we see that the area of the sector is to the area of the circle as the angle 60 degrees is to 360 degrees. 60 is $\frac{1}{6}$ of 360. The area of the sector must be $\frac{1}{6}$ the area of the circle so $\frac{16}{6}\pi = \frac{8}{3}\pi$ square inches. This approximately 8.4 square inches.

15. **Challenge:**

A **sector** of a circle is the part of the interior of the circle between two radii, like a slice of pie. A circle has radius 4 inches, and two radii make a sector with a 60° angle. Find the exact area of the sector these radii enclose.



Section 9.6 – Three-Dimensional Shapes

Big Idea:

Explore properties and volumes of prisms and pyramids.

Key Objectives:

- Learn the vocabulary associated with volume of three-dimensional figures, including prisms and pyramids.
- Develop the formula for the volume of a rectangular prism.
- Generalize the volume formula for other types of prisms.
- Use the math learned in this section to solve everyday problems, including conversions.

Materials:

Poster board, Ruler, Tape, Unit cubes, Boxes and cylinders, Yard sticks

Pedagogical/Orchestration:

- This lesson sets the groundwork for three-dimensional measurement.
- The section starts by using volume to discover conversion formulas. It concludes by developing the general formula for cubes and prisms.

Activities:

"Figure Out My Volume", "Guess My Solid" and "How Many Ice Cubes?"

Exercises:

Make sure students understand that volume is measured in cubic units, and that their answers to the exercises need to include correct units.

Vocabulary:

cube, volume, polyhedron, face, vertices, edges, prism, rectangular prism, regular, space diagonal, cone, pyramid, solid figure, right triangular prism

TEKS:

6.4(B); 6.12(A)

New 6.8(C, D)

Launch for Section 9.6:

One of the most useful tools for a geometry teacher is a model of a cubic foot. This would come in handy both for this Launch and the Exploration 2 from this section. Exploration 2 asks students to determine how many cubic inches are in one cubic foot. Today's Launch starts off simpler. Ask students to discover how many cubic feet are in a cubic yard. Tell them they can make a guess, but they must actually demonstrate their answer. Have plenty of yardsticks so students can fashion a cubic yard. They can measure a square yard on the floor and outline it with tape. Then different students can hold up four yard sticks on each corner. Another student can then take the cubic foot and move it to different locations within the cubic yard to determine how many cubic feet it would take to fill the cubic yard. Ask students if their initial guess was correct. This is a case when our intuition sometimes lets us down, as often times students will guess that there are 3 or 9 cubic feet in a cubic yard when there are actually 27. Let students know, "Today is all about 3-dimensional shapes and discovering their properties."

Figure Out My Volume

Objective: Students will use the formula to find the volume of rectangular prisms and cubes.

Materials:
Index cards with dimensions (length, width, height) of rectangular prisms and
Cubes
Die
Timer/stop watch
Unifix cubes
Chart with formulas (optional)

Activity Instructions:

1. Students throw a die to figure out who goes first, etc.
2. Teacher prepares index cards with L,W,H (dimensions in inches, feet, cm, etc.)
3. Students shuffle the dimensions index cards and leave them in a pile.
4. Player1 draws one card from the top of the pile. Player1 writes the dimensions given in his card to figure out the volume of the 3-D shape.
5. The other players check Player1's answer
6. If Player1 figures out the volume correctly, he/she gets 5 points. Then he/she gets 1min. to build his 3-shape using unifix cubes. If the shape is done on time, Player1 gets 5 more points. Players keep their score on a scoring sheet (plain sheet of paper).
7. If Player1 figures out the volume incorrectly, he/she continues to step8.
8. Player 2 gets a turn and repeat Steps 4-7
9. Activity continues until all the cards have been played.
10. The player with the most points wins.

Guess My Solid

Objective: Students will identify, describe, and classify attributes and properties of three-dimensional figures.

Materials:
3-D models (square pyramid, triangular prism, hexagonal prism, cylinder, cone, cube of different colors)
Brown paper bag
Die

Activity Instructions:

1. Students use a die to figure out who goes first, etc.
2. Each player1 pulls out one 3-D solid from brown bag without letting others see it.
3. The other players try to guess both the solid and the color of the 3-D shape drawn by player 1.
4. The player who thinks he/she knows the shape and color, takes a guess. If correct, that player keeps the 3-D solid.
5. Players continue taking turns, asking each other one question at a time about the mystery 3-D solid to try to guess (and keep) their opponent's solid.
6. The game continues until the first player who gets 5 solids wins the game.

How Many Ice Cubes?

Objective: The students will use their knowledge of volume to design a cooler that will hold ice for a family picnic. The problem is open-ended and encourages students to use their math skills and their creativity.

Materials:
Copy of "How Many Ice Cubes?" worksheet, one per student
Access to centimeter cubes, if they want them
Rulers, or any straight edge

Activity Instructions:

1. Pass out a copy of the worksheet, one per student.
2. Encourage students to work alone, but allow them to collaborate with peers if they need a little help or just want some reassurance that their ideas are working.
3. If students are struggling to find the necessary volumes, encourage them to use the centimeter cubes to model the problem.
4. It is important to stress to your students that there is not ONE correct answer to this problem. You are interested in them using correct math, but also interested in their creative ideas in the design of their cooler.
5. When checking students' work, remember that they are looking for a cooler size that will maximize space for cubes that are 3 centimeters on each edge. The cooler should be large, but not too large. It is my belief that a cooler in the shape of a rectangular prism would work best, but don't discourage students from using other shapes if they have a valid reason why they think it will work best.
6. It would be fun to have students share their designs with the class when finished. If you don't have time for all students to share, maybe you could pick your top 5 favorites and let them do a little presentation in front of their peers.

Name ___________________________________

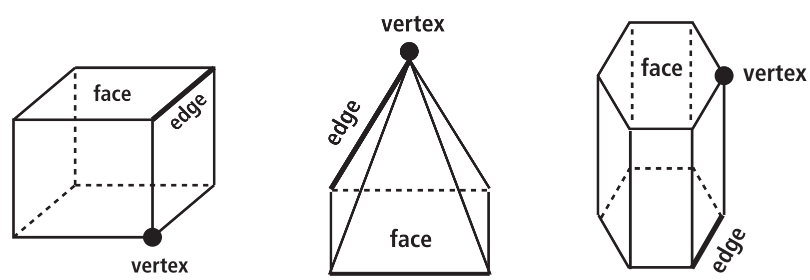
How Many Ice Cubes?

Joselyn's family is going on a picnic and wants to pack one of their coolers full of ice. This cooler will only contain ice, as they will be outside all day long and it is expected to be very hot. Her family wants to make sure that they have plenty of ice to keep their drinks cold all day long. Joselyn prefers ice that is shaped in cubes, and the ice trays at her house make perfect cubes that are 3 centimeters on each edge. In the space below, design a cooler that you would recommend for Joselyn and her family to take on this picnic for their ice. After you design the cooler, be sure to label its dimensions and find its volume. Then, in the space at the bottom of the page (or on the back of this sheet), explain in detail why you think your cooler will work best for this situation. In your explanation, please include the capacity (volume) of your cooler and the number of ice cubes that your cooler will allow them to take with them on their picnic.

SECTION 9.6 THREE-DIMENSIONAL SHAPES

In the previous sections, you studied shapes in two dimensions: triangles, squares, rectangles, parallelograms, trapezoids and circles. In this section, you will learn about three-dimensional shapes. Some of these shapes appear as familiar objects like beach balls, blocks, paper towel rolls or cardboard boxes. In this section, you will learn some mathematical terminology and ways to measure volume. We will start with the easiest three-dimensional shapes.

A basic kind of three-dimensional figure is called a polyhedron. This word comes from the Greek words *poly*, meaning "many," and *hedra* meaning "faces." So a **polyhedron** is a three-dimensional figure with many faces. The plural form of polyhedron is polyhedra. Each **face** of a polyhedron is a polygon. The vertices of the polygons are the **vertices** of the polyhedron. The **edges** are the borders of the faces that are also the line segments that join the vertices.

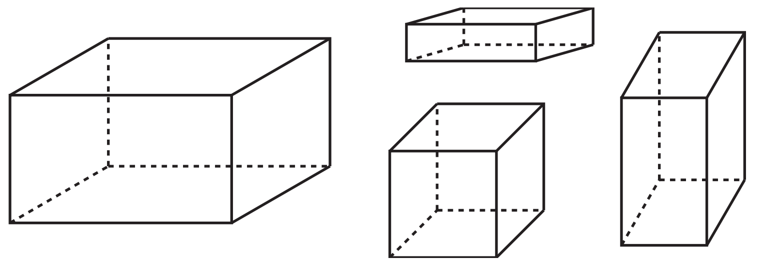


A box shape is an example of the most common type of polyhedron called a **prism**. In a prism, two of the faces, called **bases**, are parallel and congruent. Prisms are named by their bases. In the case of a box, the polyhedron is a **rectangular prism**, because the bases are rectangles. The faces that connect the two bases are parallelograms, and in this case rectangles. They are called **lateral faces**. In

PROBLEM 1

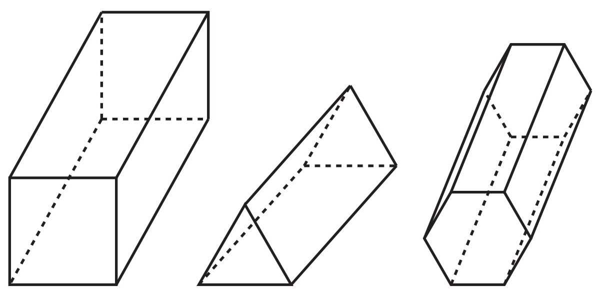
Square or rectangle, hexagon; rectangle lateral faces; rectangular prism, triangular prism, hexagonal prism.

Rectangular pyramid, Square pyramid, Pentagonal pyramid fact, although the faces of prisms are not always rectangles, in rectangular prisms all the faces are rectangles. How many total faces are there in rectangular prisms? How many are bases? How many are lateral faces?

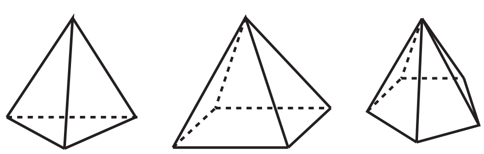


PROBLEM 1

Pictured below are three different prisms. What shape are the base faces? What shape do the non-base or lateral faces have? Give each prism an appropriate name.



Another type of polyhedron is called a **pyramid**. A pyramid consists of a polygon for a base and lateral faces that are triangles that meet at a point called the **apex**.



The pyramids are named by their base. Identify the names of the pyramids above.

PROBLEM 2

In a rectangular prism, there are 6 faces, 8 vertices, and 12 edges.

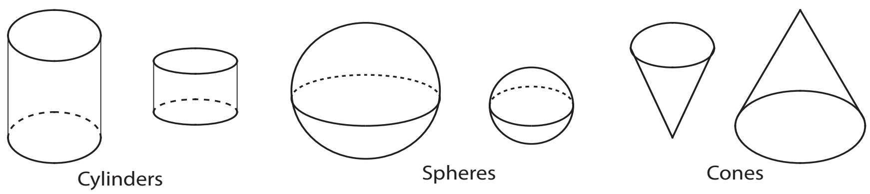
In a rectangular pyramid there are 5 faces, 8 edges, 5 vertices

PROBLEM 2

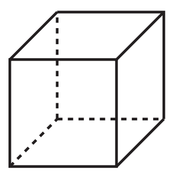
You found that rectangular prisms have 6 faces. How many edges and how many vertices does a rectangular prism have? Consider a rectangular pyramid. How many faces, edges, and vertices does it have?

Other common three-dimensional shapes include cones, cylinders, and spheres.

Notice that **cones** are related to pyramids but with a circular base. A **cylinder** in a similar way is related to a prism with circular bases. A **sphere** is a three-dimensional version of a circle, a figure formed by all points of a fixed distance from a fixed point, called a center. Examples of cones, cylinders, and spheres are below:

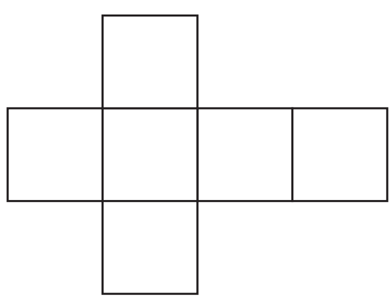


The simplest three-dimensional shape to measure is a **cube**, two parallel congruent square bases connected by four perpendicular congruent squares.



A cube is a regular rectangular prism. A cube is **regular** because each of its faces has equal sides and angles. In other words, all the faces are congruent to each other. All the cube's faces are squares.

A cube one unit long, one unit wide and one unit high has a volume of one cubic unit. Recall that the area of a two-dimensional figure is measured by the number of unit squares needed to cover it.

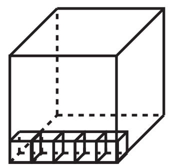


EXPLORATION 1

Note that the groups will need to divide the work to quickly make 30 one inch cubes, or you may wish to make these before class begins.

Just as you can cut a cardboard box open by cutting along some of the edges, we can cut a cube along some of the edges and flatten the cube to create a **net** similar to the figure above. How many square units is the net? Can you think of other ways in which the cube can be cut to create a net? You will study more about nets when you examine surface area.

The **volume** of a three-dimensional shape is measured by the number of unit cubes needed to fill it. For example, if each side of a cube is 1 foot long, the volume of the cube is 1 cubic foot, written 1 cu. ft. or 1 ft^3.



EXPLORATION 1

How many inch cubes (also called cubic inches) are there in a cube that is 2 inches long on each side? How many cubic inches are there in a cube that is 3 inches long on each side?

Use centimeter cubes to cover a rectangular area that is of length 2 cm and width 2 cm. Discuss the number of cubes used. Record your data as follows:

Area of first layer = 2 cm x 2 cm = 4 sq. cm.

On top of the first layer which is called the base, add an identical layer of centimeter cubes. How many total cubes are used? To find the answer you can take the answer for the number of cubes used in the base and multiply it by the number of layers. We now have 2 cm x 2 cm x 2 cm = 8 cubic centimeters. Common abbreviations for cubic centimeters are cu cm and cm^3.

Now investigate using a cube with sides of 3 cm.

FORMULA 9.6: VOLUME OF A CUBE
The volume of a cube with each side of length s inches is s^3 cubic inches. $V = s^3$.

PROBLEM 3

216 cm^3

EXPLORATION 2

How many cubic inches are there in one cubic foot?

In order to think about this problem, let's begin by reviewing how to change units in computing areas. A square that is one foot long on each side has area one square foot. Thinking in terms of smaller units, each side of the square foot is 12 inches long. Using this ratio of 1 foot to 12 inches, we have:

1 square foot = (1 foot)(1 foot) = (12 inches)(12 inches) = 144 square inches.

Using the same pattern,

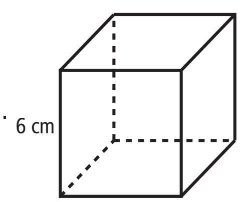
$$\begin{aligned}1 \text{ cubic foot} &= (1 \text{ foot})(1 \text{ foot})(1 \text{ foot})\\ &= (12 \text{ inches})(12 \text{ inches})(12 \text{ inches})\\ &= 1728 \text{ cubic inches.}\end{aligned}$$

Another way to compute the volume of this cube is to use the formula you just learned. Since one foot = 12 inches, each side of the cube is 12 inches long. The volume of your cube then is $12^3 = (12)(12)(12) = 1728$ cubic inches.

In three-dimensions, conversions to smaller units sometimes make volumes seem much larger even though the shape and size have not changed at all!

PROBLEM 3

Find the volume of the cube with sides of length 6 cm.

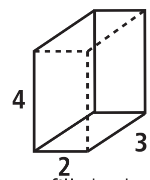


EXAMPLE 1

Make a rectangular prism with edges that are 2, 3 and 4 units long. Find its volume.

SOLUTION

The first step is to draw the two base rectangles. For example, make the bases 2×3 rectangles. Place these rectangles 4 units apart to make the height of the box.



How many unit cubes does it take to fill the box? The 2 x 3 base rectangles have area 6 square units so there are 6 cubes in the first layer. Because the rectangular prism is 4 units in height, we have a second layer of 6 cubic units, a third layer of 6 cubic units, and finally a fourth layer of 6 cubic units.

In other words, to find the volume, multiply the area of the base, $2 \cdot 3 = 6$, by the height 4 to get $6 \cdot 4 = 24$ cubic units.

In general the volume of a prism is equal to the area of the base times the height. This formula is often written as $V = B \cdot h = Bh$. The variable B is the area of the base. This general formula is true for any prism, regardless of the shape of the base, whether it is a rectangle, a triangle, a hexagon or any polygon.

FORMULA 9.6: VOLUME OF A PRISM

The volume formula for a prism can be written by

$$\text{volume} = \text{area of base} \cdot \text{height}$$

$$\mathbf{V = Bh}$$

with B = area of the base and h = height of the prism. In particular, for a rectangular prism, B = lw so **V = lwh**.

All the sides of a cube are equal, that is l = w= h. If we call the side of a cube s, then the volume formula for a cube based on the volume formula for a rectangular prism becomes $V = Bh = lwh = s \cdot s \cdot s = s^3$. Volume of a cube is given by $\mathbf{V = s^3}$.

Remember, the volume in cubic inches of three-dimensional shapes is the number of one-inch cubes it takes to fill the shape exactly. Because some shapes cannot be easily filled with one-inch cubes, the volume might be a fraction or a decimal part of a unit cube. As in the case of prisms, you can examine volumes and arrive at formulas that will make the computation much easier than counting blocks every time.

Note that in the technical definition, cylinders need not have bases as circles. Most middle school grades assume that cylinders are right circular ones.

EXERCISES

1. A - vertex B - edge C - face

2. a. triangular prism; faces - 5, edges - 9, vertices - 6
   b. rectangular pyramid; faces - 5, edges - 8, vertices - 5
   c. hexagonal prism; faces - 8, edges - 18, vertices - 12
   d. triangular pyramid; faces - 4, edges - 6, vertices - 4

EXAMPLE 2

Determine the volume of a cube that is 2 units long, 2 units wide and 2 units high in cubic units. Then double each of its dimensions. Predict the volume of this new cube. Verify your prediction by calculating the volume of the larger cube.

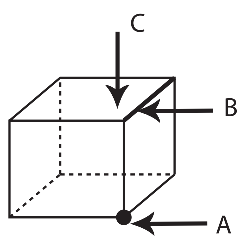
Make a prediction about what happens to the volume of any cube when its dimensions are doubled. What is the relationship of the volume of the original cube to the volume of the enlarged cube?

SOLUTION

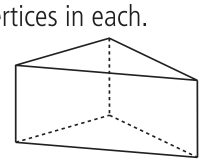
The original cube's volume is 8 cubic units, the product of the area of the base B, which is $2 \cdot 2 = 4$ square units, and the height, which is 2 units. When each of the cube's dimensions is doubled, the volume of the resulting larger cube is (4 units)(4 units)(4 units) = 64 cubic units. The original volume was 8 cubic units. The new volume is 64 cubic units which is 8 times the original volume.

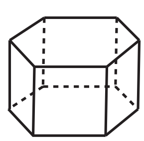
EXERCISES

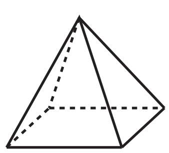
1. Identify the parts of the polyhedron shown below.

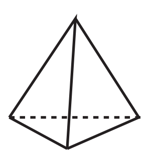


2. Name the following polyhedra. Identify the number of faces, edges and vertices in each.

a. 

c. 

b. 

d. 

3. a. $V = (3 \cdot 3 \cdot 3) - 2 = 25\ u^3$
   b. $V = (4 \cdot 4 \cdot 4) - 2 = 59\ u^3$

4. 60 in^3

5. $V = 216\ in^3$

6. 7 cm

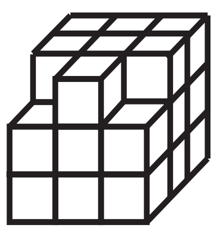
7. Prism C has the largest volume, Prism A has the longest side.

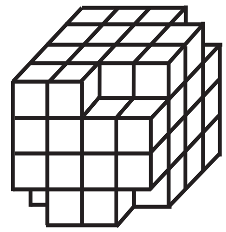
8. 700 cubic yards

9. 1331 $inches^3$

10. 5 inches

3. Find the volume of the shapes below.

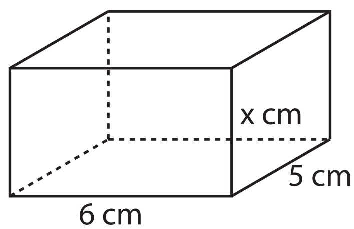
a. 

b. 

4. What is the volume of a rectangular prism with edges 3 ft, 4 ft and 5 ft?

5. What is the volume of a cube with side lengths of 6 inches?

6. A rectangular prism has volume 210 cm^3 and edges with lengths 5 cm, 6 cm and *x* cm. What is the value of *x*? Refer to the picture below.



7. Consider the following rectangular prisms. Prism A with dimensions: 15 inches x 8 inches x 1 inch. Prism B with dimensions: 10 inches x 10 inches x 3 inches. Prism C with dimensions: 7 inches x 7 inches x 7 inches. Which is the biggest rectangular prism? Explain your reasoning.

8. A swimming pool has dimensions 35 yards x 10 yards x 2 yards. Determine the volume of the pool.

9. The ice cream will be stored in a cube-shaped container with sides of length 11 inches. What is the volume of the container?

10. If a cube has a volume of 125 cubic inches, what would be the length of the cube's sides?

11. c

12. 70°

11. The table below shows the length and area of several rectangles. All these rectangles have a width of 5 cm. Which of the following equations best represents the relationship between the length, L, and area A of these rectangles?

Rectangles

Length (L)	Area (A) cm^2
4	20
6	30
9	45
11	55

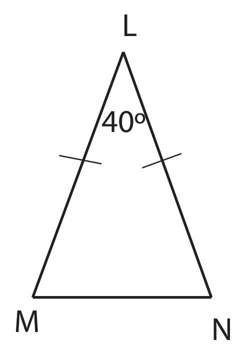
a. $A = \frac{L}{5}$

b. $A = L^2$

c. $A = 5L$

d. $A = L + 5$

12. Triangle LMN is an isosceles triangle. If the measure of angle L is 40°, what is the measure of angle N?



13. **Ingenuity:**

Suppose we fold the faces in the diagram below and glue the faces along their, edges to form a polyhedron.

Ingenuity

13. (a) We can think of the resulting polyhedron as a pentagonal prism (whose faces are the five rectangles), with pentagonal pyramids on the top and bottom (formed by the five triangles on each end). The number of faces of this polyhedron is clearly 15, the same as the number of faces in the figure we folded up. The number of edges is 25: there are 5 edges that run the length of the pentagonal prism, 5 edges of the pentagonal prism on each end, and 5 edges from the apex of each pyramid to the pentagonal prism. So the total is $5 + 2 \times 5 + 2 \times 5 = 25$. The number of vertices is 12: the pentagonal prism has 10, and the apex of each pyramid adds one more.

(b) The resulting polyhedron would actually simply be a hexagonal prism, with the hexagon on each end divided into six triangles. If we draw a regular hexagon, and place an equilateral triangle inside the hexagon on each side, the triangles will meet exactly at the center of the hexagon, filling it completely. So if we fold the faces of this diagram together, the triangles at each end will lie flat rather than jutting out to form a pyramid.

Investigation

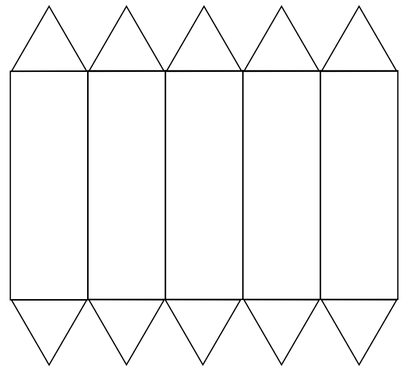
14. (a) Only 1; the cube in the middle will not have any faces painted.

(b) There is one cube in the interior of each face that has exactly one face painted red. There are 6 such cubes.

(c) There is one cube in the middle of each edge of the large cube that has two faces painted red. There are 12 such cubes.

(d) The 8 corner cubes will have three faces painted red.

(e) This time, there would be



a. How many vertices, how many edges, and how many faces will the resulting polyhedron have?

b. The figure above consists of five pieces, each of which is a rectangle with triangles on both ends. Suppose that the figure instead had six such pieces. What would the figure look like if we glued the faces together to form a polyhedron?

14. **Investigation:**

A certain 3 x 3 x 3 cube is made of 27 unit cubes. The surface of the 3 x 3 x 3 cube is then painted red.

a. How many of the 27 unit cubes have no faces painted red?

b. How many of the 27 unit cubes have exactly one face painted red?

c. How many of the 27 unit cubes have exactly two faces painted red?

d. How many of the 27 unit cubes have exactly three faces painted red?

e. How would the answers to (a) through (d) change if we replaced the 3 x 3 x 3 cube with a 4 x 4 x 4 cube made using 64 unit cubes?

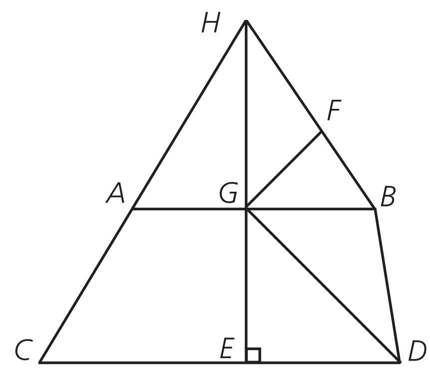
1. a. Answers will vary. For example $\angle HEC$ and $\angle HED$
   b. Answers will vary. For example $\angle HBD$ and $\angle HGD$
   c. Answers will vary. For example $\angle HGF$ and $\angle EGD$
   d. Answers will vary. For example $\angle EGD + \angle EDG = 90°$
   e. Answers will vary. For example $\angle HEC$ and $\angle HED$ are supplementary.

2. a. $x = 180° - 153° = 27°$
   b. $n = 180° - 27° = 153°$

3. a. Obtuse Isoceles triangle b. Right Isoceles triangle c. Equilateral triangle d. Obtuse Scalene
   e. Right triangle f. Acute Isoceles triangle

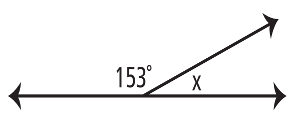
REVIEW PROBLEMS

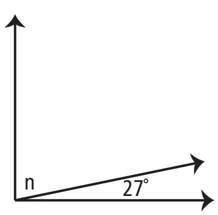
1. In the following figure, lines $\overline{AB}$ and $\overline{CD}$ are parallel. Answer the following questions using the figure. Label all answers carefully.



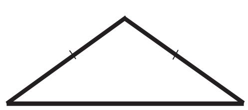
   a. Name 2 right angles
   b. Name 2 obtuse angles
   c. Name 2 acute angles
   d. Name 1 pair of Complementary angles
   e. Name 1 pair of Supplementary angles

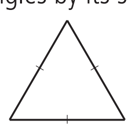
2. Find the measures of the indicated angles:

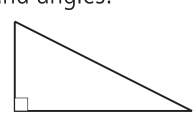
   a. 

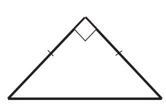
   b. 

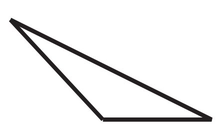
3. Classify each of the following triangles by its sides and angles:

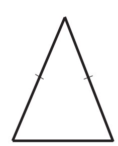
   a. 

   c. 

   e. 

   b. 

   d. 

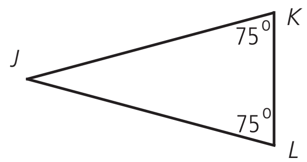
   f. 

4. 50°

5. $\angle ABC = 180° - (56° + 41°) = 83°$, $\angle ABD = 34°$ $\angle DBC = 49°$
$\angle EFG = 36°$

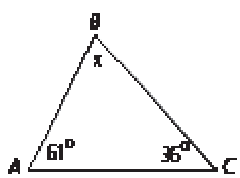
6. a. Rhombus b. Trapezoid c. Parallelogram (if the opposite sides are parallel) d. Rectangle
e. Square (if all angles are 90°) f. quadrilateral

4. In the figure below, fill in the missing angle measure with the information you are given.

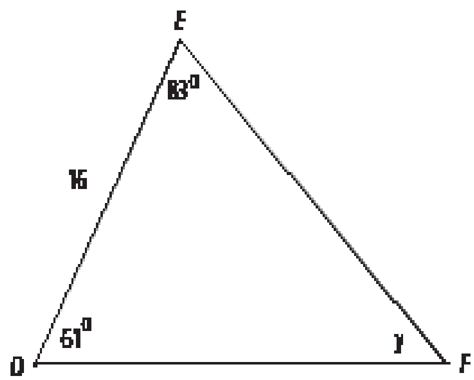


5. Find the measure of the missing angles x and y in the triangles below.

a.

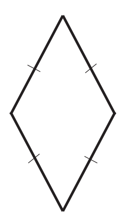


b.

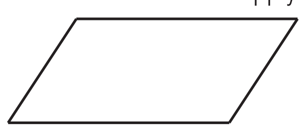


6. Classify each quadrilateral. Give all names that apply.

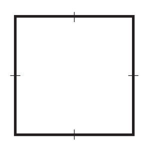
a.



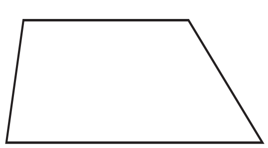
c.



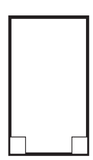
e.



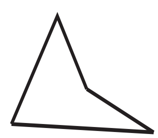
b.



d.



f.

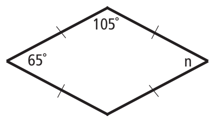


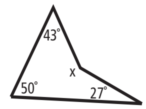
7. a. 65º and 105º b. 53º c. 240º d. 120º

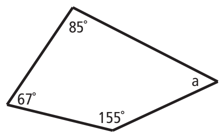
8. Faces = 5, edges = 9, vertices = 6

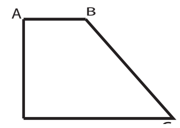
9. a. 6.8 in b. 28.8 m c. 45 ft d. 2.5 m e. 31.1 km

7. Find the missing angle or angles in each of the following quadrilaterals:

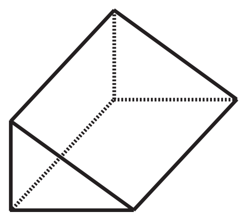
a.


c.


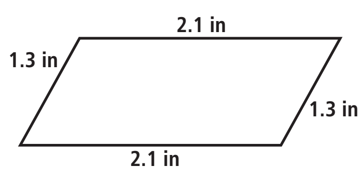
b.


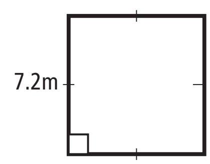
d.


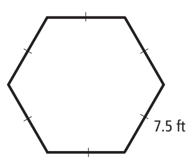
8. We have a right triangular prism as pictured. How many faces, edges and vertices does it have?

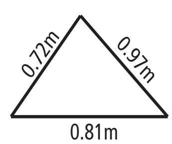


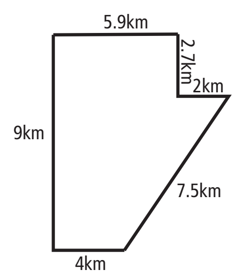
9. Find the perimeter of each figure. Remember to label your answer with the appropriate unit of measure.

a.


b.


c.


d.


e.


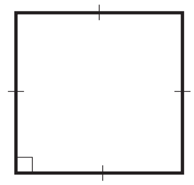
10. a. 289 cm^2 b. 56 yd^2 c. 40 km^2 d. 38 mm^2

11. a. 135 ft^2 b. 42 m^2 c. 241.5 mm^2 d. 296 ft^2

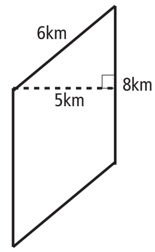
12. a. 216 ft^2 b. 740 in^2

10. Find the area for each quadrilateral. Label the unit of measure appropriately.

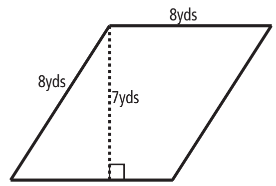
a.



c.



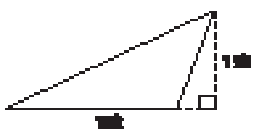
b.



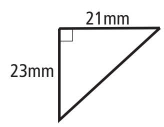
d.

11. Find the area for each triangle. Label the units of measure appropriately.

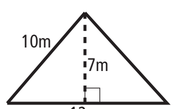
a.



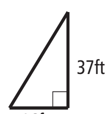
c.



b.

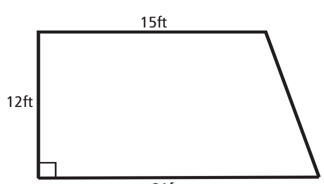


d.

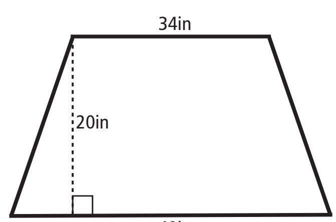


12. Find the area of each trapezoid. Label the unit of measure approriately.

a.



b.



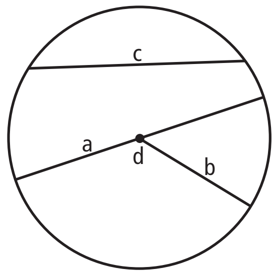
14.

13. a is diameter, d is center of the circle, b is radius, and c is chord

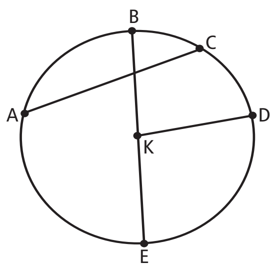
14. a. BE b. KB, KD, KE c. AC, BE d. Circle centered at K

15. a. A = 254.57, C = 56.57 b. A = 113.14 C = 37.71 c. A = 201.14, C = 50.29
d. A = 7857.14, C = 314.29

13. Name the parts of the circle in the diagram below (The center of the circle is at d):

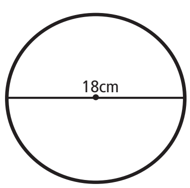


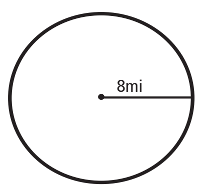
14. Identify the following from the diagram below (The center of the circle is at K):

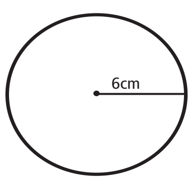


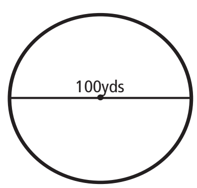
a. Name the diameter
b. Name the radii.
c. Name all the chords.
d. Name all the circle.

15. Find the area and circumference for each circle. Use $\pi = \frac{22}{7}$

a.


c.


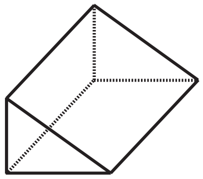
b.


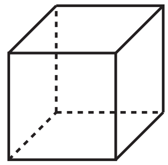
d.


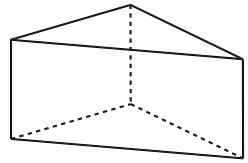
16. a. triangular prism b. Rectangular Pyramid c. cube d. Hexagonal Prism
e. triangular Prism f. triangular pyramid

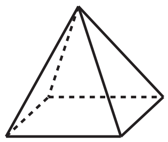
17. a. 512 b. 117 c. 240 d. 1728

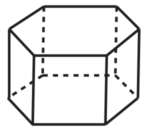
16. Name each 3-D shape. Tell the number of faces, edges, and vertices in each.

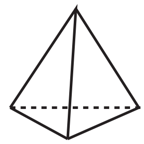
a. 

c. 

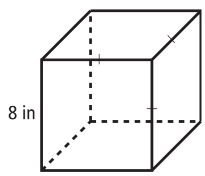
e. 

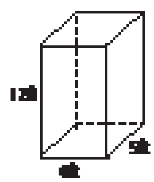
b. 

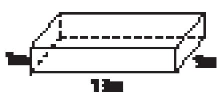
d. 

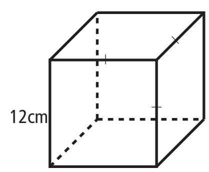
f. 

17. Find the volume of each figure. Label the unit of measure appropriately. Assume figures a and d are cubes.

a. 

c. 

b. 

d. 

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MATH EXPLORATIONS

Data Analysis

Note: the page numbers above correspond with the Student Edition.

CHAPTER PREVIEW

Section 10.1 examines different measures of central tendency including mean, median, and mode. Dot plots, box plot, and stem-and-leaf plots are introduced as ways to represent data. Section 10.2 includes histograms, bar, line, box plots, and circle graphs both to describe and summarize data and to have students construct them from given data. Appropriateness of various data representations is also discussed. Section 10.3 is an introduction to basic concepts of probability including sample space and events through commonly occurring experiments such as coin toss, number cube rolls, and card picks. Empirical and theoretical probabilities are introduced and distinguished through simple and compound experiments. Rule of product and rule of sum are discussed in Section 10.4 as important results useful in the context of independent events. The distinction between the words "or" and "and" in mathematical contexts are emphasized as well.

Section 10.1 – Measures of Central Tendency

Big Idea:

Recognizing and using measures of central tendency

Key Objectives:

- Review the vocabulary of data sets.
- Recognize and use the mean.
- Recognize and use the median.
- Recognize and use the mode.
- Find range of a given set of data
- Select which measure of central tendency is best to use in a given situation.
- Create dot (line) plots
- Create stem-and-leaf plots
- Create box plots

Materials:

Calculators (optional)

Pedagogical/Orchestration:

This section is a relatively traditional presentation of the measures of central tendency. Arguably the most important practical lesson is *which* measure is preferable, given a particular situation. Because many of your students will be reviewing the definitions and the skills to identify the measures, you might concentrate on the more analytical question of relative utility.

Activity:

"Comparing Median and Mode", and "What's Your Shoe Size?"

Internet Resources:

Rags to Riches Game: Mean, Mode, Median and Range
http://www.quia.com/rr/51667.html
http://www.quia.com/rr/85370.html

Vocabulary:

data analysis, data, data point, measure of central tendency, range, mean, arithmetic mean, average, median, frequency, mode, skewed, sample, dot plot., box and whisker, stem and leaf, outlier

TEKS:

6.10(B) New: 6.12(A, B, C, D); 6.13(A, B)

Launch for Section 10.1:

For this lesson, discuss with your students the following questions: "What would be a good way to summarize a whole set of data like the heights of every student in this class? What if you wanted to compare the heights of the students in this class with the heights of students in other math classes?" Let students give their ideas, and if no one mentions a measure of central tendency, lead them by asking, "How could we use just one number to represent your heights? Would the highest number be a good value to represent the set of heights?" Allow for responses, listening for mentions of mean, median or mode. Let your students know that there could be more than one number that could represent the data, and that different measures offer different information about the data. The mean, median and mode all describe the center of the data in different ways, and are called measures of central tendency. Have your students stand in front of the room in a line and ask how they could arrange themselves so that they could figure out the median and mode for their heights. The students can stand from shortest to tallest, and then measure and record their heights in inches. A physical demonstration of having the two students on each end take a step forward and working their way to the center is an excellent visual for finding the median. Something similar can be done for the mode having students of identical heights step forward and recording the data. Keep track of all measures on the board or on chart paper. Have students sit down and figure the mean using the heights of the students that were written on the board or chart paper. Tell your students that their data will be compared with students in other math classes, and that today they will be exploring the use of measures of central tendency and will learn to choose the measure that is preferable for a given situation.

Comparing Median and Mean

Objective: Students will compare the median and mean of sets of data. They will determine how the distribution of the grades affects whether the mean will be greater than, less than or equal to the median.

Materials:
Comparing Median and Mean Handout
Pencil
Colored Pencils
Calculator (optional)

Activity Instructions:

1. Find the median and mean of each set of test data, and plot the (median, mean) as ordered pairs for each set. There should be 6 ordered pairs when done.
2. Explain to students what the graph of y=x would look like and help them graph it on the grid. Have a discussion on what it means for y to equal x in this context.
3. Students will answer questions #1-4, and create their own set of data for question 5.

Answers to Activity:
The (median, mean) ordered pairs for the tests are as follows:

Test 1(90,78); Test 2(92,92); Test 3(90,87.8); Test 4(22, 37.4); Test 5(75, 75.4); Test 6(95,83.2)

1. The line y = x will be a diagonal line that includes points (1,1), (2,2), etc. If data point is above the line, this means that the mean is greater than the median (y>x). If data point is below the line, then the mean is below the median (y<x). If the data point falls on the line then the mean equals the median (y=x).
2. The median and mean for Tests 2,3 and 5 are close to or equal to each other. The grades for these tests are evenly distributed about the median. For instance, on Test 5 one test is 5 below the median and the other test is 20 below the median for a total of 25 below the median. The two tests above the median add up to 27 points above the median. This caused the mean to be just slightly above the median.
3. Tests 1 and 6 have means that are much less than the median. The grades higher than the median on Test 1 add up to 15 points higher than the median, whereas the points lower add up to 75 points. This is due to the outlier, 20, which had a great affect on the mean, and caused the mean to be much lower than the median.

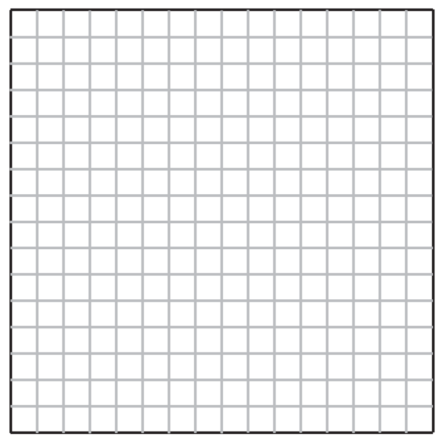
4. Test 4 has a mean much greater than the median. The grades below the median range only 7 points whereas the grades above the median range 73 points. This unequal distribution caused the mean to be greater than the median.
5. Answers will vary, but the point when plotted should be at least 10 units above the line y=x. The data should be chosen so that the test grades above the mean have a much greater difference with the median than the test grades below the median.

Comparing Median and Mean

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
Student 1	95	90	90	35	70	60	
Student 2	100	91	75	95	55	62	
Student 3	85	92	82	22	95	95	
Student 4	90	93	100	20	82	99	
Student 5	20	94	92	15	75	100	
(median, mean)							

These are the grades of five students in your class for the last 6 tests that have been given. Calculate the median and mean for each of the 6 tests, and then plot the data as points on a coordinate grid with median on the x-axis and mean on the y-axis.

1. Graph the line y=x and compare the points you plotted to this line. What does it mean if your data point is above this line? What does it mean if it is below the line? What does it mean if the data point is on the line?



2. For which tests are the median and mean close to or equal to each other? What do you notice about the grades for these tests that caused the mean to be close to the median?

3. For which tests are the mean much less than the median? What do you notice about the grades for these tests that caused the mean to be less than the median?

4. For which test is the mean much greater than the median? What do you notice about the grades for this test that caused the mean to be greater than the median?

5. Create a set of test grades for Test 7 so that the mean is at least 10 points greater than the median. Plot this point on the grid. Where is the point in comparison to the line y=x? Describe how you chose the data in order to make the mean greater than the median.

What’s Your Shoe Size?

Objective: Students will collect and record data about their class shoe sizes to find mean, median, mode and range of a set of data.

Materials:
Scratch paper for tracing
Measuring tape/ruler (one per group)
Large chart paper (optional)

Activity Instructions:

1. Each student will trace his/her foot on a sheet of paper, leaving their socks ON! Students will use measuring tape or rulers to measure their foot length-wise (toe to heel) and record it in cm. Students will record this data and their actual shoe size (in either whole or half sizes), and then give this data to their teacher as well.
2. The teacher will collect the data from each student and display this data in a place in the room where all students can access the information. Each student will record the data for the entire class, both their foot lengths and their shoe sizes.
3. Students will figure out the mean, median, mode, and range of the foot lengths and shoe sizes and record this data in the attached table.
4. Have students compare lengths to shoe sizes, checking to see if the measurements are consistent.
5. Have a Winning Prize for the student with the lengthiest foot or largest shoe size, or the smallest (outliers) etc…for fun!!

What's Your Shoe Size? Worksheet

LENGTH OF FOOT

Class Data for Length of Foot

MEAN	MEDIAN	MODE	RANGE

Class Data for Shoe Size

MEAN	MEDIAN	MODE	RANGE

SHOE SIZE

EXPLORATION 1

This is an excellent opportunity to discuss measurement and the conversion from mixed feet/inches to inches.

You may wish to gather other data about the students in your class along with their height. Perimeter of feet or shoes or the students' hand print area is an opportunity to discuss irregular shapes and finding their area and perimeter.

DATA ANALYSIS 10

SECTION 10.1 MEASURES OF CENTRAL TENDENCY

Sets are useful for grouping interesting and related numbers. One such set is the heights of all of the people in your class. In order to use these sets, we need to analyze the numbers, or data, in context. The first step in **data analysis**, the process of making sense of a set, is collecting data. In data analysis, the idea of a data set is slightly different from that of a set. Unlike regular sets, data sets can have repetition of elements, and the order or arrangement matters.

EXPLORATION 1

Measure each person in your class in inches and record their name, age in months, and height in inches in a table like the one below. The numbers below were taken from another class, so your own class will have different results. Try to find ways to summarize the information in the table so that you can share your results with a friend without showing her the whole table. Would your strategy still work if there were 100 people in the survey? 1000 people?

Notice that the range is a single number. For example, in Exploration 1, students will naturally want to say that the range of heights is from 49 to 62, but the range = 62 – 49 = 13.

Name	Height (in)	Age (months)
Sophia	52	113
Rhonda	51	112
Edna	57	112
Danette	61	115
Hesam	55	117
Eloi	62	110
Vanessa	58	113
Michelle	60	108
Mari	58	125
Calvin	56	129
Moises	57	124
Amanda	57	120
Hannah	55	131
Tricia	55	129

Name	Height (in)	Age (months)
Kristen	57	130
Max	52	135
Jim	50	142
Karen	57	136
Diane	49	138
Tiankai	58	138
Oscar	51	137
Jenny	60	138
Bence	59	142
Pat	53	134
Teri	59	135
Sally	57	139
Will	57	140

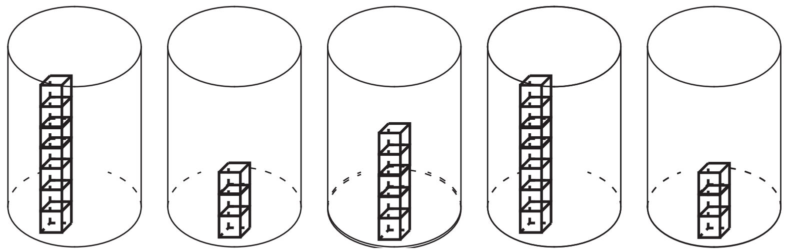
The entire collection of numbers is called the **data** and each individual piece of information is called a **data point**. Data is plural for datum.

A major goal of data analysis is to find a simple measure of the data, called a **measure of central tendency**, that summarizes or represents, in a general way, the majority of the data. There are three common measures of central tendency: the *mean*, *median*, and *mode*. The mean, median and mode are different ways to identify the location or center of the data. We are also interested in how spread out our data is. The **range**, the difference between the largest and smallest values of the data, provides a simple measure of how much the data varies.

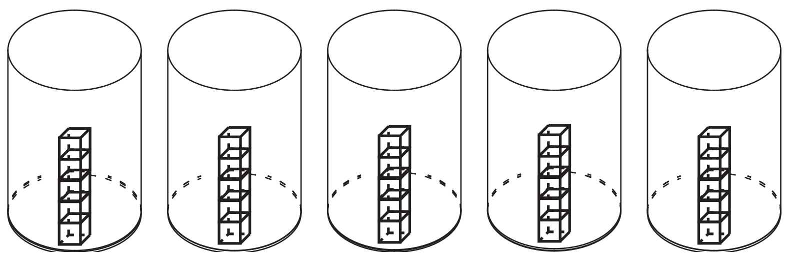
The **mean**, also called the **arithmetic mean** or **average**, is the sum of all the data values divided by the number of data points. For a visual example, suppose we have five containers, each containing a certain number of blocks:

:

Median: Arranging the data set in decreasing order may help students find the range more easily to carry out the needed subtraction.



These data can be grouped into a data set: {7, 3, 5, 7, 3}. We will notice the importance of the order of arrangement and the repetition. There are 25 blocks total. The *mean* number of blocks in a container is the number of blocks each container has if these 25 blocks are distributed evenly among the 5 containers: $\frac{25}{5} = 5$.



EXAMPLE 1

Find the mean of the following data set.

{91, 100, 83, 76, 37, 98}

SOLUTION

Add the data points. Divide the sum by the number of data points.

Sum = 91 + 100 + 83 + 76 + 37 + 98 = 485.

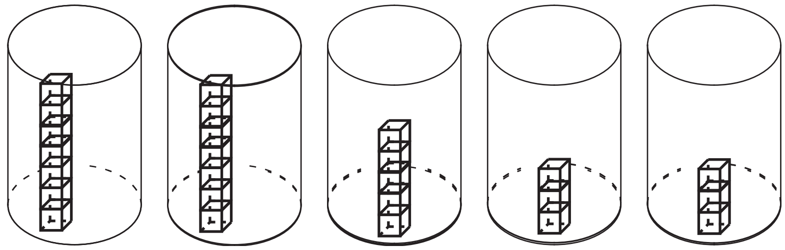
Mean = 485 ÷ 6 = 80.83

The **median** is the value of the middle data point when the values are arranged in numerical order. If the data set has an even number of data points, the median is the average of the two middle values. To find the median value for the container example, order the data, with the largest number of blocks first and the smallest number last:

While the mean and median only make sense for quantitative data, the mode can be computed for quantitative and quantitative and categorical data. However, the mode is not a very useful measure when quantitative data can take many distinct values.

Weave in conversations about when to use each measure of central tendancy:

Mean	with average grades, temperature, rainfall
Median	with large range when the mean is affected by outliers.
Mode	with elections or selecting favorites



The *median* is the number of blocks in the middle, or third container *with respect to the sorted ordering*. The median is a helpful measure of central tendency because half of the values are less than or equal to the median and the other half of the values are greater than or equal to it.

EXAMPLE 2

Find the median of the following data set.

{91, 100, 83, 76, 37, 98}

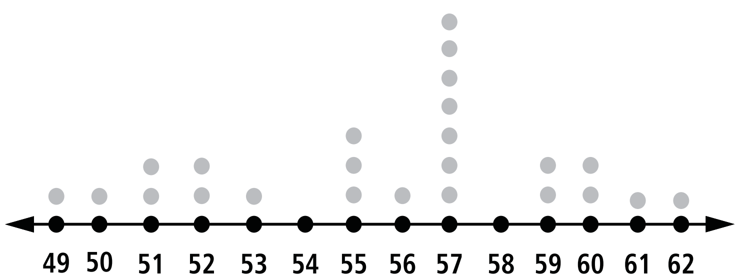
SOLUTION

Arrange the data points in decreasing order

100, 98, 91, 83, 76, 37

Because there are an even number of data points, take the middle two points, 91 and 83 and find their mean. (91 + 83)/2 = 174/2 = 87. The median for the data set is 87.

Frequency is the number of times a data point appears in a data set. For example, if there are 4 people in the class who are 56 inches tall, then the frequency of the height 56 inches in the class is 4. The **mode** is the value or element that occurs the most often or with the highest frequency in the data set. One way to illustrate the frequency of a relatively small data set is to use a **dot plot**. The line plot below uses the data set from Exploration 1



Note that in a stem and leaf plot, there are typically no commas used. The leaf portions are, therefore, the single right most digits of each data point.

In Exploration 1 the mode in our data is 57 because it appears 7 times, or the most times in the data. What is the range for this data set? The line plot is helpful in seeing the largest and smallest values. The range for Exploration 1 is the difference between the largest value of 62 and smallest values of 49 in the data set. The range is 62 – 49 = 13 for this data set.

A set of data can have more than one mode. The modes for the containers of blocks discussion earlier are 3 and 7. Sometimes a set of data has no repeated data points. What would the mode be in such a data set? If you said no mode, you were correct. It would be incorrect to say that the mode was zero.

EXAMPLE 3

Create a stem and leaf plot for the heights of students from the table in Exploration 1.

SOLUTION

Stem	Leaf
4	9
5	0112235556777777788899
6	0012

As you can see, this graph shows how most of the data is in the 50's and indicates a shape that peaks in the 50's and well distributed among the other values.

The stem and leaf plot can show how the distribution of large number of data is shaped. Some data can be very clustered around certain numbers as in example 3 where most of the numbers were in the 50's. Other sets of data can be spread out, or even skewed towards one end or another. It can even show that there may be a few data points that are very different from the others.

One way to see the contrast in shape and distribution is to examine two sets of data as in the exploration below.

EXPLORATION 2

Data on daily temperatures in two cities are given:

Daily average temperature in a Texas city in April.

61	59	65	68	82	72	77
76	73	65	65	62	70	67
57	50	62	61	70	69	64
80	77	82	75	79	71	79
87	80					

Daily average temperature in a New York city in April.

41	57	59	40	34	33	40
51	54	64	65	45	47	63
63	54	59	57	42	45	46
70	72	48	48	43	51	59
70	89					

1. Create a stem and leaf plot for each of the two cities.

2. Find the median of each of the data sets.

3. Find the mean of each of the data sets.

4. Describe some features of the individual stem and leaf that you notice. Write down differences and similarities between the two data.

5. Did you notice the lowest and the highest temperatures for each of the cities?

6. What is the range of each of the data sets?

We close this section with the **box plot,** sometimes called the box and whisker plot, which is another way of organizing data.

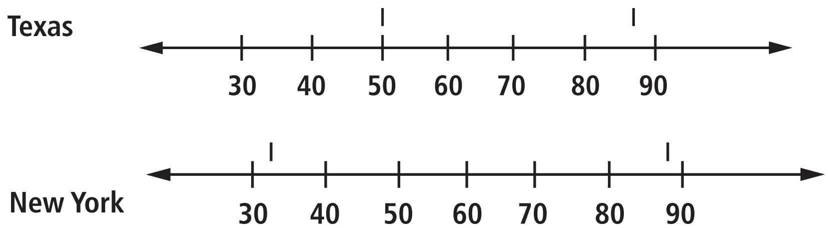
The first step is to order the data in increasing order. As an example, we use the data from Exploration 2 for the two cities.

Texas: 50, 57, 59, 61, 61, 62, 62, 64, 65, 65, 65, 67, 68, 69, 70, 70, 71, 72, 73, 75, 76, 77, 77, 79, 79, 80, 80, 82, 82, 87

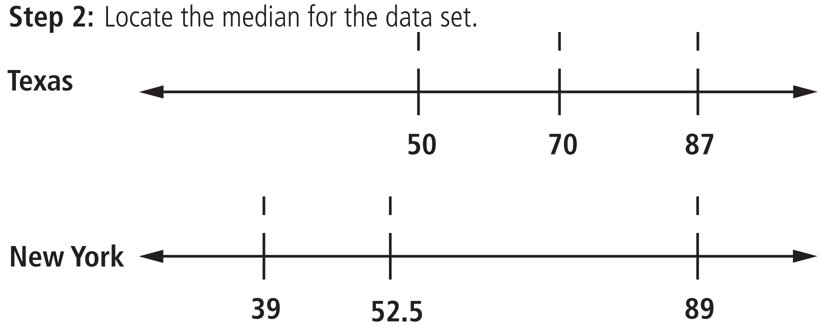
.**New York:** 33, 34, 40, 40, 41, 42, 43, 45, 45, 46, 47, 48, 48, 51, 51, 54, 54, 57, 57, 59, 59, 59, 62, 63, 63, 64, 65, 70, 70, 89

The instructions for constructing the box plot is as follows:

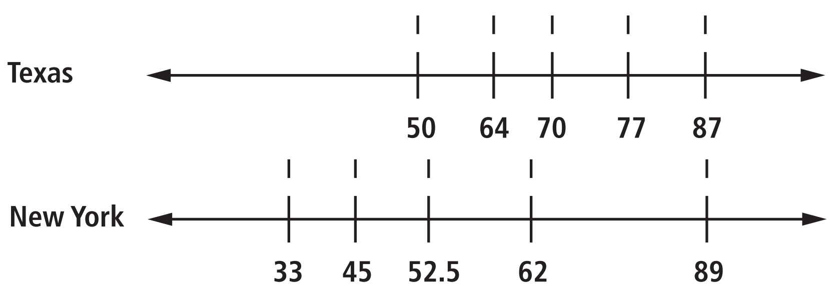
Step 1: Place the largest and smallest values on the respective number lines and put notches above those numbers as shown below.



Step 2: Locate the median for the data set.



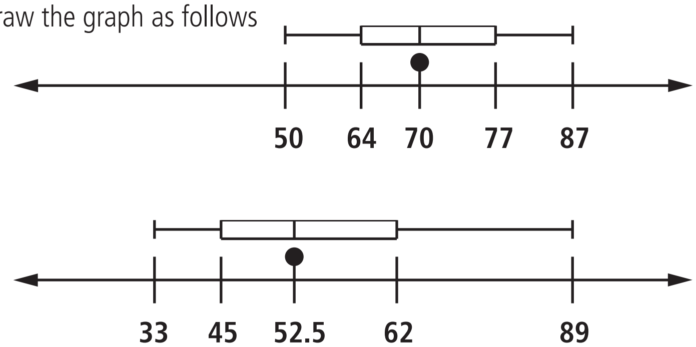
Step 3: Locate the median for the lower half called the lower or first quartile and the median for the upper half called the upper or 3rd quartile.



Step 4: Draw the graph as follows

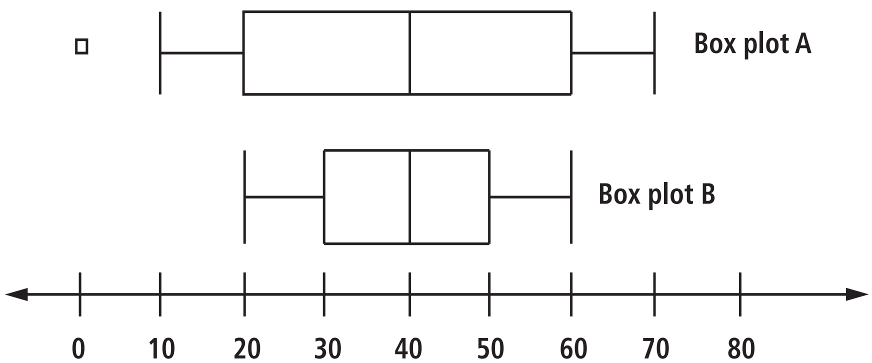
Texas

New York



Along with the range and medians that are summarized in the box plot, another number referred to as the Interquartile Range (IQR) tells us how the distribution of the 50% is concentrated. The IQR = upper quartile – lower quartile.

Two boxplots below represent two datasets.



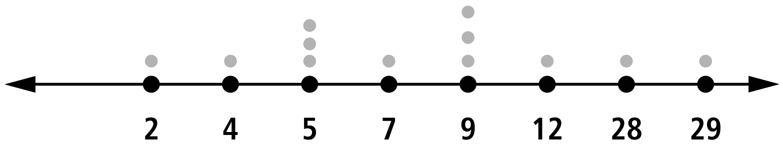
A summary for each boxplot is given below:

Box plot A
Minimum value is 10
Maximum value is 70
The range is 70 – 10 = 60
The median is 40
First quartile is 20
Third quartile is 60
IQR is 60 – 20 = 40
Outlier(s) at 0

Box plot B
Minimum value is 20
Maximum value is 60
The range is 60 – 20 = 40
The median is 40
First quartile is 30
Third quartile is 50
IQR is 50 – 30 = 20
No outliers

PROBLEM 3

The mean is 130/13 = 10 and the median of the ordered data set {2,4, 5, 5, 5, 6, 7, 9, 9, 9, 12, 28, 29} is the middle number 7. The modes are 5 and 9 because they both appear 3 times times in the data set. The range is 29 – 2 = 27.

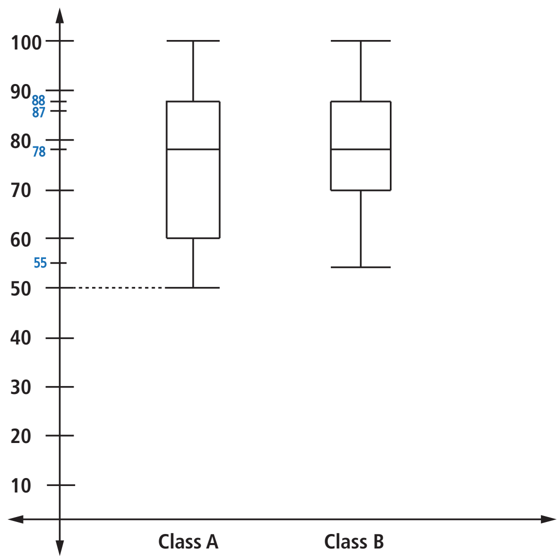


Notice that the median for both data sets is 40. The data set for A has a greater spread than B because the range of A is 60, while the range of B is 40. The IQR that shows the range of the 2nd and 3rd quartiles is 40 for data set A, while B has a smaller IQR of 20. This suggests that more data points are concentrated near the center for B. Another difference between the two data sets is that A has an outlier while B does not.

In our example above, the box plots were situated horizontally. The box plots can also be situated vertically as in the problem to follow.

PROBLEM 1

Consider the two box plots below for the test grades from two 6th grade classes. Describe the center, spread, and shape of the data distribution using the ideas of range, median, upper and lower quartiles, and the interquartile range.



PROBLEM 2

Create a box plot for the data set of heights used in exploration 1. Use the box plot to summarize your observations.

PROBLEM 4

The mean is given by (95 + 30 + 98 + 93 + 100)/5 = 83.2.
The median is the middle value when the data is ordered (30, 93, 95, 95, 100), so it is 95.

PROBLEM 5

Answers will vary with data per class.

EXPLORATION 3

Original mean = 56.04 inches; new mean = 68.32 inches.
Original median = 57 inches; new median = 57 inches.
The mean has increased, but the median has not changed.

EXAMPLE 4

Find the mean, median, mode, and range of the following data set. Create a dot plot with the given data.

$$\{2, 8, 4, 8, 8, 6, 5, 7, 9, 3, 7, 5\}$$

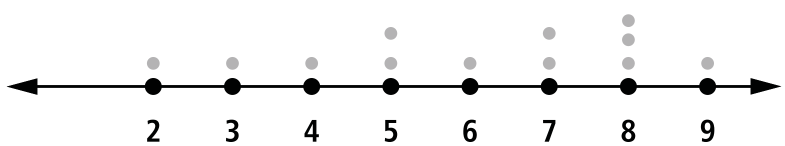
SOLUTION

The mean is found by adding the values together and dividing by the number of values. The sum of the values is $2+8+4+8+8+6+5+7+9+3+7+5=72$. The number of values in the set is 12. The mean is $\frac{72}{12}=6$.

Putting the data set into order from smallest to greatest value results in $\{2, 3, 4, 5, 5, 6, 7, 7, 8, 8, 8, 9\}$. Because there are an even number of values in the set, the median is the average of the two middle values. The median is the average of 6 and 7, $\frac{(6+7)}{2}=\frac{13}{2}=6.5$.

The most commonly occurring value in the data set is 8, so 8 is the mode.

The range is the difference between the highest and lowest value. The range is $9-2=7$.



PROBLEM 3

In the following data set, what is the mean? the median? the mode? the range? Include a dot plot of the given data.

$$\{4, 9, 9, 12, 5, 9, 2, 5, 6, 7, 28, 29, 5\}$$

The mean depends on all the numbers in the data, but the median only depends on the value of the data point in the middle position. That does not, however, suggest that the mean is a better measure of central tendency than the median.

PROBLEM 6

1. mean = 79
2. median = 86
3. mode = 87
4. range = 37
5. Check line plots

PROBLEM 4

Find the mean and median of the following six weeks test grades:

{95, 30, 98, 93, 100}.

Compare the value of each as a measure of the data.

PROBLEM 5

Find the mean, median, mode, and range of your class data.

Another way to view a data set is to use a **Stem and Leaf Plot**. The leaf is usually the last digit of the numbers in the data set and the stem is the rest of the numbers to its left arranged in a vertical numerical order. Let's look at an example. This type of display lists all data points in a condensed form.

EXPLORATION 3

Using the data from Exploration 1, compute the mean and the median of the heights of the class. Then, imagine that a giant who is 400 inches tall joins the class. Compute the new mean and find the new median. How has each changed?

PROBLEM 6

You are given a data set represented by the following stem and leaf plot:

Stem	Leaf
10	0
9	6442
8	98777553
7	98841
6	73

Use the information to determine the following, if possible, round any value to the nearest one:

1. The mean of the data set.
2. The median of the data set.
3. The mode of the data set.
4. The range of the data set.
5. A dot plot of the data set.

If the data is **skewed**, or uneven, a median value is a more accurate picture of the representative value than the mean is. Exploration 2 had a very tall giant join the class. The mean was affected by this **outlier**, a term used to refer to a value that is drastically different from most of the data values. The median, however, was not affected. The mean is usually more influenced by extreme values than the median.

A distribution that has a behavior in the lower quartile similar to the upper quartile would be symmetric about the median. The distribution would be shaped symmetric to the median. We will discuss this some more in the next section.

Let us review the ways in which we summarized data in this section.

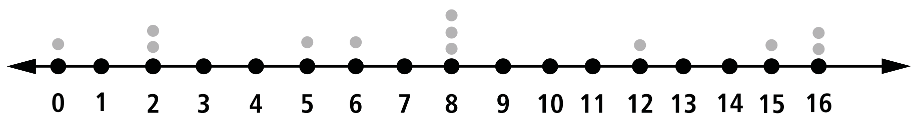
If we have a set of n values, then we can find the following measures:

- Find the mean by adding the values and dividing by n.
- Find the median by ordering the values and finding the value that is in the middle, if n is odd, or taking the average of the middle values, if n is even.
- The mode is the most frequent value that occurs. There could be two or more such values. There could also be no mode for a data set.
- The range is the difference between the largest and the smallest values in the set.
- The interquartile range (IQR) is the difference between the median of the upper half (the third or upper quartile) and the median of the lower half (the first or lower quartile).

EXERCISES

1. a. mean = 8.17 median=8 mode=8 range=16
   b. mean=68.17 median=68.5 mode=71 range=14

2.



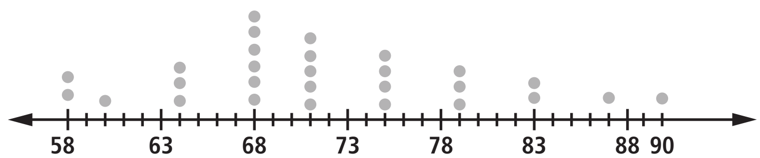
3. 6 | 665430
   7 | 4433111

5. a. mean b. mode c. range d. median

6. a. median=83 b. mode=87 c. range=18

EXERCISES

1. Find the mean, median, mode, and range of the following data sets:
   a. {8, 16, 0, 8, 6, 2, 15, 2, 12, 8, 16, 5}
   b. {71, 66, 74, 64, 66, 73, 71, 60, 71, 65, 63, 74}
2. Take the data from problem 1a and create a dot plot.
3. Take the data from problem 1b and create a stem and leaf plot.
4. Summarize the information given by the dot plot below representing the pulse rate, in beats per minute, for a group of 28 students.



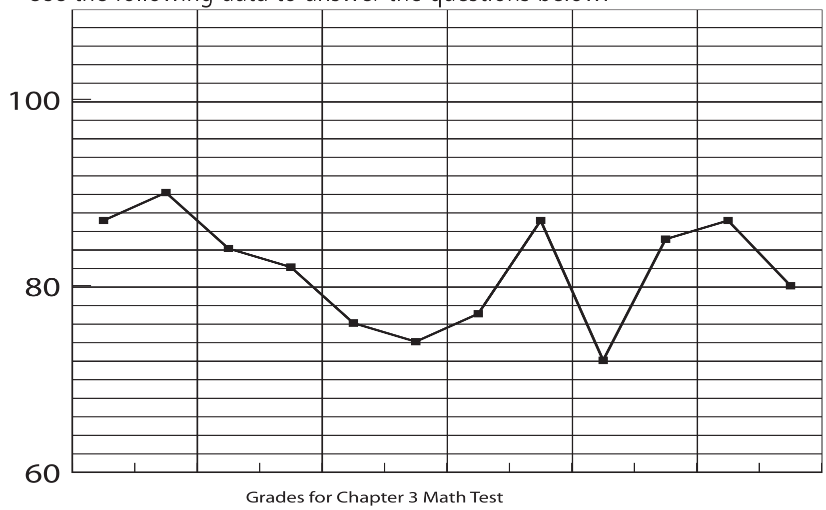
Determine:

   a. the range of the data set
   b. the mode of the data set
   c. an estimated normal pulse for this class
5. Which measure of central tendency is most helpful in representing the following situations? Choose from mean, median, mode, and range.
   a. Determining your report card grade in a subject. mean
   b. The number of bowling pins knocked out the most often during a bowling game. mode
   c. The high and low temperature of a city in one day.
   d. The midpoint age among a class of college students. me

7. 63 inches

8. Yes, pack different clothes. The range of temperatures in Castolon means you would need a wider variety of clothes for that location. Galveston is right next to the Gulf of Mexico while Castolon is far inland. The large body of water is why Galveston has a smaller temperature range; the water moderates the daily swing in temperature.

6. Use the following data to answer the questions below:



a. What is the median of the grades for the Chapter 3 Math Test? mean
b. What is the mode of the grades for the Chapter 3 Math Test?mode
c. Identify the range of the grades for the Chapter 3 Math Test?

7. The total height of all the students of a class of 15 is 945 inches. What is the mean height of the class?
8. The January mean daily temperatures for Castolon, TX and Galveston, TX are approximately the same. However, their ranges are quite different. The temperature data, in degree Fahrenheit, from the national oceanic and atmospheric administration (NOAA) are:

City	Maximum	Minimum	Mean	Range
Galveston	61.9	49.7	55.8	12.2
Castolon	67.7	33.6	50.7	34.1

Even though Galveston and Castolon have about the same daily mean temperature for January, would you consider packing different clothes for the two places? Which measure of central tendency influenced your decision? Why?

9. The height seems to be going up about 3 inches per year. So the median height for 15-year-olds might be 68 inches. However, at some point, growth stops. So the mean height for 24- and 25-year-olds is probably the same.

9. On the right are estimated national median heights in inches for 9- through 14-year-olds in 2000, according to the National Center for Health Statistics (NCHS). Based on this data, what is your estimate for the median height for 15-year-olds? Do you think the median heights for 24-year-olds and 25-year-olds are that much different? Explain.

Age group	Height (in)
9-year-olds	52.5
10-year-olds	54.5
11-year-olds	56.5
12-year-olds	59.0
13-year-olds	62.0
14-year-olds	65.0

10. Use the data in the stem-and-leaf to draw a box plot. Summarize your observations (e.g. range, median, etc.)

Stem	Leaf
3	9
4	12334444556789
5	01

11. Sam and Matt went fishing in the Gulf of Mexico. Over the week each caught a total of 13 fish. They recorded the length, in inches, of each fish that they caught.

Sam: 13, 12, 9, 5, 8, 14, 20, 16, 14, 9, 6, 12, 12

Matt: 11, 12, 10, 13, 15, 11, 12, 8, 13, 12, 9, 9, 14

Draw a box plot for the two data sets. Summarize and compare their median, range, interquartile range, and their upper and lower quartiles.

12. Spiral Review (6.8C) Answer: 50 degrees.

13. Spiral Review (6.1A) Answer: c

Ingenuity

14. If Ling ends up with an average of 90 over seven grades, then she will have earned a total of 90 x 7 = 630 points. So far, she has earned 85 + 94 + 96 + 87 + 98 + 89 = 549. So she needs to earn at least a score of 630 - 549 = 81 on her last assignment.

Another way to approach this problem is to think of Ling's previous grades in terms of whether they are higher or lower than 90, the grade she wants to meet or exceed. So far, her grades are 5 below, 4 above, 6 above, 3 below, 8 above, and 1 below. If we add all of these (taking into account whether they are above or below the threshold Ling wants to meet), the total of the differences is 9 above the threshold. So Ling can afford to be 9 below the threshold on her final assignment. So she needs at least an 81. This is a very powerful technique for dealing with averages of numbers that are relatively large, but all fall close to a certain central number.

Investigation.

15. a. The median is 1. The mean clearly must be greater than 1, since every number is at least 1, and some numbers are greater than 1. So the median is less than the mean.

b. The median is 8.5; we should expect the mean to be smaller, since the outlier of 1 drags down the average. Indeed this is the case; the mean is 7. So the median is greater than the mean.

c. The median and mean are both 13, so they are equal.

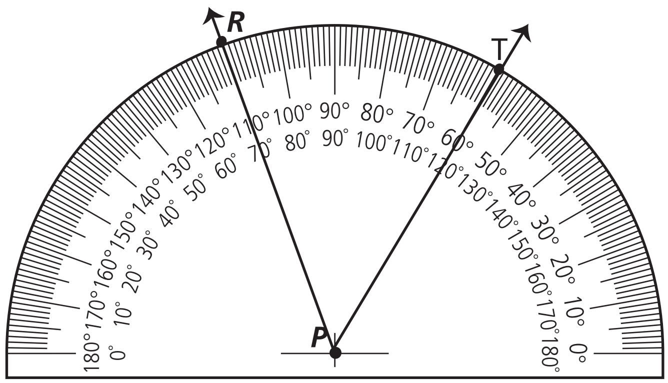
d. The median and mean are both 45, so they are equal. (We can easily tell that the mean is 45 because the numbers are symmetrically distributed about the number 45.)

e. The median is 5, and the mean must be greater for the same reasons explained in (a). So the median is smaller than the mean.

f. The median is 80. The mean is smaller because the numbers less than 80 on the list are farther from 80 than the ones greater than 80. So the median is greater than the mean.

Spiral Review:

12. What is the measure of $\angle RPT$?



13. Which statement is not true?

a. $\frac{4}{5} > 70\%$ c. $40\% > \frac{2}{5}$

b. $-1\frac{3}{5} < -\frac{4}{5}$ d. $1\frac{5}{6} > 1\frac{2}{3}$

14. **Ingenuity:**

Ling's grades in her math class are 85, 94, 96, 87, 98, and 89. If Ling has one more assignment to do, and she wants to make sure that she has an average of at least 90 in the class, what grade does she need to make on the assignment?

15. **Investigation:**

For each of the following sets of numbers, determine whether the median is greater than, less than, or equal to the mean. Can you find a way to do this without actually calculating the median and mean of the numbers?

a 1, 1, 1, 1, 2, 2, 3

b. 1, 8, 9, 10

c. 13, 13, 13, 13, 13

d. 42, 43, 44, 45, 46, 47, 48

e. 5, 5, 5, 5, 5, 5, 11, 11

Challenge.

16. Answer: 5

f. 42, 57, 75, 80, 84, 89, 91

16. **Challenge:**

    A class of 11 students was given the following extra credit question on a test:

    Pick a positive integer between 1 and 10, inclusive: ______

    (The least integer greater than or equal to the nonnegative difference between the mean and the median of the answers given will be added to everyone's score.)

    What was the maximum bonus that the class could have earned?

Section 10.2 - Graphing Data

Big Idea:
Graphing data

Key Objectives:
- Construct and read a bar graph.
- Construct and read a histogram.
- Construct and read a circle graph.
- Analyze data using graphs.

Materials:
Grid paper, Protractor, Compass or handouts with pre-made circles, Large area for making human graphs, String or yarn

Pedagogical/Orchestration:
This section constructs histograms, percent bargraphs, and pie graphs from given data. We also talk about how to interpret the data the relative use of each type. The exercises are rich with applications involving graphing data.

Activity:
"Our Class Data"

Vocabulary:
bar graph, pie graph, line graph, stem and leaf

TEKS:
6.10(A,D) New: 6.12(A, B, C, D); 6.13(A, B)

Launch for Section 10.2:
Remember that a Venn Diagram is one way to show a relationship between sets. For example, you may have one set consisting of all students in our class who owns a pet. Raise your right hand and keep it up. Another set can be the set of all students in our class who have a sister or a brother. Raise your left hand if you do. Can you see that some of you have both hands up? Can you use words to describe those people. Let's use a Venn Diagram to illustrate these two sets.

Our Class Data

Objective: The students will collect a variety of data from their classmates, and use their knowledge of displaying data to make a bar graph and a pie chart.

Materials:

Paper and pencil to record data collection
Rulers
Protractors
Markers, crayons or colored pencils
White paper

Activity Instructions:

1. Divide your class up into groups of two. Explain to each group that they are to collect data from each of their classmates about what type of pets live in their house.
2. After collecting data, groups should brainstorm about how they are going to organize their data into both a bar graph and a pie graph.
3. After brainstorming, students will pick up two pieces of white paper and their rulers and protractors and start creating their graphs. Groups can either work together, or they can divide the work and make one graph each.
4. After each group has created two graphs, students can use their markers, crayons or colored pencils to make their graphs more creative.

EXAMPLE 1

Emphasize that 12 is somewhat arbitrary. Any integer slightly larger than the highest frequency, 10, will do. Have students discuss their answers. While we can make a bar graph with any order of colors, alphabetical or in order of frequency are the most common.

SECTION 10.2 GRAPHING DATA

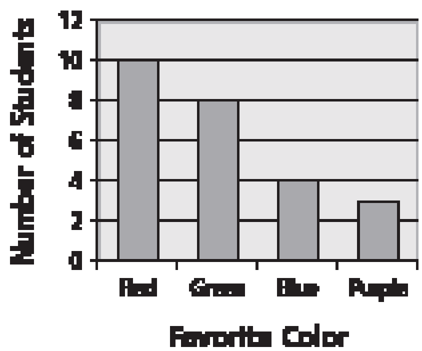
When collecting data, it is often useful to draw a picture or graph to represent the data that has been collected. A graph of the data gives a quick, easy way to see what the data represents.

EXAMPLE 1

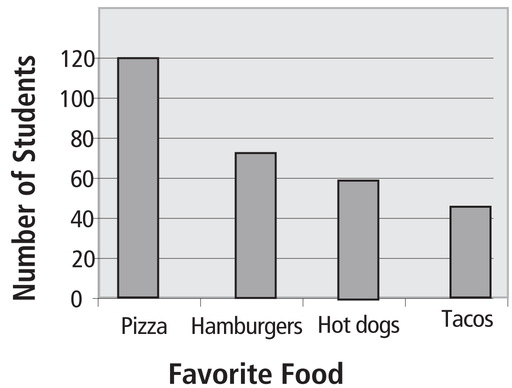
Ms. Garcia's class has twenty-five students. Each student was asked which color they like best. The survey shows that 10 students prefer red, 8 students prefer green, 4 students prefer blue and 3 students prefer purple. What are the best ways to represent or display this information?

SOLUTION

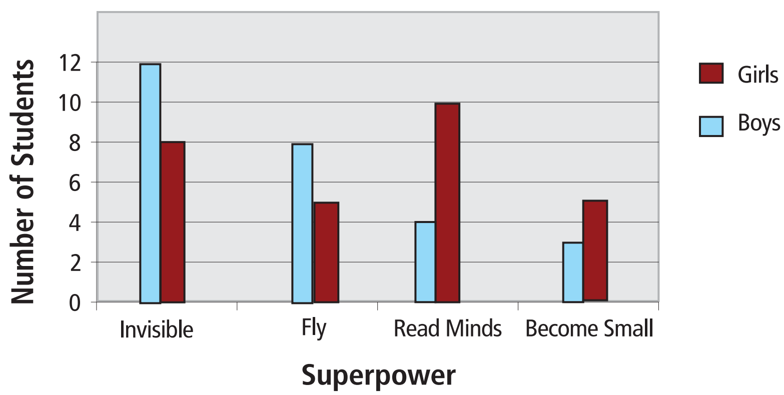
One way to display the data is to make a special kind of graph called a **bar graph**. A bar graph is generally used when the categories are not numerical as in this case when we examine kinds of color. To construct a bar graph, draw an x- and y-axis, subdivide the horizontal or x-axis into four equally-spaced intervals and label the intervals with the categories Red, Green, Blue, and Purple. Then label the vertical or y-axis with points from 0 to 12. For each color, draw a vertical bar equally separated from the other bars. The height of each bar represents the number of people who liked a particular color best. The bar graph should look like this:



PROBLEM 1



EXAMPLE 2



Explain why the vertical axis has a number scale from 0 to 12. Why is there no number scale on the horizontal axis? Explain whether the order of the colors is important.

PROBLEM 1

The cafeteria at Summit Ridge Middle School took a survey to determine what were student's favorite cafeteria foods. The results of the survey were as follows: 120 liked pizza; 75 liked hamburgers; 60 liked hot dogs; 45 liked tacos. Construct a bar graph to represent the survey results.

A double bar graph is another useful way of representing certain kinds of data. Notice in the previous examples the data gathered were for color or food categories. If there is more than one category, such as boys and girls, then a double bar graph might be called for.

EXAMPLE 2

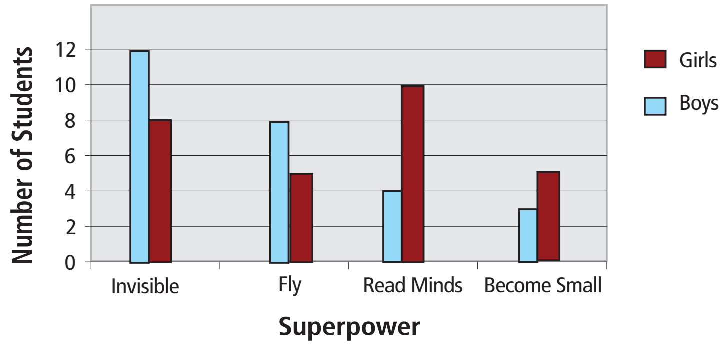
A survey asked boys and girls in Ms. Lowrance's class what one super power they would choose if they were superheros. There were four choices of super powers and the results of the survey is as follows:

Be invisible was chosen by 12 boys and 8 girls
Be able to fly was chosen by 8 boys and 5 girls
Be able to read people's minds was chosen by 4 boys and 10 girls
Become small was chosen by 3 boys and 4 girls.

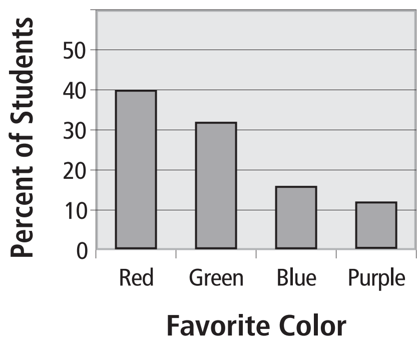
SOLUTION

Just as with bar graphs, we let the vertical axis be frequency in this case from 0 to 12, and the horizontal axis the four categories of super powers. But instead of just one bar, we put the boys and girls next to each other and are able to compare the categories as well as compare the boys and girls.

PROBLEM 2



PROBLEM 3 Discuss with students what settings might you make one choice of data representation over another and why.



Creating Pie Graph Activity

Objective: Create a bar graph. Convert bar graph to circle graph.

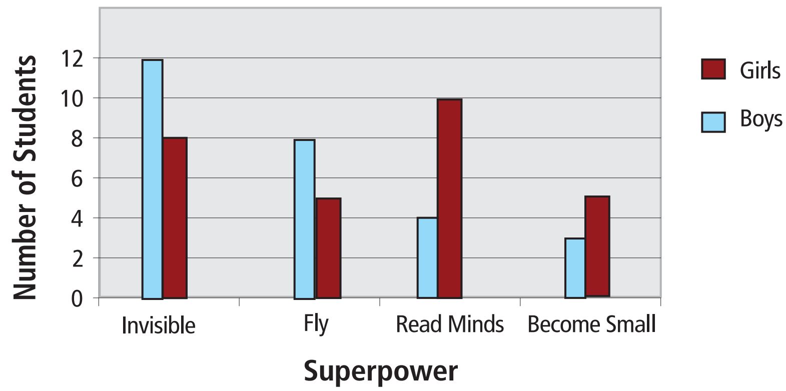
Materials: 1-inch graph paper, scissors, tape

Activity Instructions:

1. Conduct a survey with the class with at least three different choices. Record the survey and represent the data as a bar graph using a 1-inch graph paper. Make the bars of different colors and each square represents a vote.
2. Once each bar is colored, students cut out each bar and tape or clue each bar onto a blank bar the size of the whole.
3. Cut out the "whole" strip and fold horizontally down the middle with the colored side on the outside.
4. Create a circle from the strip and determine, as best as possible, the circle of the square. Draw the radii from the center of the circle to the line that marks the different colors.

Extension: Have each student create/complete a table of data that shows each survey result or pie graph as a fraction, decimal, and percent of the set of data points.

Technology Connection: Have students use the Excel spreadsheet to enter data and to create various graphical methods such as the bar graph and pie graph. The students can also include the fractional, decimal, and percent interpretation of the data.



PROBLEM 2

A survey was conducted in Ms. Soto's class to determine favorite sports among boys and girls. The results were as follows:

Football	18 boys	3 girls
Baseball	10 boys	8 girls
Volleyball	3 boys	12 girls
Bowling	8 boys	10 girls

Construct a double bar graph for the data.

Another way to represent the data from example 1 is by using percents. Because there are 25 students in all, $\frac{10}{25}$ of the class likes the color red. Convert $\frac{10}{25}$ to the decimal 0.40 and then to the percent 40%. Similarly, $\frac{8}{25} = 0.32 = 32\%$ of the class likes green.

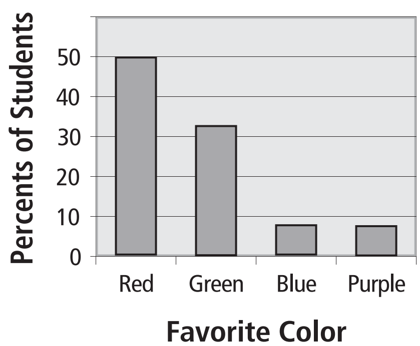
PROBLEM 3

Calculate the percents of the other two colors from example 1. Use these percents as data on the vertical axis to build another bar graph. Label the axes and draw the graph.

The bar graph shows the actual frequency in numbers while the circle graph shows only the percent of frequency certain data occur. The whole in the circle graph is clear but some possible choices that did not occur disappear while the bar graph shows that some outcomes did not occur.

PROBLEM 4

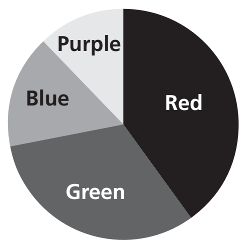
Since the total number of students is different in the two classes, the bar graph with counts is deceiving. It is preferable to use percentages. This is one of the reasons we would use percentages instead of counts.



Although the shape of the bar graph is the same, the percentage graph gives a picture of the relative quantity of the class' preference for each color, not the number of students directly. Instead, the bar graph shows the relative number or percentage of students immediately.

Another way to represent the percentage data is to use a **circle graph** or a **pie graph** of the data. Use your protractor or compass to draw a circular outline for the circle graph. You have already computed the percentage of each color. The proportion of the circle graph with a given color corresponds to the percentage of students who prefer that color. The larger the sector of the circle graph, the greater the percentage of people who liked the color.

The completed pie graph on the next page clearly represents which color students like best and makes the result of the survey visually obvious.

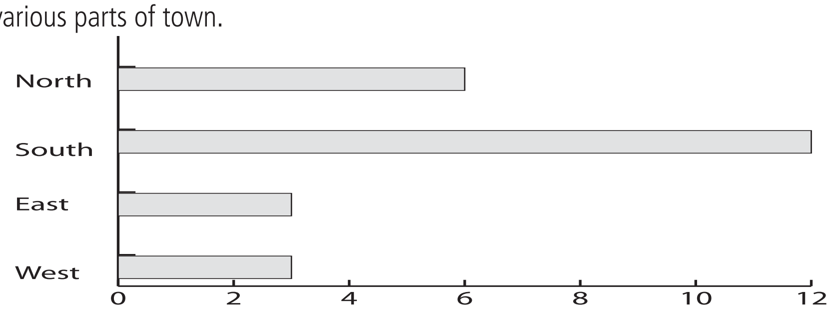


PROBLEM 4

Mr. Ruiz asked his 12 students which color they liked best as well. In his class, he found that 6 students prefer green, 4 students prefer red, 1 student prefers blue, and 1 student prefers purple. Make a bar graph to compare the data from Mr. Ruiz's class to the data from Ms. Garcia's class. In which class is green more popular?

PROBLEM 5

The bar graph represents the number of students in Mr. Mungia's class who live in various parts of town.



PROBLEM 5

1. 6 students live in the north part of town
2. 25% of students live in the north part of town.
3. 75% of students do not live in the north part of town.
4. percentages for pie graph: 50% for south, 12.5% west, 25% north, 12.5% east.

Use the information to determine the following:

1. How many students live in the northern part of town?

2. What percent of the students in Mr. Mungia's class live in the northern part of town?

3. What percent of the students in Mr. Mungia's class do not live in the northern part of town?

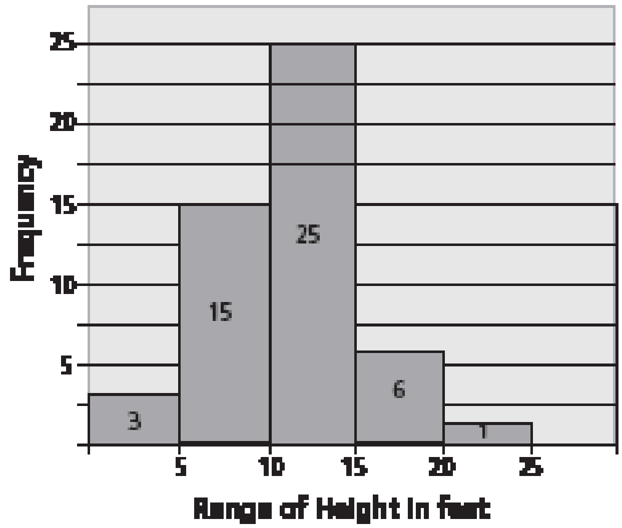
4. Use the bar graph to create a corresponding circle or pie graph.

Recall that bar graphs were used to represent data with non-numerical categories such as color. A **histogram** is a graphical representation of data with numerical categories. Histograms are drawn as in a bar graph with the positive x-axis indicating numerical categories of numbers or range of numbers. The heights of the "bars" can be either the frequency or numbers in each category or they can be percents of the data. The bars are also drawn touching each other whereas the bar graph generally has the bars not touching each other. We use the following example to demonstrate these concepts.

The data shows the results of a survey taken by the city to determine the heights of the crape myrtle trees in the city.

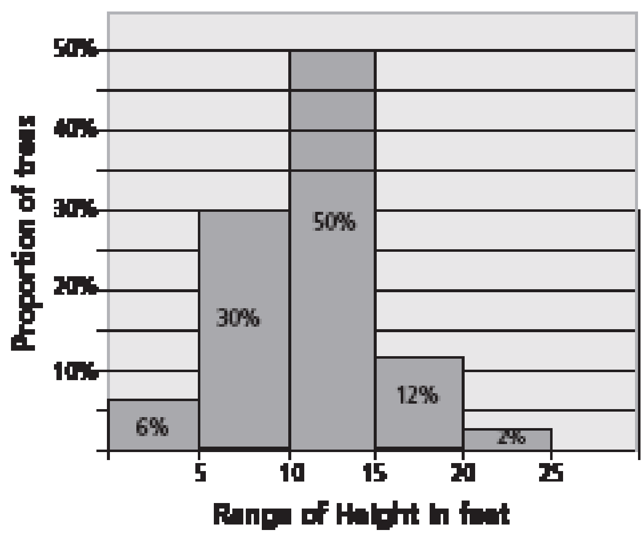
Height of tree (in feet)	Number of trees
Between 0 and 5	3
Between 5 and 10	15
Between 10 and 15	25
Between 15 and 20	6
Between 20 and 25	1

The histogram using the range of heights of the trees on the x-axis and the frequency of that size tree along the y-axis we have the following histogram:



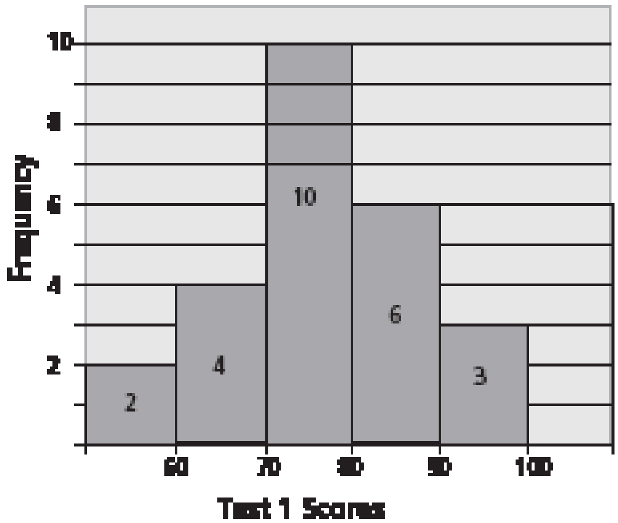
We can also create a percent bar graph for both bar and histograms.

This histogram can also be constructed using the proportion of trees in each category. The histogram would then look like the following:

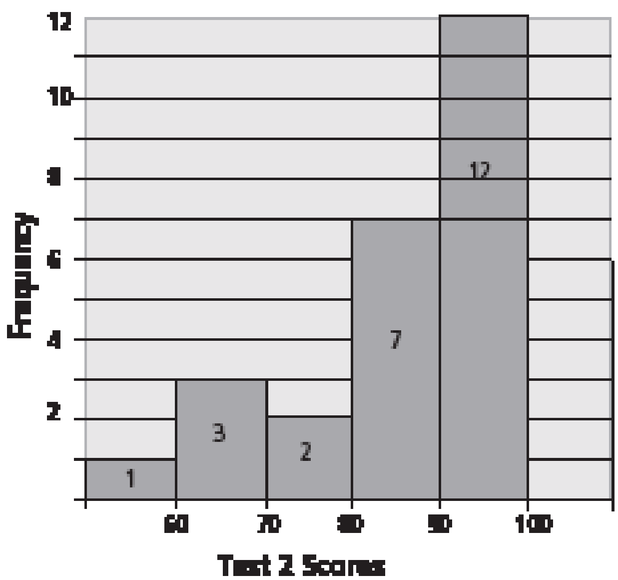


PROBLEM 6

Histogram for Test 1 scores



Histogram for Test 2 scores



PROBLEM 6

The test grades on the Test 1 and Test 2 are given below. Create a histogram for each test using the data given below.

Range of test grade	Number of Students TEST 1	Number of Students TEST 2
Between 50 and 59	2	1
Between 60 and 69	4	3
Between 70 and 79	10	2
Between 80 and 89	6	7
Between 90 and 100	3	12

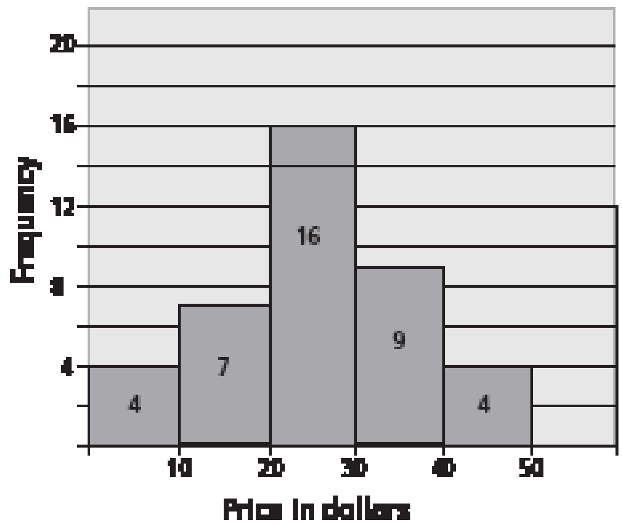
Use the histogram to make observations about the shape and distribution of the data. Is there enough information to determine the mean or the median? Explain why.

Histograms give visual representation to a set of data and can reveal information about what is similar or different about the sets and their distribution and variability. The idea of variability is an important part of statistics. The data that you want to study may have a great deal of variability such as the time that you were born. On the other hand, there is very little variability to the information regarding the time you are dismissed from school.

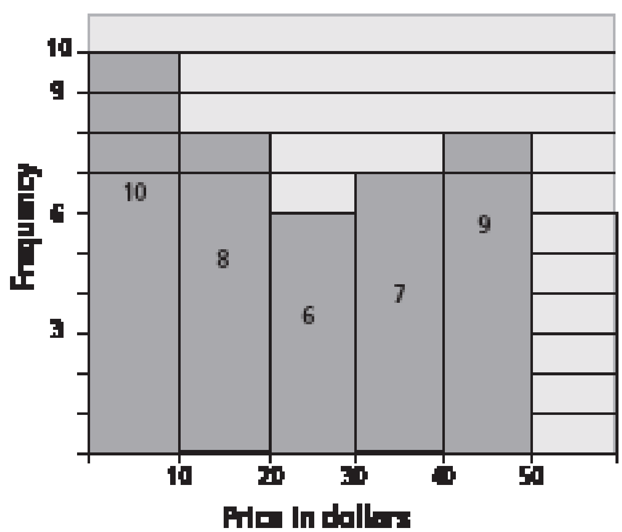
For example, in the three histograms below, list what you observe about their ranges, means, medians, and their shapes (symmetric or skewed).

The histograms are of a sample of the costs of 40 items sold at the three stores and the number of items in that price range.

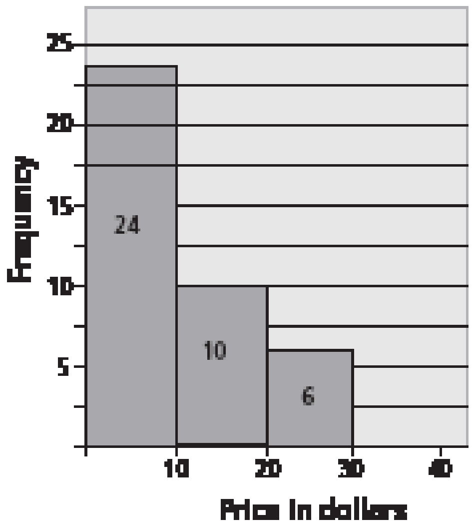
Store 1



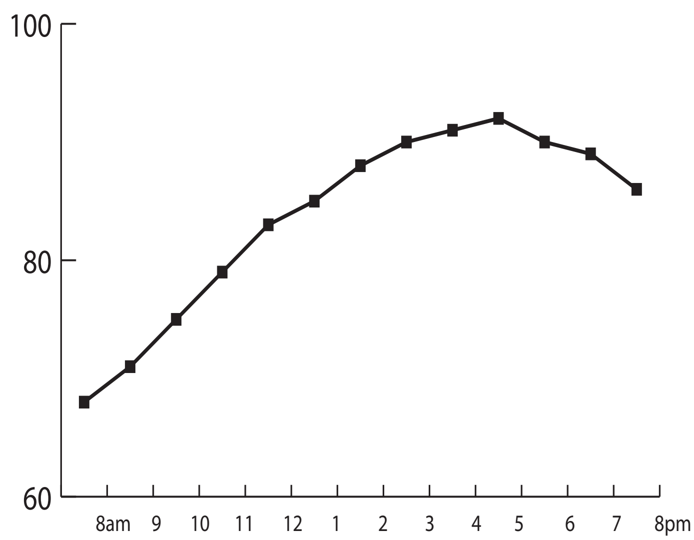
Store 2



Store 3

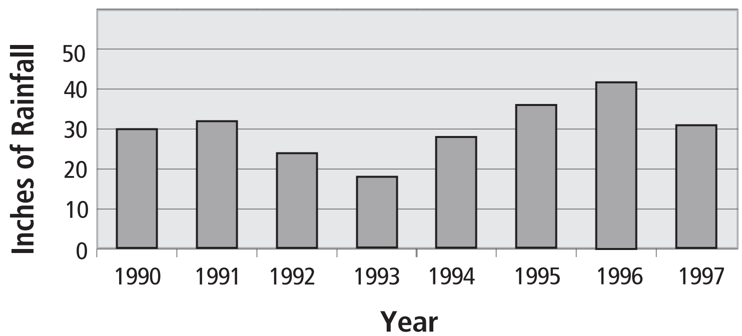


A **line graph** for a set of data points is often used to show changes in the data over a period of time. For example, hourly changes in the temperature for one day using a line graph shows the rise and fall of the temperature. While only hourly changes are recorded, the points are usually connected from point to point as in a portion of a line graph below:

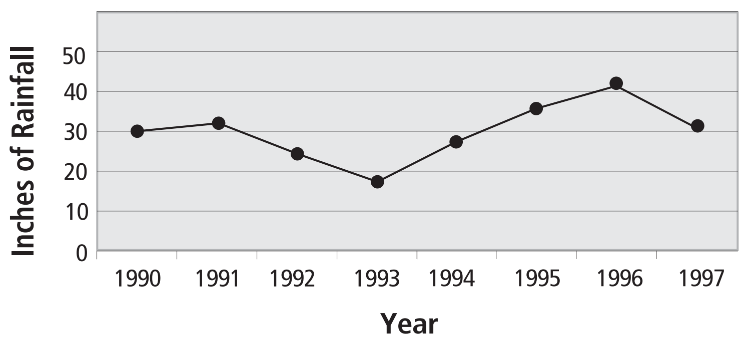


EXPLORATION

a.



b.



EXERCISES

1. a. 17 cars sold b. $293,000

EXPLORATION

The rainfall record for a region over an 8-year period from 1990 to 1997 is listed to the right.

Year	Rainfall
1990	30 inches
1991	32 inches
1992	24 inches
1993	18 inches
1994	28 inches
1995	36 inches
1996	42 inches
1997	31 inches

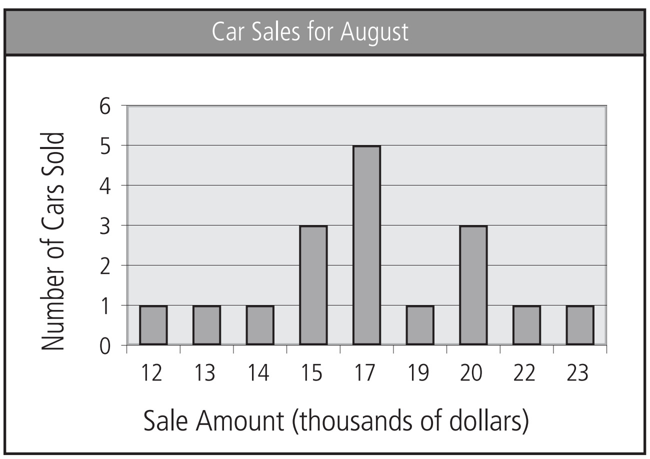
a. Make a bar graph to represent this data.

b. Plot the data on a coordinate plane. Label the horizontal axis to represent time in years, and the vertical axis to represent inches of rainfall. To convert the set of points to a **line graph**, connect the points sequentially with straight lines.

c. What differences do you notice between the line graph and the bar graph?

EXERCISES

1. Connie was working on important data when her computer crashed and she lost all the numbers. Luckily, she had printed a bar graph of her data earlier.



Based on the data from the bar braph:

a. How many cars were sold?

b. What is the total sales amount for the month of August?

2. Create bar graph

3. Line graph.

4. a. summer b. depends on class data

5. 30.125 inches

6. 30.5 inches

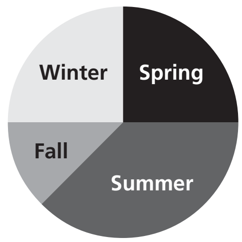
Use the information from the table to answer questions 2-3.

2. Create a bar graph from the Birthday Data Table.
3. Create a line graph from the Birthday Data Table.

Birthday Data Table

Month	Birthdays
January	2
February	3
March	0
April	3
May	5
June	2
July	4
August	3
September	2
October	3
November	0
December	3

4. Karla polled her homeroom class and asked them what their favorite time of year is, Winter, Spring, Summer or Fall. She compiled the data into a pie chart.



a. What was the class' favorite season?

b. Poll your own class, and ask them what their favorite month is. Use the

make your own pie chart to display the information.

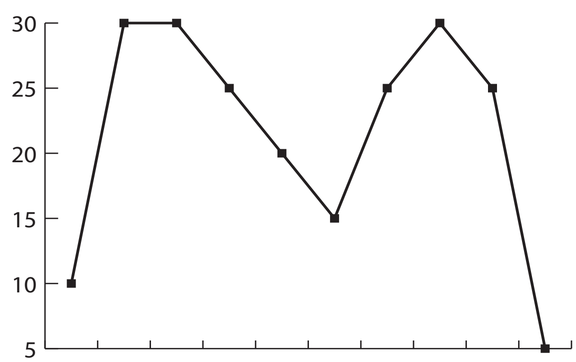
Use the data from the table of Yearly Rainfall Totals to answer questions 5-7.

Yearly Rainfall Totals

Year	Rainfall
1990	30 inches
1991	32 inches
1992	24 inches
1993	18 inches
1994	28 inches
1995	36 inches
1996	42 inches
1997	31 inches

5. Find the mean of the yearly rainfalls for the years recorded.

6. What is the median rainfall amount of the years recorded?



7 a. 25, 30 b. 25 c. 21.5 cars

8\. a. January $= \frac{2}{30} = \frac{1}{15} = 6\frac{2}{3}\%$ or 6.67%; May $= \frac{5}{30} = \frac{1}{6} = 16\frac{2}{3}\%$ or 16.67%

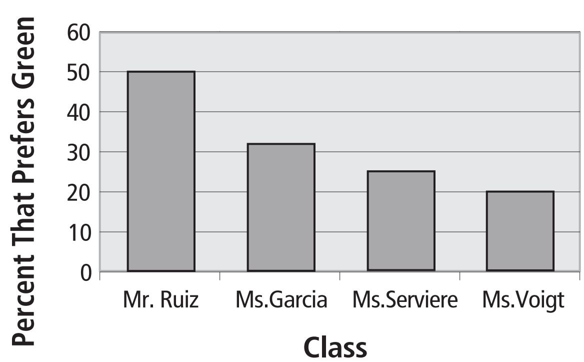
b. Because $\frac{5}{30} = \frac{1}{6}$, we take $(360)(\frac{1}{6}) = 60$ degrees.

c. $\frac{9}{30} = \frac{3}{10} = 0.3 = 30\%$

9 a. February column is taller than the January column

b. answers will vary

10\. A pie chart is inappropriate for this data because the percentages are from different classes, so the "whole" is different for each class.



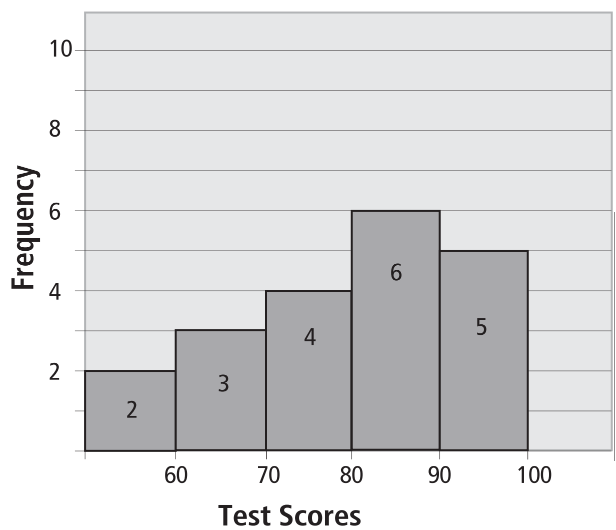
7. The parking attendant recorded data on how many cars are in the parking lot at school at any given time during the school day from 8a.m. to 5p.m. Use the given line graph to find out
   a. What is the mode number of cars in the parking lot?
   b. What is the range number of cars in the parking lot?
   c. What is the mean number of cars in the parking lot?

Use the Birthday Data Table to answer questions 8-9

8. Create a pie chart using the Birthday Data Table to answer the following questions.
   a. What percent of the students were born in January? May?
   b. What angle corresponds to May birthdays?
   c. What percent of the students were born in the summer months of June, July and August?
9. Use the bar graph you created in exercise 2 to answer the following questions.
   a. What is the relation between the heights of the January column and the February column?
   b. Which graph do you prefer for this data, bar or circle graph? Explain.
10. A pie chart cannot be made in all cases where data is presented in percentages. All four seventh grade teachers in a school asked their students which color they liked best. The table shows the results for what percentage in each class liked green best. Make a bar graph. Explain why you couldn't make a pie chart for the data in the table.

Class	Percent that prefer green
Ms. Garcia	32%
Mr. Ruiz	50%
Ms. Serviere	25%
Ms. Voigt	20%

12. Histogram for Ms. Lowrance's class

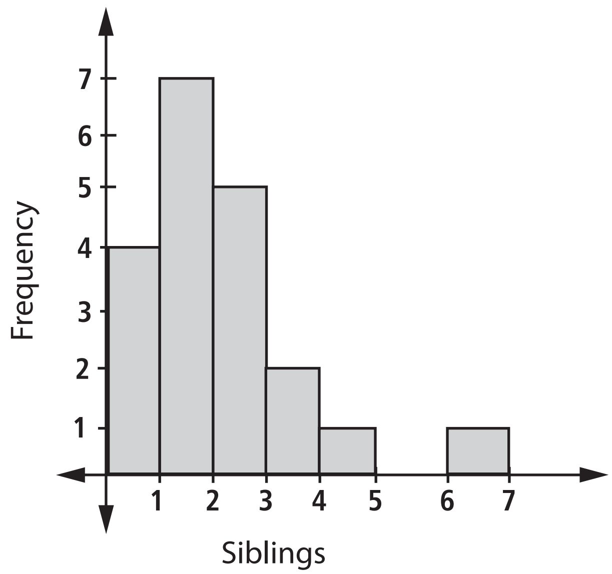


Mean: 80 **Median: 82.5** **Mode: 84**

Sprial Review

14. a has greater area. Its area is 36 square units. The area of b is 30 square units.

11. The histogram below represents the number of brothers and sisters that students in Ms. Gonzalez's class have.



Summarize the information given by the histogram. Include the range, mode, mean, and median, if possible. If not, explain why why you can't.

12. The test results in Ms. Lowrance's class were as follows: 55, 59, 63, 65, 67, 73, 75, 77, 77, 81, 84, 84, 84, 87, 88, 90, 92, 95, 97, 99. Group the test scores by 10s (for example 55 and 59 make up the 50's group) and create a histogram that shows how many students got the particular grade. Summarize the test results by finding the mean, median, mode, and range of this data.

13. Observations are made of the number of cars passing through a fast food drive through as follows:

8 am up til 10 am	8
10 am up til noon	15
noon up til 2 pm	13
2 pm up til 4 pm	4
4 pm up til 6 pm	9
6 pm up til 8 pm	18

15. **Ingenuity**

We know that the sum of the angles around the center of a circle must be 360 degrees, so each student should be represented by 360 [divided by] 30 = 12 degrees of the pie graph. So the region corresponding to the students who prefer math should be 12 x 10 = 120 degrees, the region corresponding to the students who like science should be 12 x 9 = 108 degrees, the region corresponding to the students who like language arts should be 12 x 7 = 84 degrees, and the region corresponding to the students who like history should be 12 x 4 = 48 degrees.

Investigation

16. Answers will vary. A line graph does not make sense because the order of the data does not matter.

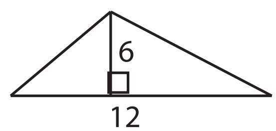
8 pm up til 10 pm	3

Use this data set to create two histograms, one with the frequency of cars and the other with percent of the cars in each time interval. Summarize observations that you made using the histogram.

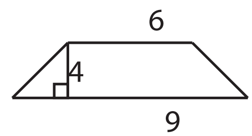
Spiral Review:

14. Which figure has a greater area?

a.



b.



15. **Ingenuity:**

Yuri polls his 30 classmates to find out what their favorite school subjects are. 10 of his classmates like math best, 9 like science best, 7 like language arts best, and 4 like history best. If Yuri were to create a pie graph to show the results of his poll, what should be the central angle of each part of the pie graph? (The central angle of a region of a pie graph is the angle of the region that is at the center of the circle.)

16. **Investigation:**

Conduct a survey of 20 people on a question of your choice that has three possible answers. Collect your data into a table, and display the results with a bar graph, a line graph, if possible, and a pie graph. Which representation seems most appropriate for your survey? Why?

Section 10.3 - Probability

Big Idea:

Exploring probability of events using sample spaces, and simple and compound experiments

Key Objectives:

- Understand events and sample spaces in an experimental setting.
- Understand the difference between a simple and a compound experiment.
- Know how to compute the probability of an event from the data.
- Understand the difference between experimental and theoretical probability.
- Learn how to display data in a compound experiment.
- Use probability in daily situations.

Materials:

Two different colored dice, coins for flipping, spinner, two different coins or two different colored chips

Pedagogical/Orchestration:

This section lays the groundwork for later compound events. It defines a simple experiment with its possible outcomes or events that make up its sample space. Our examples often suggest each outcome is equally likely, though depending on the experiment and the sample space, each outcome may not be equally likely. As an example: Roll two dice and find the sum of the two rolls. The sample space consists of {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Note that the outcomes are not all equally lkely. After many rolls, 7 is far more likely an outcome than 2. In order to make this idea clearer, Probability is defined mathematically. There is also a discussion of theoretical versus empirical (or experimental) probability. Finally, the discussion shows some of the most efficient ways to display compound event, like rolling two number cubes, including tree diagrams and data tables.

Activities:

"Drawing to Win", "Eating Lunch" and "Waking Up Early"

Internet Resources:

Battleship Game: Probability

http://www.quia.com/ba/51817.html

Vocabulary:

experiment, outcomes, sample space, simple experiment, compound experiment, probability, theoretical probability, empirical (experimental) probability, equally likely

TEKS:

6.3(C); 6.9(A,B); 6.11(D); 6.12(A); 6.13(A,B)

New: 6.4(B); 7.6(A, C); 7.7(A, B); 7.8(A, B, D, E)

Launch for Section 10.3:

Lesson 10.3 is an introduction to probability. Tell your students, "Today we will be playing a probability game that involves flipping two coins." The students need to get into groups of three, and each group will be handed two distinctive coins, such as a penny and a nickel. Player A gets a point if the two coins land on Heads when flipped, Player B gets a point if the two coins land on Tails when flipped, and Player C gets a point if one coin lands on Heads and the other on Tails. They are to flip the coins 20 times each. Each person in the group has a role. Two can be assigned to flipping the coins and the third person can record the results of the flips, also known as the outcomes. Once the game is concluded, have groups share their results. It is likely that Player C won in all groups, so ask your student why they think Player C dominated. Once students have given their opinions, tell them they will be examining the outcomes from each group. Ask each group the following information and record it on the board or chart paper: "How many outcomes were two heads? How many outcomes were two tails? How many outcomes were one of each?" Add up the totals of each outcome for the class, and tell them these are the results of the experiment that they have just conducted and they will be comparing these "empirical" outcomes to all the possible outcomes known as the "theoretical" outcomes. Tell them this list of all the possible theoretical outcomes is also known as the sample space. Ask them to make the sample space for the data. It helps if they are to specify the coins as Coin A and Coin B in a table such as this:

Coin A	Coin B
H	H
T	T
H	T
T	H

Students should recognize that the sample space signifies why Player C dominated the game, and that the game as set up was not a fair game. Look at the empirical outcomes again and see if they are similar to the sample space. Tell your students that today they will be learning more about probability and how it allows us to make educated guesses about what might happen in the future.

Drawing Blocks!

Objective: Students will discover that the ratio of the number of favorable outcomes to the number of possible outcomes is equal to the theoretical probability of the outcome. Students will understand the relationship between experimental probability and theoretical probability.

Materials:
25 Colored blocks (10 red, 12 yellow, 3 blue, note that each block should be the same size and shape), bucket or container to hold the blocks (bucket should not be transparent).

Activity Instructions:
Teacher fills each container with the same number of blocks as specified above for groups of 3 to 4 students. Without looking, each student draws a block from the container. Repeat the process 20 times. *Note that the students should not know the number of each of of the colored blocks in the container. The students should be informed the different color blocks in the container. After they draw, students will return their block to the container. Students will keep a tally of the results and record the color chosen. At the end, the class will compare and discuss their results to complete (Part 1), Experimental Probabilities as shown below:

Part I: Experimental Probability

$$P(\text{red}) = \frac{\text{total red blocks drawn}}{\text{total number of trials}}$$

$$P(\text{blue}) = \frac{\text{total blue blocks drawn}}{\text{total number of trials}}$$

$$P(\text{yellow}) = \frac{\text{total yellow blocks drawn}}{\text{total number of trials}}$$

After all the Experimental Probabilities are found, the teacher should reveal all of the blocks that were in the container. The class will use this new knowledge to find the Theoretical Probabilities (Part II), as shown below.

Part II: Theoretical Probability

$$P(\text{red}) = \frac{\text{total red blocks in the container}}{\text{total number of blocks}}$$

$$P(\text{blue}) = \frac{\text{total blue blocks in the container}}{\text{total number of blocks}}$$

$$P(\text{yellow}) = \frac{\text{total yellow blocks in the container}}{\text{total number of blocks}}$$

Eating Lunch

Objective: Students will flip a coin to collect data and find experimental probabilities of events.

Materials:
Coin
Chart or table for data collected

Activity Instructions:
Teacher will give students the following situation:

"Fernando always eats lunch at school with his girlfriend Terry. His favorite food is pizza, but his girlfriend is encouraging him to eat healthier. So every day, he flips a coin. If the coin falls on heads, he eats pizza and if it falls on tails, he eats a healthy salad".

Students will conduct an experiment to predict how many days in April Fernando is going to eat pizza. They will record their data in a table for each day in April, one toss per day. Below is a sample of how their table should look.

Coin Toss Data

Day	Result of Toss (H or T)	Number of Heads so Far	Fraction of Heads so Far	Percent of Heads so Far
1				
2				

EXTENSION: Teacher will ask students to explain what happens to the percent of tosses that are heads as we add more data. To do this, the class will gather data from all the groups in class and answer the following questions:
- What percent of the total number of tosses for your class is heads?
- As you add more data, what happens to the percent of tosses that are heads?
- Based on your results for April, how many times do you predict Fernando will eat pizza in May? Students will explain their reasoning.

Waking Up Early

Objective: Students will collect data by using a spinner to determine the probability of an event.

Materials:
Probability spinner
Paper clips

Activity Instructions:
Teacher will ask students: "During the summer, how early would you be willing to wake up if your parents paid you for getting up early? Your parents will decide "how early" using the spinner that is attached.

Students are given a copy of the attached spinner divided into the spaced sections labeled 8:00, 9:00, and 10:00. The teacher should ask the students which time of the morning they think will occur most often in this experiment. Students will vote for the most popular time (probably 9:00 am).

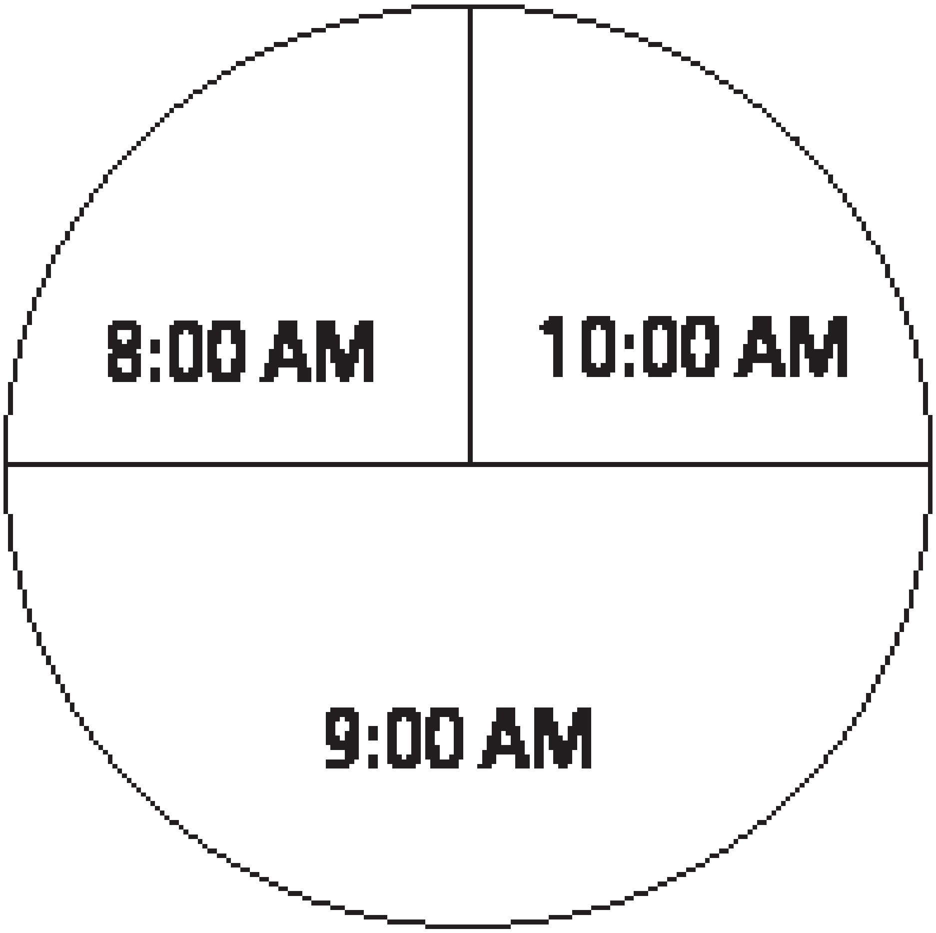
Each student uses a paper clip and spins as many time as decided by the teacher (or as time permits) to figure out the probability that the paper clip will land on 8:00, 9:00, or 10:00. Then, they will answer the following questions once they have gathered their data.

1. What is the experimental probability that the clip falls on 8:00, 9:00, and 10:00?
2. What do you think will happen if you spin the clip 100 times?

Students determine the fraction of each section of each time allotted in this spinner to answer the following questions:

3. What is the theoretical probability that the clip falls on 8:00, 9:00, and 10:00. Explain your answer.
4. Based on the theoretical probability, which time(s) is/are most favorable? Does your answer change if you spin the spinner 100 times?
5. Compare the experimental and theoretical probabilities?

Probability Spinner



Contrary to popular belief, 80% chance of rain does not mean that 80% of the area will get rain. It means that in the past, when the area had similar climatic situations, it rained four out of five times, or 80% of the time.

The number cubes are just dice. Two different colors are preferable in order to distinguish a compound event of (1, 6) and (6, 1) as a roll of 2 dice.

Not all students are familiar with the characteristics of a deck of cards. Because cards are often used as examples in probability, it may be a good idea to introduce the suits, denominations, and establish how the all will be considered (e.g. face card, number, high or low etc.)

SECTION 10.3 PROBABILITY

The study of probability allows us to make educated guesses about what might happen in the future based on past experience, to determine how likely different outcomes are. This knowledge can help us make the best choices.

You have heard people say, "There is a 50-50 chance of getting heads on a coin flip," or "There is a 80% chance of rain today." What do statements like these mean? How can we determine the chance that some event will happen?

The following activity will help us to set the stage for carefully studying probability.

Activity:

1. a. Take a six-sided number cubes and examine the numbers on all six sides. Jot these down as possibilities of what can come up in a roll.
   b. Roll the number cube 20 times. Record the number that comes up for each roll. Call this experiment: **Roll One**. Keep a careful record.
2. a. Take two six-sided number cubes of different colors such as one red number cube and one green number cube. What are the possible rolls that you could get, where a roll is noticing what you got on the red number cube and also what you got on the green number cube?
   b. Roll the two number cubes 20 times. Record the numbers that come up for each roll on a piece of paper, being very careful to record which number came up with which color number cube. Call this experiment: **Roll Two**. Did you get any rolls you thought you could get?
3. a. Take a standard deck of cards. Examine how many cards are in your deck and other distinguishing features of the deck. Describe all you notice including how many there are with the feature you observed. If you picked one card from the deck, what are all the possible selections you could get?
   b. Shuffle the cards carefully. Select a card and record its color and then return the card back to the deck and shuffle. Do this action 20 times and keep a record of the color of the card for each time

Coins:
Abraham Lincoln is on the penny
Thomas Jefferson is on the nickel
Franklin Roosevelt is on the dime
George Washington is on the quarter

you select. Call this experiment: **Pick a Card.**

4. a. Take a coin and observe the two sides. We call one side Heads and the other side Tails. Can you see why? Usually on a US coin one side of a coin has the face of a US president. This side is usually called heads. You might check to see which of the presidents are on the penny, nickel, dime, and quarter. They are the first, third, sixteenth, and thirty-second presidents.
   b. Flip a coin 20 times and record the results of your toss. Call this experiment: **Coin Toss**
5. As another experiment, flip the coin twice and record the first and second toss in that order. Repeat this 20 times. Call this experiment: **Two Flip.**
6. Take a spinner with at least 4 different possibilities. Spin 20 times and record where the spinner lands each time. Call this experiment: **Spin Once.**

We will refer back to some of these activities during this section.

We begin our study of probability with some useful and important concepts and vocabulary. An **experiment** is a repeatable action with a set of **outcomes**. For example, the experiment of flipping a coin has two possible outcomes, either heads or tails. The set of all possible outcomes of an experiment is called the **sample space**. Flipping one coin is called a **simple experiment** because only one thing happens. One important characteristic of an experiment is that it must be repeatable, with similar possible outcomes.

In studying an experiment, the question is, “What are all the possible outcomes?” If you flip a coin once, you will observe only one of the two possible outcomes, heads or tails. The key to finding any probability is to determine the likelihood of each possible outcome.

It is often helpful to use abbreviations or draw **tree diagrams** to represent the outcomes of the experiment. For example, when flipping a coin, we often write H for the outcome of getting a head and T for the outcome of getting a tail. The sample space is the set {H, T}; there are only two possible outcomes.

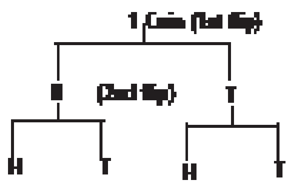
Some student might pick the sample space {all heads, one of each, all tails}. This is a possible sample space but each outcome is not equally likely because {one of each} can occur two different ways. We prefer that the sample space consist of all possible simple and equally likely outcomes.

Have students look up the many definitions of the word “compound”.

EXAMPLE 1

In the experiment, Two Flip, you flipped a coin once and observed the outcome then flipped it a second time and observed the out come. You then recorded the outcome such as: first flip head and second flip tail. What is the set of all possible outcomes? How many outcomes show no tails?

SOLUTION



The order of outcomes is important. The outcome of getting a head and then a tail, denoted by HT, is a different outcome from getting a tail and then a head, denoted by TH. This experiment has the sample space {HH, HT, TH, TT}. Notice that there are four possible outcomes. When the experiment involves flipping a coin twice, {H} is an impossible outcome. The simple event {HH} is the subset containing the outcome that both flips show heads and is the only outcome that shows no tails.

The outcome of getting at least one tail could be described as {TH, HT, TT}. There is more than one possible way that you can get a tail when tossing a coin twice. This is called a **compound experiment**. Explain why.

In both the Coin Toss and Two Flip activities that we now call experiments, each outcome in the sample spaces has the same chance of occurring as any other outcome. Each outcome is then said to be **equally likely**.

Remember the Coin Toss has sample space that we can write as {H,T}. If the coin is a fair coin, then the chance of getting an H is the same as the chance of getting a T.

In Two Flip with the four possible outcomes, {HH, HT, TH, and TT}, there is as good a chance of getting an HH as TT. You may have observed this in an earlier activity.

One way to show your class the distinction between P(A) = 0 or P(A) = 1 is to ask the following two questions. If the day of this lesson is Monday, ask, "What is the probability that tomorrow is Tuesday?" It is one, of course. It has to be Tuesday. Then ask, "What is the probability that tomorrow is Saturday?" Unfortunately, that is impossible, so P(A) = 0.

DEFINITION 10.1: EVENT, SIMPLE EVENT AND COMPOUND EVENT

An **event** is any subset of the sample space. A **simple event** is a subset of the sample space containing only one possible outcome of an experiment. A **compound event** is a subset of the sample space containing two or more outcomes.

The words simple and compound are used to describe both events and experiments. The main thing to remember is that a simple event has just one outcome while a compound event has more than one outcome listed. A simple experiment has just one action, such as pick a card, roll a number cube, or flip a coin. A compound experiment may roll two number cubes or flip a coin more than once.

Call E the event of "getting at least one head" in the Two Flip experiment introduced earlier. The outcomes HH, HT, and TH satisfy the criteria for being in E, so E = {HH, HT, TH}. Because E contains three possible outcomes, it is a compound event.

In order to study a compounded event E, a technique that is often used is to find the possible outcomes that are not in E. We call this set the **complement of E**, or E^c.

For example, in the Two Flip experiment, we have the sample space S = {HH, HT, TH, TT}. We called an event E in this experiment as "getting at least one head." This can be written as E = {HH, HT, TH}. The complement of E consists of all the outcomes in the sample space when you do not get at least one head. This leaves only TT where there are no heads. We then write E^c = {TT}.

In Example 1, the probability of getting no tails is $\frac{1}{4}$, because 1 of the 4 equally likely outcomes shows no tails. The probability of getting at least one tail is $\frac{3}{4}$.

DEFINITION 10.2: PROBABILITY

In an experiment in which each outcome is equally likely, the **probability of an event A**, written **P(A)**, is $\frac{m}{n}$, where m is the number of favorable outcomes and n is the total number of outcomes in the sample space S.

Empirical is a big word that means what we experience. Empirical knowledge is knowledge we gain through our experience.
Sometimes in middle school texts and materials, empirical probability is called experimental probability.

To simulate 6 experiments that involve flipping a fair coin 50 times, run the FLIP 50 program, from the CD, on a TI-83

Notice that the probability of an event from an experiment is always a number between 0 and 1. Explain why this is true, using the Coin Toss example if necessary. Explain why $P(S) = 1$, where S is the sample space for an experiment.

EXAMPLE 2

Consider the experiment of rolling one number cube.

a. What is the probability of getting a 3?
b. What is the probability of not getting a 3?
c. What is the probability of getting an even number?
d. What is the probability of getting a number greater than 4?

SOLUTION

Identify the sample space, S = {1, 2, 3, 4, 5, 6}, for the experiment of rolling one number cube.

a. Let A = the event of rolling a 3 in this experiment. Written as a subset, we have, A = {3}. The probability of A is the fraction with the numerator equal to 1, the number of outcomes in A and the denomina -tor equal to 6, the number of total outcomes in the sample space S. Therefore, $P(A) = \frac{1}{6}$.

b. Let B = the event of not rolling a 3 in this experiment. Written as a subset, we have B = {1,2,4,5,6}. Notice that B is the complement of event A because the outcomes in B are "not 3." The probability of not getting a 3 = $P(B) = \frac{5}{6}$. Another way to think about the probability of

"not an event " is 1 – probability of the event. A notation for this is:

$P(B) = P(A^c) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$.

c. Let T = the event of getting an even number. Written as a subset, we have T = {2,4,6}. $P(T) = \frac{3}{6} = \frac{1}{2}$.

d. Let R = the event of getting a number greater than 4. Written as a subset, we have R = {5,6}. $P(R)= \frac{2}{6} = \frac{1}{3}$.

EXAMPLE 3

Recall that this is a question from Section 12.1 without the notation.

You may wish to discuss with the students that distinguishing the two number cubes is important, for example, if we wanted to know how often the sum of two rolls is 2 as compared to the sum being 7. The students should see that there are more ways to get 7 than there are to get 1. The chart is one good way to see all the possibilities. A tree diagram is also a good way to show that there are 36 different outcomes. The students should notice that in the table, (4,5) and (5,4) are being distinguished because the dice are of two colors. This table could work if one rolled the same die, but twice. One then distinguishes the first roll from the second roll.

When we “consider” an experiment of rolling one number cube, we do not actually roll a number cube. Instead, we think about what could possibly happen if we rolled a number cube. This is called a thought experiment and is used in **theoretical** probability. If we really rolled a number cube and used the observed outcomes as in the first activity, that would be an example of **empirical (or experimental)** probability.

Gather some more empirical data for the Coin Toss by performing the experiment many times.

In most instances, when you see “experiment,” that means a thought experiment that leads to the theoretical probability of an event happening. If the experiment refers to some empirical data, then of course, the problem would most likely use the gathered information.

EXAMPLE 3

List the possible outcomes of rolling a red number cube and a green number cube. Create a table to organize your data.

SOLUTION

What is a possible outcome? You might roll a 3 and a 4. As with tossing a coin twice, the order matters. Rolling a 3 and a 4 is different from rolling a 4 and a 3. To see this more easily, think about rolling one red number cube and one green number cube. Rolling a red 4 and a green 3 is a different outcome from rolling a red 3 and a green 4.

To list all the outcomes of this thought experiment, abbreviate the outcome red 3 and green 4 as (3, 4). The sample space for the two-number cube experiment can be listed in several ways. One convenient method is to make a table like the one below

(R, G)	1	2	3	4	5	6
1	{(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)}

PROBLEM 1

A = {(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1)}, so P(A) = $\frac{11}{36}$. B has 6 outcomes and C has 6 outcomes, so P(B) = $\frac{6}{36} = \frac{1}{6}$; P(C) = $\frac{6}{36} = \frac{1}{6}$.

Flipping a coin twice or flipping two coins have equivalent sample spaces. So do rolling a number cube twice and rolling two number cubes. However, the experiments are different, just as flipping a coin is different from running a computer simulation of flipping a coin, even if the probabilities are the same. Extend the idea a bit further: it's a lot easier to keep track of an experiment in which you flip a single coin one thousand times than one in which you flip one thousand coins once each; but on the other hand, if you've flipped one thousand coins, you don't have to record the results until the end, since at the end, you can see exactly how each of the coins flipped.

1. a. 1/2 b. 1/6 c. 5/9 d. 0 e. 1/12 f. 4/9

2.. a. 3/13 b. 1/2 c. 9/13 d. 3/26

PROBLEM 1

Consider again the experiment of rolling two number cubes. What is the probability of each of the following events?

a. A = {getting at least one 6}

b. B = {getting a double (both number cubes have the same number)}

c. C = {the sum of the two number cubes is 7}

What is the difference between flipping a coin two times and flipping two coins simultaneously? What is the difference between rolling a number cube two times and rolling two number cubes? Run the thought experiments for both the coins and the number cubes to see the difference or similarities.

EXERCISES

1. In the Roll One experiment, using a standard number cube, answer the following questions.

   a. P(even number)
   b. P(number less than 2)
   c. P(6 or 1)
   d. P(0)
   e. P(prime number)
   f. P(NOT 5 or 6)

2. A standard deck of playing cards has 52 cards broken into 4 suits with 13 cards in each suit. The cards are either red or black. Using this data, answer the following probability problems.

   a. P(face cards)
   b. P(red cards)
   c. P(number cards)
   d. P(red face card)

3. a. 1/3 b. 3/4 c. 1/2 d. 0

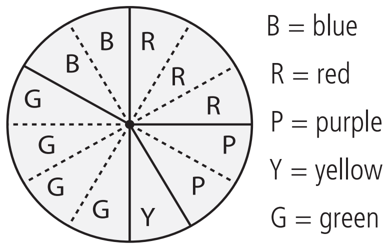
4. a. see student's work. Sample space given in part b.
   b. TTT TTF FTT FTF
      TFF TFT FFT FFF
c. 1/8

5.

(coin, die)	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

6.a Theoretical =20/80 = 1/4. Experimental = 19/80= approx 23%. So experimental < theoretical.
  b. approximately 5 green, 35 not green. Probability of not green is approximately 35/40.

3. Use the spinner below to answer the following questions.

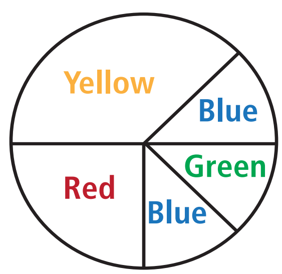


a. What is the probability of landing on green?
b. What is the probability of NOT landing on blue or yellow?
c. What is the probability of landing on purple or green?
d. What is the probability of landing on orange?

4. The last 3 questions on Mrs. Garcia's test were true or false items.
a. Create a tree diagram to show the sample space of the possible answers for the last three questions.
b. List all the possibilities from the tree diagram.
c. What is the probability of getting all three answers to be true?

5. Perform the experiment of first flipping a coin and then rolling a die. What is the sample space of this experiment? Create a table to illustrate the possible outcomes. How many outcomes does it have?

6. Use the spinner to answer the following questions.



a. Israel conducted an experiment of spinning 80 times. The spinner landed on blue 19 times. What is its predicted theoretical probability? Was the experimental probability greater, less than or equal to the theoretical probability of this spinner?
b. Iliana spins 40 times. Predict how many times Iliana's spinner lands on green in 40 spins? Predict how many times Iliana's spinner will not land on green. What is the probability that the spinner will NOT land on the green section of the spinner?

7. a. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
   b. 1/2 c. 2/5 d. 1/2
   e. 4/10 + 5/10 = 9/10. The probability is not 1 because the number 1 is neither prime nor composite.

8. Spiral Review (6.8A) 57 inches

9. Spiral Review (6.8A) $32.50

Ingenuity

10. a. It may seem that the probability is 1/8, since there are eight tables, and once Cody has "chosen" a table, we want Edward to choose the same table. However, there is a subtle problem with this reasoning. Once Cody has been assigned to a seat, there are 7 other seats at that same table, and 63 other seats overall. Therefore, the probability that Edward will get one of the 7 other seats at Cody's table is 7/63 = 1/9.

    b. We know that once Cody has been assigned to a seat, there are 7 other seats at that table. We are assuming that Edward is sitting in one of them. Of these 7 seats, 2 are next to Cody, so the probability that Edward is sitting next to Cody is 2/7.

    c. Suppose that Cody has already been assigned to a seat. This time, there are several possibilities that will lead to the three sitting in consecutive seats.

    We could have Edward sitting immediately to Cody's right, and Sam sitting immediately to the right of Edward.

    We could have Sam sitting immediately to Cody's right, and Edward sitting immediately to the right of Sam.

    We could have Edward sitting immediately to Cody's left, and Sam sitting immediately to the left of Edward.

    We could have Sam sitting immediately to Cody's left, and Edward sitting immediately to the left of Sam.

    We could have Edward sitting immediately to Cody's right, and Sam sitting immediately to Cody's left.

    We could have Sam sitting immediately to Cody's right, and Edward sitting immediately to Cody's left.

    All of these possibilities are mutually exclusive (no two of these scenarios can happen at the same time), and the probability of each is 1/7 x 1/6 = 1/42. This is because once Cody has been seated, we want Edward (or Sam) to occupy a particular one out of seven remaining positions at the table, and then we want Sam (or Edward) to occupy a particular one out of six remaining positions.

    Since there are six mutually exclusive possibilities, each with a probability of 1/42, the probability that the three friends sit together is 6 x 1/42 = 1/7.

7. Kassandra draws one card from a set of cards numbered 1 through 10.
   a. Write the sample space for this experiment.
   b. What is the probability of getting an even number?
   c. What is the probability of getting a prime number?
   d. What is the probability of getting a composite number?
   e. Add the results of parts c) and d). What do you notice? Explain your observation.

Spiral Review:

8. A circular serving plate has a diameter of 18 inches. Find the circumference of the plate to the nearest inch.

9. Tickets for the student play at Abell Junior High cost $8.75 for adults and $5.00 for students. If a mom, dad, and 3 students attend the play, what is the cost for the family to watch the play?

10. **Ingenuity:**

    Cody and Edward are at a banquet. At the banquet, seats are assigned randomly, with each guest assigned to one of eight round tables, each of which has eight seats. Each guest receives a card telling him which table and which seat is his.

    a. What is the probability that Cody and Edward are assigned to the same table?
    b. Assuming that Cody and Edward end up at the same table, what is the probability that they are seated next to each other?
    c. Suppose that Cody, Edward, and their friend Sam are all seated at the same table. What is the probability that they occupy three consecutive seats?

11. Investigation

a. The red number cube has a 2/6 = 1/3 chance of coming up 1, a 1/3 chance of coming up 6, and a 1/3 chance of coming up 8. Similarly, the green number cube has a 1/3 chance of coming up 2, a 1/3 chance of coming up 4, and a 1/3 chance of coming up 9. We can make a table of the possible outcomes of rolling the two number cubes, and record the winner in each case:

Belle / Anna	2	4	9
1	Belle	Belle	Belle
6	Anna	Anna	Belle
8	Anna	Anna	Belle

Since each outcome is equally likely, Anna has a 4/9 chance of winning, and Belle has a 5/9 chance of winning.

b. We can make a table similar to the one we made in part (a):

Belle / Anna	3	5	7
1	Belle	Belle	Belle
6	Anna	Anna	Belle
8	Anna	Anna	Anna

This time, Anna has a 5/9 chance of winning, and Belle has a 4/9 chance of winning.

c. We make one more table:

Belle / Anna	3	5	7
2	Belle	Belle	Belle
4	Anna	Belle	Belle
9	Anna	Anna	Anna

This time, Anna has a 4/9 chance of winning, and Belle has a 5/9 chance of winning.

d. Belle should consider that the green number cube has a slight advantage in a face-to-face match against red, blue has a slight advantage in a face-to-face match against green, and red has a slight advantage in a face-to-face match against blue. She should wait for Anna to choose a number cube and choose her own number cube accordingly. These three number cubes are an example of an interesting mathematical phenomenon called "nontransitive number cubes"; they demonstrate that if a strategy A has an advantage over strategy B, and strategy B has an advantage over strategy C, it does not follow that strategy A has an advantage over strategy C.

Challenges:

12. 6; 36; $\frac{1}{6}$

11. **Investigation:**

Anna and Belle play a number cubes game involving three different number cubes. The three number cubes are colored red, green, and blue, and each number cube has a number on each side. The numbers on the number cube are as follows:

Red number cube: Two sides labeled 1, two sides labeled 6, two sides labeled 8

Green number cube: Two sides labeled 2, two sides labeled 4, two sides labeled 9

Blue number cube: Two sides labeled 3, two sides labeled 5, two sides labeled 7

In order to play the game, each player chooses a number cube and rolls it once. The player who gets the higher number wins.

a. Suppose Anna chooses the red number cube, and Belle chooses the green number cube. Which player has a better probability of winning? What is the probability that player will win?

b. Suppose Anna chooses the red number cube, and Belle chooses the blue number cube. Which player has a better probability of winning? What is the probability that player will win?

c. Suppose Anna chooses the green number cube, and Belle chooses the blue number cube. Which player has a better probability of winning? What is the probability that player will win?

d. Suppose Anna chooses a number cube first, and then Belle is allowed to choose her number cube from the two remaining number cubes. Explain what Belle should do in order to have the best possible chance of winning.

12. **Challenge:**

Arthur rolls two standard six-sided number cube then multiplies the two numbers that he gets. How many ways are there for the result to be prime? How many possible outcomes are there (prime or not)? Compute the probability (i.e. ratio of desired outcomes to all outcomes) that the result is prime.

Section 10.4 - Independent Events

Big Idea:

Understanding probability of independent events

Key Objectives:

- Understand what makes an event independent.
- Use tree diagrams to represent the sample space and study compound (composite) events.
- Learn the Rule of Product and when it applies.
- Learn the Rule of Sum and when it applies.
- Solve probability problems involving compound events.

Materials:

Decks of cards, Number cubes, Coins, Paper bags, Index cards for Launch

Pedagogical/Orchestration:

This section combines the elements of probability to understand independent events. It explains when to sum and when to multiply and why the two operations are appropriate at different times. It is a very active, experimental section.

Vocabulary:

independent, root, mutually exclusive

TEKS:

6.9(A,B); 6.13(A,B)

New: 6.4(B); 7.6(A, C); 7.7(A, B); 7.8(A, B, D, E)

Launch for Section 10.4:

Give each group two paper bags: Bag X includes cards 1 through 4 and Bag Y includes cards A, B, and C. Show your students that you are selecting a number and a letter from the bag and recording them on a piece of paper. Do not show them which letter and number you picked, and return your selections to the bags. Make a show of hiding the paper in a place inaccessible to the students, and tell students there will be a prize for the group that ended up with that outcome the most times. Tell your students they will be conducting an experiment and one student from each group will need to record the outcomes. Have them blindly draw a card from each bag, record the outcomes and replace the cards in the bags. They are to repeat this process 20 times. Once they have completed the experiment, ask students if they thought all of the outcomes were equally likely? At this time, take the paper from its hiding place and give the prize to the group that matched your outcome the most times. Tell them that today they will be figuring out the sample space for this experiment later on in the lesson and will compare it to their experimental (empirical) outcomes

EXAMPLE 1

Have your class conduct a thought experiment like this, either as a class or in groups. This experiment is a combination of two independent experiments: (1) choose the number and (2) choose the letter.

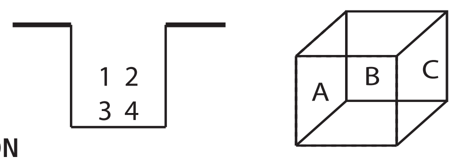
Because the table is a 4 by 3 table, the number of outcomes is $4 \cdot 3 = 12$. We will come back to this in a bit.

SECTION 10.4 INDEPENDENT EVENTS

One of the major goals of mathematics is to find simple underlying ideas to explain how and why things work. To do this, mathematicians often analyze problems by breaking them into simpler steps.

EXAMPLE 1

Suppose you have a hat and a box. The hat contains the numbers 1, 2, 3, and 4, and the box contains the letters A, B and C. Imagine the following experiment: Without looking, reach into the hat and pull out one number, and then reach into the box and without looking, pull out one letter. What is a possible outcome? How can you represent all the possible outcomes? How many possible outcomes are there?



SOLUTION

One possible outcome is selecting a 2 from the hat and the letter B from the box. We could write this combined outcome of a hat choice that has 4 possible outcomes, {1,2,3,4} and a box choice of {A,B,C} as {2,B} or 2B for short. Always try to write the sample space with some sort of order, if possible, so you do not miss a possible outcome. In this experiment, the outcomes can be listed as

$$S = \{1A, 1B, 1C, 2A, 2B, 2C, 3A, 3B, 3C, 4A, 4B, 4C\}.$$

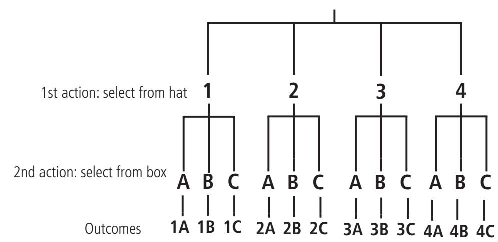
The outcomes can also be listed in a table:

box hat	A	B	C
1	1A	1B	1C
2	2A	2B	2C
3	3A	3B	3C
4	4A	4B	4C

Sample space ; {1A, 1B, 1C, 2A, 2B, 2C, 3A, 3B, 3C, 4A, 4B, 4C}

How many outcomes are in the sample space? How does the arrangement in the table help to count the number of outcomes?

A tree diagram is another method to visually represent the sample space:



The first action in the experiment of choosing from {1,2,3,4} is depicted by the four branches in the tree. Each branch splits into three smaller branches, representing the second action in this experiment, choosing from {A, B, C}. There are 12 branch endings at the bottom. To obtain the 12 outcomes in the combined experiment, follow the branches from top to bottom, reading the labels to obtain the 12 outcomes in the combined experiment listed in the table.

The selected number has no effect on the letter picked, so the two events of choosing a number and choosing a letter are said to be independent. When two events are independent, the two actions can occur either in succession or simultaneously, it doesn't matter which. Using the table model, the 4 rows represent the 4 choices in the hat. The choices correspond to the first 4 branches in the tree model. Each row has 3 columns that represent the 3 choices in the box. The rows correspond to the second level of branching. For each of the first 4 choices, there are 3 second choices. So to count the number of members in the array, or the number in the sample space, compute the sum of 4 rows with 3 outcomes in each row: $3 + 3 + 3 + 3$. This is the same as the area model for multiplication. The total number of outcomes is $4 \cdot 3$.

The process discussed above can be summarized in a formal rule in counting called the Rule of Product.

THEOREM 10.1: THE RULE OF PRODUCT

If one action can be performed in m ways and a second independent action can be performed in n ways, then there are $m \cdot n$ possible ways to perform both the first action and the second action.

PROBLEM 1

a. see students' diagrams

b. Sample Space = {(soccer, game), (soccer, sports), (soccer, comics), (basketball, game), (basketball, sports), (basketball, comics), (football, game), (football, sports), (football, comics)}

You can assume this is the same as putting all seven elements together in one container. This makes choosing each element equally likely.

Go over deck of cards: 52 cards, 4 suits (hearts, diamonds, clubs, spades) with 13 cards each, each suit has 3 face cards (Jack, Queen, King), one Ace, and the numbers 2-9. Diamonds and hearts are red. Clubs and spades are black.

The idea of equally likely outcomes is very important. When we roll a die, we are assuming that the outcome of 1 is the same as 2, 3, 4, 5, or 6. We say that the probability of rolling a 1 is $\frac{1}{6}$. Similarly, the probability of rolling a 2 or a 3 or a 4 or a 5 or a 6 is each $\frac{1}{6}$. In flipping a coin, heads is as likely to come up as tails so we say the probability of a H, or P(H) = $\frac{1}{2}$ and P(T) = $\frac{1}{2}$.

When we say that two actions are mutually exclusive, we mean that the two events cannot occur at the same time. For example, in a roll of a die, the sample space is {1, 2, 3, 4, 5, 6}. Consider the event of getting an even roll, {2, 4, 6}, and the event of rolling an odd, {1, 3, 5}. The two events are mutually exclusive because you cannot get a roll that is both even and odd at the same time.

PROBLEM 1

Marty was going to buy a ball and a magazine. He had to choose between a soccer ball, a basketball, or a football. He could choose a game magazine, sports magazine, or comics magazine.

a. Construct a tree diagram to show the sample space of all the possible outcomes.

b. List all the possible outcomes

EXAMPLE 2

Suppose, with the same hat of numbers and box of letters, we change the experiment to the following: Without looking, either reach into the hat and pull out one number, or reach into the box and, without looking, pull out one letter. How many possible outcomes are there for this experiment?

SOLUTION

This is the same as choosing one item from the combined sample space {1, 2, 3, 4, A, B, C}. The combined sample space has seven elements, or possibilities.

The main difference between the earlier situation and this one lies in the change of one word. In the first example we chose from the hat *and* the box, while in the second we chose from the hat *or* the box. This illustrates the importance in mathematics of reading words carefully, especially words like "**and**" and "**or.**" The following rule captures the number of ways to perform one action "or" another:

THEOREM 10.2: THE RULE OF SUM
If one action can be performed in m ways and a second action can be performed in n ways, then there are $(m + n)$ ways to perform one action or the other, but not both.

Discuss distinction between "or" and "and." One example: Suppose a reward of getting a ticket depends on:

1. clean room and earn an A in math
2. clean room or earn an A in math.

What is the difference in the two situations? Explain with the students.

PROBLEM 2

a. 5/26 b. 3/26 c. 4/13

PROBLEM 3

a. $\frac{2}{12} = \frac{1}{6}$.

b. $\frac{6}{12} = \frac{1}{2}$.

c. This is the complement of the event in part b. $1 - \frac{1}{2} = \frac{1}{2}$.

d. This is the complement of the event of drawing neither an odd number nor a B. $1 - \frac{2}{4} \cdot \frac{2}{3} = 1 - \frac{4}{12} = \frac{8}{12}$ $= \frac{2}{3}$.

EXERCISES

1. a. 9 possible outcomes: {(Molly swim, Billy beach), (Molly swim, Billy zoo), (Molly swim, Billy carnival), (Molly movie, Billy beach), (Molly movie, Billy carnival), (Molly movie, Billy zoo), (Molly golf, Billy swim), (Molly golf, Billy beach), (Molly golf, Billy zoo)}
   b. 6 possible outcomes: {swimming, beach, zoo, carnival, movie, golf}

PROBLEM 2

George will win if he draws either a red even number or a red face card from a standard deck of cards. Find the following favorable outcomes:

a. even red cards

b. red face cards

c. possible outcomes that will allow him to win.

The rule of sum and the rule of product provide a powerful way to examine experiments made up of several actions. By carefully using the rule of sum and the rule of product, it can often be far easier to compute the number of possible outcomes in such a compound experiment.

PROBLEM 3

Recall the experiment in example 1: A hat contains the numbers 1,2,3, and 4. A box contains the letters A, B, and C. The experiment is to pick a number from the hat and then pick a letter from the box. Use the sample space to compute the following probabilities:

a. The event of drawing an even number and an A.

b. The event of drawing neither a 1 nor an A.

c. The event of drawing a 1 or an A.

d. The event of drawing an odd number or a B.

EXERCISES

1. Molly and Billy are going to choose an activity to do on Saturday afternoon with their dad. Molly wants to go swimming, see a movie, or go play miniature golf. Billy wants to go to the beach, go to the zoo, or go to the carnival.

   a. If they each choose one of their own activities, what are the possible outcomes for the day? Draw and label a tree diagram representing all the possible outcomes.

   b. If Molly and Billy decide to spend the day together doing the same activity, what are the possible outcomes?

2. $(4)(3)(3) = 36$ possible outfits.
   Have students use tree diagram or chart method to construct sample space.
   Emphasize to students that it is important to be systematic so they don't skip an event.

3. a. 25 elements: { JJ, JK, JL, JM, JN, KJ, KK, KL, KM, KN, LJ, LK, LL, LM, LN, MJ, MK, ML, MM, MN, NJ, NK, NM, NN}
   b. 5/25 or 1/5
   c. 15/25 or 3/5

4. a. $(4)(4)(4) = 64$. Some of these words are also English words: e.g. mat, ham, hat. Other words are not English: e.g. thm, hma, aht. Discuss the merits of the following methods: make a tree, a table (which can be tricky with more than two categories), or just use the rule of product.
   b. The number of words that start with m = $(1)(4)(4) = 16$. The probability = $\frac{16}{64} = \frac{1}{4}$.
   c. The number of words that do not have an m = $(3)(3)(3) = 27$. The probability = $\frac{27}{64}$.
   d. This is the complement of the probability in part c. $1 - \frac{27}{64} = \frac{37}{64}$.

5. 18 elements in sample space. {thin small pep, thin small ssg, thin small sup, thin med pep, thin med ssg, thin med sup, thin lg pep, thin lg ssg, thin lg sup, pan small pep, pan small, ssg, pan small sup, pan med pep, pan med ssg, pan med sup, pan lg pep, pan lg ssg, pan lg sup}

6. a. 2/3 b. 1/3

7. a. 4 b. 16 c. 64 d. 256 e. 1024

2. Jennifer bought some new clothes for her summer wardrobe. She bought 4 shirts, 3 pairs of shorts and 3 pairs of sandals. How many different outfits can she make with her new clothes? (An outfit is different from another outfit if just one of the articles of clothing is different.) Make an organized list to show the sample space.

3. Use the letters J, K, L, M and N, to make passwords.
   a. Make a list of all two-letter passwords you can make?
   b. If you pick one of the new passwords at random, what is the probability that it starts with the letter J?
   c. If you pick one of the new passwords at random, what is the probability that it has a M or N?

4. You have only the following words to work with: {m, a, t, h}?
   a. How many 3-letter passwords can be made from the four letters?
   b. What is the probability that the word starts with m?
   c. What is the probability that the word does not contain m?
   d. What is the probability that the word contains at least one m?

5. You are ordering a pizza. You must decide between thin crust or pan crust. You have three choices of sizes: small, medium, or large. Your topping choices are pepperoni, sausage, or supreme. Make a tree diagram of the possible pizza combinations you can order.

6. Using the list you created, answer the following questions.
   a. What is the probability of having a pizza with no pepperoni?
   b. What is the probability of ordering a large pizza?

7. Given a four-letter alphabet, find the following:
   a. How many one-letter code words are possible?
   b. How many two-letter code words are possible?
   c. How many three-letter code words are possible?
   d. How many four-letter code words are possible?
   e. How many five-letter code words are possible?

7
f. 4^{10}

8. $6 + 4 + 26 = 42$.

9. $(6)(6)(6) = 216$

10. Only 1 is neither even nor prime, so the probability is $\frac{5}{6}$. The probability of each event is $\frac{3}{6}$, but the events are not mutually exclusive because a 2 is both even and prime.

11. $(52)(26) = 1352$

12. $4 + 12 = 16$

13. Spiral Review (6.10 C): Percentages for chart: Class A 6.25%, Class B 25%, Class C 50%, Class D 6.25%, Class E 12.5%

14. Spiral Review (6.11 A): We need to know the number of square feet each tile covers or the dimenstions of each tile.

f. Use a pattern that you observe above to determine how many ten-letter code words are possible.

8. Think about the experiment of rolling a standard die or choosing a digit or choosing a letter from the alphabet. How many outcomes are possible?

9. Consider the experiment of rolling three dice: 1 green, 1 blue and 1 red. How many outcomes are there in the sample space?

10. Think about the experiment of rolling a standard die. What is the probability of getting an even number or a prime number when you roll the die? Is this answer different than you might expect? Explain your reasoning.

11. Kristen draws a card from a standard 52-card deck and then selects a letter from the alphabet. How many outcomes are possible?

12. Chuck draws one card from a standard 52-card deck. How many ways can he draw a 10 or a face card?

Spiral Review

13. Ms. Tate recorded the time she spent observing students in the classroom at Emerson Elementary. The results are shown in the table below. Draw a circle graph representing the percentage of time Ms. Tate spent observing in each classroom.

Classroom	Time (hours)
A	$\frac{1}{2}$
B	2
C	4
D	$\frac{1}{2}$
E	1

14. Dr. Pascoe tiled his rectangular patio using square tiles. Each box of tile contained 25 square tiles. The patio measures 35 feet x 20 feet. What else do you need to know in order to find the number of boxes of tile Dr. Pascoe should buy?

15. **Ingenuity**

a. To choose a license plate, we need to choose five digits. Each digit can be one of ten possible digits, so the number of possible license plates is 10 x 10 x 10 x 10 x 10 = 10^5 = 100,000. (Once again, note that students may answer 10 + 10 + 10 + 10 + 10 rather than 10 x 10 x 10 x 10 x 10. This would be a good time to revisit the idea of using tree diagrams to represent multiple choices made in succession.)

b. Suppose we choose the four digits of a license plate freely. There are 10 x 10 x 10 x 10 = 10,000 ways to do this. Once we have chosen the first four digits, there is only one value for the fifth digit that will yield a good license plate. For example, if our first four digits are 1729, then we have 1 + 7 + 2 + 9 = 19, and the only fifth digit that will give us a good license plate is 1. So once we have chosen the first four digits, we are stuck using a particular digit as the fifth digit. Thus there are only 10,000 good license plates.

16. **Investigation**

The idea of this Investigation is to demonstrate that untrained humans are not very good at coming up with random binary sequences, such as sequences of heads and tails. Usually, when amateurs try to come up with random binary sequences, they do not have long runs of consecutive heads or tails (because long runs do not seem "random" enough to the human eye), and they have more "switches" (consecutive head-tail or tail-head pairs) than they should, because switching seems like a "random" thing to do.

Answers to this Investigation will vary, and some students may come up with heads-tails sequences that look very close to random ones.

15. **Ingenuity:**

In the country of Mathylvania, each license plate has five digits on it. The digits can be any of the digits from 0 to 9, and the same digit may be used more than once.

a. How many license plates are possible?

b. A license plate is said to be *good* if the sum of the five digits on the license plate is a multiple of 10. How many good license plates are possible?

16. **Investigation:**

For this Investigation, you will need a fair coin.

a. Without using the coin, write down a sequence of 50 letters, with each letter either an H or a T. Try to make your sequence as random as possible.

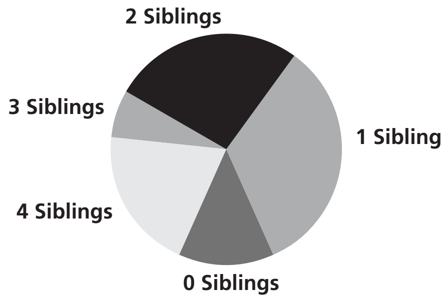
b. After you have written your sequence, toss the coin 50 times. Each time you toss the coin, keep track of the result. If you get a head, write an H; if you get a tail, write a T. Continue until you have a sequence of 50 letters written down.

c. What is the longest run (sequence of consecutive heads or consecutive tails) that you have in the sequence of letters you wrote down in part (a)? What is the longest run in the sequence you wrote down in (b)? Which sequence has the longest run?

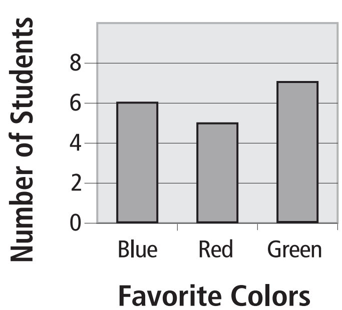
d. A "switch" in a sequence of H's and T's is a pair of consecutive letters in the sequence that are different. For example, the sequence HHTHTTTH has 4 switches: between the second and third letters, between the third and fourth letters, between the fourth and fifth letters, and between the seventh and eight letters. Count the number of switches in the sequences you made in parts (a) and (b). Which sequence has the most switches?

e. Do you notice any other differences between the sequences you wrote down in (a) and (b)?

1. Mean: 2; Median: 2; Mode: 1

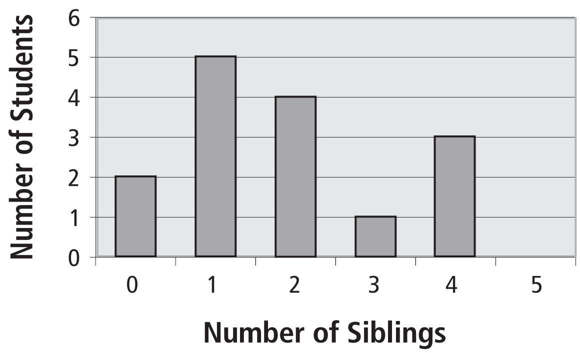


2.

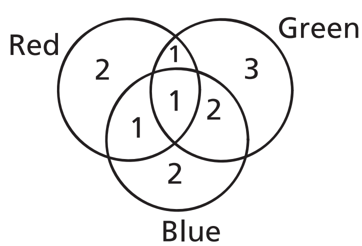


REVIEW PROBLEMS

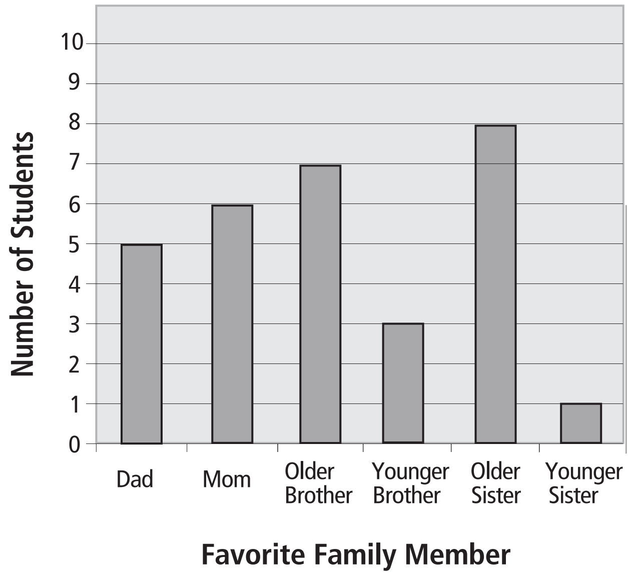
1. Ms. Murphy's class recorded the number of siblings each member of the class has in the bar graph below. Calculate the mean, median, and mode of the data. Can you do this without turning the data into tabular form? Graph the data in a pie chart.



2. Ms. Carnett's class is asked what their favorite colors are. However, many of her students can't decide and have multiple favorite colors. Their preferences are recorded in the Venn diagram. Make a bar graph that represents the number of hands that will be raised if Ms. Carnett asks her class if their favorite colors are blue, red, or green.



3.



P(Parent): $\frac{11}{30}$; P(Female): $\frac{15}{30} = \frac{1}{2}$

4. Possible outcomes = $6 \cdot 6 = 36$

   Ways to get sum of 5: (1, 4), (2, 3), (3, 2), (4, 1) = 4 so P(sum of 5) = $\frac{4}{36} = \frac{1}{9}$

   Ways to get sum of 3: (1, 2), (2, 1) = 2 so P(sum of 3) = $\frac{2}{36} = \frac{1}{18}$

5. Possible outcomes = $6 \cdot 6 = 36$

   Ways to get product of 5: (1, 5), (5, 1) = 2 so P(product of 5) = $\frac{2}{36} = \frac{1}{18}$

   Ways to get product of 12: (2, 6), (3, 4), (6, 2), (4, 3) = 4 so P(product of 12) = $\frac{4}{36} = \frac{1}{9}$

6. Possible code words = $4 \cdot 4 \cdot 4 = 64$

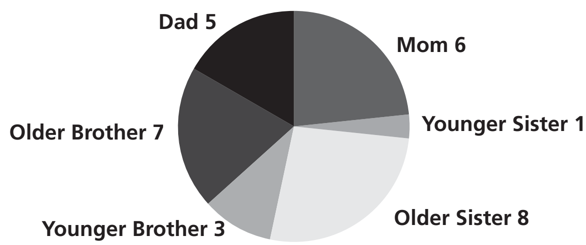
   Possible code words that start with P = $1 \cdot 4 \cdot 4 = 16$

   P(starts with P) = $\frac{16}{64} = \frac{1}{4}$

7. {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH}

   The number of possible outcomes = $2 \cdot 2 \cdot 2 = 8$

3. Ms. Gaeta asks her class who their favorite member of their immediate family is and records it in a pie chart. Graph this data as a bar graph. What's the probability that a student's favorite family member is a parent? What's the probability that the favorite family member is a female? Is it easier to figure this out from the pie chart or the bar graph?



4. If you roll two six-sided die, what is the probability that the sum of the two numbers rolled is 5? What is the probability the sum is 3?

5. If I roll two six-sided die, what is the probability that the product of the two numbers rolled is 5? What is the probability the product is 12?

6. Suppose we form 3-letter code words from the alphabet {a, o, p, t}. How many of these code words are there? How many of these code words start with p? What is the probability that one of these code words starts with p?

7. Write out the set of all the possible outcomes when you flip a coin three times. How do you know that you've listed them all?

8. a. Mean: 171.46; b. Median: 156 c. Mode: {156, 250};

8. The table shown lists 24 people and the day of the year they were born.

Day of the Year Born			
Wilby	335	Jose	301
Bence	12	Jeffrey	123
Danette	107	R.J.	260
Terry	156	David	74
Trisha	92	Jacob	103
Kristen	237	Karen	250
Vanessa	155	Maja	13
Teri	352	Jennifer	156
Wesley	274	Tiffany	293
Donovion	50	Kaitlyn	45
Ben	250	Rhonda	216
Andrew	43	Michelle	218

Find the following:

a. mean of the data set

b. median of the data set

c. mode of the data set

Section 11.1 – Types of Charge Cards

Big Idea:

Key Objectives:

-
-
-
-
-

Materials:

Pedagogical/Orchestration:

Activity:

Vocabulary:

TEKS:

WARM-UPS for Section 11.1

Launch for Section 11.1:

MATH OF FINANCE 11

SECTION 11.1 TYPES OF CHARGE CARDS

As you become older, you will be making purchases that are both large and small. Most small purchases can be made using the cash that you carry in your pocket or wallet. For larger purchases you might have noticed that adults often use a plastic credit card. The store clerk often asks a customer if the card is a charge card or a debit card. Another customer might write a check for the amount. These are some of the alternatives to paying with cash.

Let's begin by exploring the difference between a credit card and a debit card. You may use either to make a purchase, but in fact they differ in one important way:

When you use a debit card, the money is withdrawn from your bank account immediately. If you do not have enough money in your account, your purchase will usually be denied

Using a credit card is like creating a loan. You will still have to pay for the some part of the purchase at the end of the month when you receive your credit card statement. If you do not pay the full amount, you will have to pay extra charges and interest on your loan.

To summarize: The main difference between a debit card and a credit card is that a debit card is "pay now" whereas a credit card is "pay later."

While you can use either type of card, the type of card used can make a big difference not just to you but also to the store where you make your purchase.

There are also different ways that your purchase might be approved by the store. One way is called signature approval. With signature approval, you only have to sign the receipt, and your purchase is approved. The other way that a purchase might be approved is through PIN approval. PIN stands for Personal Identification Number. A PIN is usually used with a debit card. When you enter your PIN, the
If t is less than one year, the fractional part of the year represents t, as above.

merchant is assured that the card is being used by the real owner. With the PIN consumers are also assured that only they will be able to charge directly from their account.

EXPLORATION 1

Ask your parents which type of card they use the most. Why do they prefer that type of card? Are there different fees associated with each? What are these fees? Do their cards also have special bonuses to encourage them to use any of their cards?

EXPLORATION 2

Compare the different advantages of debit cards offered by two different local financial institutions. Which features are most important to you?

EXPLORATION 3

Compare the different costs of debit cards offered by these two different financial institutions. Which card seems to offer the best value?

Some of the costs you find might include monthly fees, interest rate charged, late fees, and transaction fees.

A **monthly fee** is the amount that the bank or credit card company charges each month (or year) to issue a credit card.

The **interest rate** is the rate charged if the bill is not paid in full on time. Some cards charge a different rate even if you do pay your bills on time. So be careful to find out what fees are charged to a card because each card company has different conditions.

If you pay your bill late, then you may be charged a **late fee**. Some companies may also charge a **transaction fee** each time you use your card.

Finally, there are special features for some cards, such as bonus cash for using the card, or cash back depending on the number of purchases made. Some credit cards might give bonus mileage on a Frequent Flyer Program, or free gas if their card is used to purchase gas, or even a discount at a certain store if when using their card. When you explored local financial institutions, did they offer some of these features? Explain.

Let's talk about some of the vocabulary that is used in finance.

For each bank or credit union account that you have, there are three types of transactions: deposits, withdrawals, and transfers.

Deposits add money to your account. You might make a deposit when you receive a paycheck or a gift for your birthday.

Withdrawal takes money out of your account. For example, you might make a withdrawal to take money out of your account that you need for buying clothes.

Transfers are special types of transaction, where money is moved or transferred from one account (a withdrawal) to another account (a deposit). You might make a transfer to take money out of your savings account (withdrawal) to pay a bill in your checking account (deposit). In this case, you make a transfer from your savings account to your checking account.

EXAMPLE 1

Sue opened an account on January 15 by depositing $250 into a new savings account. On January 18, she made a withdrawal of $100, and on January 24, she transferred $75 from her savings account to her checking account to pay the balance she owed in that account. What is her new savings account balance?

SOLUTION

Sue made one deposit of $250. She made one withdrawal from savings of $100. There is one transfer, which is the same as a withdrawal from savings, of $75. So her new savings account balance is: $250 - 100 - 75 = \$75$.

EXERCISES

1. What are the different types of credit cards? How are they different? Which do you prefer and why?

2. Are all debit cards the same? What are some of the differences you discovered?

3. Compare the different features offered by at least two local financial institutions on their debit cards. Then check on line to see whether you can find a debit card with more attractive features. Explain your findings.

4. Compare the different costs of least two local financial institutions' debit cards. Then check on line to see whether you can find a debit card with lower costs. Explain your findings.

5. What are some of the differences that various credit cards have?

6. Compare the different features of at least two local financial institutions' credit cards. Then check on line to see if you can find a credit card with more attractive features. Explain your findings.

7. Compare the different costs offered by at least two local financial institutions on their credit cards. Then check on the web to see if you can find a credit card with lower costs. Explain.

8. What could happen if you fail to pay your credit card bill at the end of the month?

9. Do you have to pay a debit card bill at the end of the month? Why, or why not?

10. Cathy's savings account has $255.25 at the beginning of the month. She makes three withdrawals, for $25, $75, and $43.11. She also makes one new deposit for $125.45, and transfers $85 from her savings account to her checking account. What is the balance in her savings account at the end of the month?

11. Suppose you use a plastic card to make a purchase totaling $300. Compare the different results of making the purchase if you

    i. Use a debit card that is tied to an account in which you have $400.

    ii. Use a debit card that is tied to an account in which you have $200.

    iii. Use a credit card for the purchase and pay the full amount on time when the payment is due the following month.

    iv. Use a credit card for the purchase but are unable to pay the full amount when the payment is due the following month, and the card issuer charges 1.5% interest per month on the amount owed.

12. Exploration: How do withdrawals, deposits, and transfers affect your balance in a checking account? Ask your teacher to help you find a couple of good internet sites.

13. Mickie's checking account has a $400 balance. She has a number of earnings and expenses listed below. Create a check register to record the transactions and keep a running balance of what you have at the end of each activity day. What is your final balance?

January 22	Baby sitting Job	$25
February 13	Valentine cards check #23	$15.25
March 9	Birthday from Grandmother	$50
April 1	Baby sitting Job	$20
May 26	Teacher's present check #24	$24.75

Check #	**Date**	**Transaction Description**	**$ Withdrawal**	**$ Deposit**	**$ Balance**

14. Pick a bank in your city and investigate what services they offer. Is there a charge for a checking account? How much is the charge? Do they offer any interest on the balance? Is there a minimum balance that must be maintained? Is there a charge for a credit card? How much charge? Is there a charge for a debit card? How much charge? What might be other questions you may have about this bank in comparison with another bank and their fees?

Section 11.2 - Credit Reports

Big Ideas:

Key Objectives:

-

Materials:

-

Pedagogical/Orchestration:

-

Activity:

Vocabulary:

TEKS:

WARM-UPS for Section 11.2

Launch for Section 11.2:

SECTION 11.2 CREDIT REPORTS

When you want to set up a credit card account, you are asking the bank to approve making you a loan. Any purchase made on the credit card is basically a loan from the bank that the cardholder will repay when paying the credit card bill.

That is why to approve a credit card for anyone, the bank must check the potential cardholder's financial history. The bank might ask several the following questions:

1. How much money do you earn?
2. Do you have other credit cards? What kind of balances do you have on them?
3. Have you ever declared bankruptcy? Bankruptcy is a legal term used when a person cannot repay his debts.
4. How old are you?
5. How long have you had your current job?
6. Do you own your house?
7. What is your social security number?
8. Will you approve a "credit report" of your credit history?

Each of these questions is designed to help the bank decide if a client is "credit worthy." This means they need to learn enough about any potential lender to ensure that extending credit to the cardholder will be a good investment for them.

If they approve a credit card, they are really approving making a loan of whatever credit limit is on the card. If they approve a credit limit of $2,000, then they are saying the client can use the credit card to make charges totaling $2,000.

The reason that the bank or credit union asks for a social security number is so that it may obtain a credit report. The credit report will list the different type of assets the client has, as well as detail any times loans have not repaid in a timely manner. That means every time consumers obtain a credit card, debit card, or checking account, they are building up a credit history. This credit history then becomes part of the credit report that the financial institution or lender will use

when deciding if they will approve a loan or credit card. The lender will then get the consumer's credit score from the different credit rating agencies. The credit score will enable the bank to decide if they think the customer is a good credit risk or not. So approval for the loan and the interest rate charged are determined by the credit score. Generally, the higher the credit score, the lower the interest rate. That is why it is so important to maintain a positive credit history.

According to www.consomuer.ftc.gov,

The Fair Credit Reporting Act (FCRA) requires each of the nationwide consumer reporting companies — Equifax, Experian, and TransUnion — to provide you with a free copy of your credit report, at your request, once every 12 months. The FCRA promotes the accuracy and privacy of information in the files of the nation's consumer reporting companies. The Federal Trade Commission (FTC), the nation's consumer protection agency, enforces the FCRA with respect to consumer reporting companies.

A credit report includes information on where you live, how you pay your bills, and whether you've been sued or arrested, or have filed for bankruptcy. Nationwide consumer reporting companies sell the information in your report to creditors, insurers, employers, and other businesses that use it to evaluate your applications for credit, insurance, employment, or renting a home.

Time limits for keeping negative financial information:

The three credit-reporting agencies mentioned above can report a consumer's most accurate negative information for seven years and bankruptcy information for ten years. However, there is no time limit on reporting information about crimi¬nal convictions.

Why do you need a positive credit history?

You need to pay your bills on time, otherwise your credit report might not be positive. If you have trouble paying your bills on time, the credit agencies will report that you are not as good a risk. In that case, you might have to pay a mean a higher credit score from the reporting agencies. This has several benefits.

First, a high credit score makes it easier to get a credit card or debit card. Further, it qualifies consumers for lower interest rates when they are approved. Finally, it enables people to qualify not only for credit cards but for loans like car or house.

EXERCISES

1. Why is it important to establish a positive credit history? Name three different benefits.

2. What information is described in a credit report? What actions make credit reports better?

3. How long is the information in a credit report kept by the credit agencies?

4. What is the difference between a credit history and a credit report?

5. How is a credit report of value to borrowers? What will a good credit report enable them to do that could not be done otherwise?

6. How does a lender use a credit report? Why is it of value to a lender?

7. What are great credit scores? Medium credit scores? Poor credit scores?

8. How much difference does a credit score make in the rate paid for a loan? Look up car loans and house loans on the web and report what you find.

9. Investigate what it means to talk about a person's credit history. Possible sites to visit are:
   http://www.ftc.gov/bcp/edu/microsites/freereports/index.shtml
   http://www.ftc.gov/bcp/edu/pubs/consumer/credit/cre03.shtm

Section 11.3 - Going to College

Big Ideas:

Key Objectives:

-

Materials:

-

Pedagogical/Orchestration:

-

Activity:

Vocabulary:

TEKS:

WARM-UPS for Section 11.3

Launch for Section 11.3:

SECTION 11.3 GOING TO COLLEGE

Why Go?

There are many reasons to consider going to college after high school. Preparation for careers that require a post-secondary education. For example, anyone who wants to a doctor, engineer, or school teacher, needs a college degree. And mathematics will be a huge help, both in getting into college and as a requirement for many degrees in science, technology, engineering, and mathematics (also known as STEM).

College graduates earn more. According to the College Board's 2010 Education Pays report, the average college graduate will earn on average $21,900 per year more than students with only a high school diploma. If you work for 30 years, how much difference would this make in your total earnings?

1. This same report shows that college graduates tend to be healthier, to smoke less, and to exercise more.

2. College graduates also are more satisfied with their jobs, and have greater job stability. In short, they are more likely to be satisfied with what they do, and less likely to become unemployed.

Junior College versus 4-year college?

Many students consider a junior college to start with. These are two-year programs that are typically easier to get into then many four-year programs and are less expensive annually than a four-year college. They tend to have smaller class sizes and often have a curriculum that is more closely tied to a specific degree. However, their curriculum might be limited, and a junior college degree alone probably will not be enough to qualify for many careers, particularly in STEM. However, for many students a Junior College Degree is great preparation for a four-year college and is a nice transition immediately after high school. Junior College Degrees also prepare students for many types of jobs that a high school degree does not.

EXPLORATION

Compare the annual salaries of different careers. Find at least one career that only requires a high school education, one career that requires only a Junior College Degree, one career that requires a four-year degree, and one career that requires

a more advanced degree, like a MA or PhD. Find the annual average salaries for each career, then calculate how much more a worker might make during a career. Use the internet to explore the different possibilities.

There are numerous advantages to either a two-year or four-year diploma over a high school diploma. One of the first questions that arises, however, is how much does a college education cost, and how can a student or family pay for college?

The expenses for all colleges are different. These expenses include tuition, fees, books, travel, room, and board. Individual students need to look at each potential college's cost. After that, the students and their families must decide how to get the money needed for college. Fortunately, there are many ways to get college money.

Let's talk about some options:

1. Savings

Some parents are able to save enough money to pay for their children's college education. Savings can be a huge help. Some parents establish a savings plan early for their children so that they have enough money to pay for college.

However, many students find it necessary to get more funding than available savings. So let's look at other ways to finance a college education.

2. Grants

Many colleges provide grants to help students who need financial help to attend them. There are federal grants and state grants. Check with the college of your choice to find what kind of grants are available. Your guidance counselor might also have information about grants that are independent of a particular college.

3. Scholarships

Colleges also provide scholarships. A scholarship acts a like a grant that is given to the student to pay certain expenses. There are many types of scholarships: sports, academic, needs-based, or other. A needs-based scholarship is given to a student based on financial needs. Check with each college you are considering to see what aid you might qualify for. If you have a special talent, like music, check with the music departments at different schools to see whether they have a scholarship that is dedicated to music majors. People establish many special scholarships to

encourage students to enter one particular area. There are many different types of scholarships available, and you might qualify for several different scholarships. The more scholarships you get, the better!

4. Student Loans

The advantage of grants and scholarships is that you don't have to repay the amount you receive. If you don't receive a scholarship or grant, however, you can apply for student loans. There are two types of loans: subsidized and unsubsidized.

A subsidized loan is a loan that is partially supported by someone else. For instance, the government might subsidize the loan, so there is no any interest on the loan until graduation.

An unsubsidized loan is not supported by anyone else. The lending agency probably begin charging interest from the time you receive the money.

However, each loan is different, so carefully check the conditions of the loan to decide which is best for you.

5. Work-Study

This option allows students to make money through a special program that is partially subsidized or supported by the government. In a work-study program, the students work a certain amount each week, usually 25 hours or less, to help pay for expenses. However, not all students are eligible for work-study. Qualifications depend on financial situation and university policy, so check the possibility, if needed.

EXERCISES

1. Is it possible to save up enough money to go to college? Why is it difficult for many families to save enough for college?
2. If a family doesn't have enough savings, what are other ways to pay for college?
3. Who generally provides grants to help students go to college? How does a student find out how to qualify for a grant?

4. How can a student find out about different scholarships? Can a student qualify for more than one scholarship?
5. What different types of loans can help pay for college? What is the difference between a loan and a scholarship?

6. What is work-study? How could it help pay for college? Can everyone qualify for work-study?

7. Compare the annual salaries of at least four different occupations, two of which require at least a college degree and two of which don't.

8. Compare the lifetime salaries, over a 30-year period, of the different professions you listed in problem 7.