# INFINITE SERIES: Convergence/Divergence Tests

## The Sandwich Theorem for Sequences:

If $a_n \le b_n \le c_n$ for all $n$ beyond some index $N$, and if $\lim a_n = \lim c_n = L$, then $\lim b_n = L$ also.

## The $n^{th}$ Term Test for Divergence:

If $\lim\limits_{n\to\infty} a_n \ne 0$, or if $\lim\limits_{n\to\infty} a_n$ fails to exist, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

## The Non-Decreasing Sequence Theorem:

A non-decreasing sequence converges if and only if its terms are bounded from above. If all the terms are less than or equal to $M$, then the limit of the sequence is less than or equal to $M$ as well.

## Comparison Test for Series of Non-negative Terms:

Let $\sum a_n$ be a series with no negative terms:

1. *Test for convergence*:
   The series $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > n_o$, for some positive integer $n_o$.

2. *Test for divergence*:
   The series $\sum a_n$ diverges if there is a divergent series of non-negative terms $d_n$ with $a_n \ge d_n$ for all $n > n_o$.

## Integral Test:

Let $a_n = f(n)$ where $f(x)$ is a continuous, positive, decreasing function of x for all $x \ge 1$.

Then the series $\sum a_n$ and the integral $\int_1^{\infty} f(x)dx$ both converge or both diverge.

## Limit Comparison Test:

1. *Test for convergence*:
   If $a_n \ge 0$ for $n > n_o$ and there is a convergent series $\sum c_n$ such that $c_n > 0$ and $\lim \frac{a_n}{c_n} < \infty$ then $\sum a_n$ converges.

2. *Test for divergence*:
   If $a_n \ge 0$ for $n \ge n_o$ and there is a divergent series $\sum d_n$ such that $d_n > 0$ and $\lim \frac{a_n}{d_n} > 0$, then $\sum a_n$ diverges.

## Simplified Limit Comparison Test:

If the terms of the two series $\sum a_n$ and $\sum b_n$ are positive for $n \geq n_o$, and the limit of $\frac{a_n}{b_n}$ is finite and positive, then both series converge or both series diverge.

## The Ratio Test:

Let $\sum a_n$ be a series with positive terms, and suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p$

Then (a) the series converges if $p < 1$

(b) the series diverges if $p > 1$

(c) the series may converge or it may diverge if $p = 1$

(The test provides no information.)

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq n_0$, and suppose that $\sqrt[n]{a_n} \to p$

Then (a) the series converges if $p < 1$

(b) the series diverges if $p > 1$

(c) the test is not conclusive if $p = 1$

## The $n^{th}$- Root Test:

Limit of the $n^{th}$ term of a convergent series.

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

$n^{th}$-term test for divergence:

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Examples:

1. $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$
2. $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$
3. $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist
4. $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \left( \frac{-n}{2n+5} \right) = \frac{-1}{2} \neq 0$

## The Absolute Convergence Theorem:

If $\sum_{n>1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

## How to Test a Power Series for Convergence:

1. Use the Ratio Test (or $n^{th}$ -Root Test) to find the interval where the series converges absolutely.

2. If the interval of absolute convergence is finite, test for convergence or divergence at each of the two endpoints. Use a Comparison Test, the Integral Test, or the Alternating Series Theorem, not the Ratio Test nor the $n^{th}$ –Root Test.

3. If the interval of absolute converge is $a - h < x < a + h$, the series diverges (it does not even converge conditionally) for $|x - a| > h$ because for those values of x, the $n^{th}$ term does not approach zero.