## **Problem Set 1**

## "Think Deeply of Simple Things" - Arnold E. Ross

Acknowledgment: These problem sets are based on those designed by Arnold Ross over a 40 year period for use in his ongoing program at the Ohio State University. We also borrow freely from the problem sets designed by Glenn Stevens for use in the PROMYS program at Boston University, and we have incorporated further suggestions by Daniel Shapiro. Warshauer, Shapiro and Stevens attended the Ross Program at Ohio State while in high school.

## Numerical Problems. (Some food for thought)

Mathematics, like all of the sciences, is based on experience. This summer we will explore a mathematical system which we have all experienced, namely the system of integers. The integers are the whole numbers

We denote the set of integers by the symbol Z. So  $Z = \{ \dots -3, -2, -1, 0, 1, 2, 3, \dots \}$ . As we all know, it is possible to add and multiply in Z. It is also possible to divide evenly in Z. For example, if we try to divide 1 by 2 in Z, then we seek an integer  $x \in Z$  such that 2x = 1. But there is no such integer x. Early in school, most of us learn how to do long division with remainders. This is a simple process known to mathematicians as the *division algorithm*. For example, if we divide 1068 by 7 then we get a quotient 152 with remainder 4.

$$152 = \text{quotient}$$
7 1068  
1064  
4 = remainder

This means that:  $1068 = 7 \cdot 152 + 4$ . More generally, if m is a positive integer, then we can divide any integer a by m to get a quotient q and a remainder r where r is one of the numbers 0, 1, 2, ..., m - 1. We summarize this by writing:  $a = m \cdot q + r$ , where  $0 \le r < m$ .

The division algorithm is a simple process. Nevertheless, as we shall see this summer, it has a large number of unexpectedly deep consequences and can be generalized in far-reaching ways. One of the most basic applications of the division algorithm is to our notation for integers. It is customary to use base ten notation to write the elements of Z. For example 1068 is our notation for  $1 \cdot 10^3 + 0 \cdot 10^2 + 6 \cdot 10 + 8$ . Note that the digits allowed in base ten are the numbers from 0 to 9 (the "one-digit numbers").

We could equally well use the base seven or some other base. For example, to write 1068 in base seven, we have to find a, b, c, d which satisfy:  $1068 = a \cdot 7^3 + b \cdot 7^2 + c \cdot 7 + d$ . Here a, b, c, d are "base seven digits", that is, numbers from 0 to 6. How can we find those digits?

Here is a technique. Divide 1068 by 7, then divide the quotient by 7, and that quotient by 7, etc.

This is what we get:

1068	=	$7 \cdot 152$	+4
152	=	7.21	+ 5
21	=	7.3	+0
3	=	7.0	+3

Substitution then shows that

 $1068 = 7 \cdot 152 + 4$ = 7 \cdot (7 \cdot 21 + 5) + 4 = 7 \cdot (7 \cdot 3 + 0) + 5) + 4.

Multiplying this out we get  $1068 = 3 \cdot 7^3 + 0 \cdot 7^2 + 5 \cdot 7 + 4$ . Therefore 1068 (in base ten) is equal to  $(3054)_7$  in base seven. The four "digits" in this base seven representation are the four remainders in the scheme above.

## Problems

1. Let N = 32071 in base ten. Write N to base five, to base nine, to base sixteen. In the last case, you might need to introduce new "digits" A = ten, B = eleven, C = twelve, etc.

2: Perform the following operations in base seven, (NOT changing to base ten as an intermediate step.)

- (a) add  $(6153)_7$  to  $(3455)_7$ .
- (b) subtract (2346)<sub>7</sub> from (4354)<sub>7</sub>.
- (c) multiply (632)<sub>7</sub> by (435)<sub>7</sub>.
- (d) divide (5602)7 by (43)7.

3: Convert each of the following numbers to base 3 notation: 3, 9, 27, 243, 1/3, 1/27. Write each of them to base 2.

4: Make an inventory (list) of properties which we assume about the set of integers Z. If this inventory suffices for every discussion of questions which we undertake, then we shall consider it an acceptable description of the integers. We want to agree on a common set of assumptions as we approach problems.

5: Internet Search

What is a prime number? a Mersenne prime?

6: According to the terms of a partial amnesty, a jailer walked a total of 100 times along a row of 100 jail cells. On the first pass he turned every lock, hence opening them all; on the second pass, he turned every second lock starting with the second, hence relocking the even numbered cells; on the third pass, he turned every third lock starting with the third; and so forth until on the last pass he turned only the last lock. Then the prisoners were allowed access to the doors. Which prisoners were freed? Which locker was opened and closed the most number of times? Explain.