Collaboratively Engaging with GCFs and LCMs

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Collaborative engagement provides an opportunity for students to construct and solidify their own knowledge and understanding of important mathematical ideas. According to Van de Walle, Karp, and Bay-Williams, "learning is enhanced when the learner is engaged with others working on the same idea" (2015, p. 52). In allowing students to work with their peers to practice problems and construct important mathematical connections, the students build on their combined prior knowledge to formulate newfound ideas and conjectures. We recognize that grouping students so that each group will function in a productive manner can often be difficult. Therefore, we have devised this activity that allows students to work together and communicate with ten different students individually. In a usual group setting, the students would get to work with one or two other students, but the format of this activity allows for more forms of mathematics communication and collaboration.

In learning about rational numbers, factoring, and operations, students must be able to find a common factor to simplify fractions or to factor expressions. They must also be able to find a common denominator to combine rational expressions. To complete these tasks, the students must first understand the concepts of greatest common factor (GCF) and least common multiple (LCM). The Common Core's sixth-grade standard suggests that students should be able to "find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12" (CCSSI 2010, p. 42). Oftentimes students view finding these values as two disjoint procedures. However, there is a close connection between the product of two numbers and the product of their GCF and LCM.

We can use this relationship to explore different approaches to calculating, justifying, and checking GCFs and LCMs. Moreover, this activity will connect students' previous knowledge of factorization to the concepts of GCF and LCM and allow the students to explore the connections between these mathematical concepts. This aligns with NCTM's *Principles and Standards for School Mathematics* (2000), which states that middle school students "should regularly engage in thoughtful activity tied to their emerging capabilities of finding and imposing structure, conjecturing and verifying, thinking hypothetically, comprehending cause and effect, and abstracting and generalizing" (p. 211).

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IMPLEMENTATION

This activity is intended for a classroom of sixth-grade or seventh-grade students and can be implemented in a single class period, with the discussion and exploration possibly extending over an additional class period. Because finding GCFs and LCMs may be a new concept for some, consider providing some real-world contexts in which the two numbers arise rather naturally. The **activity sheet** includes an opportunity for students to consider what the GCF and LCM can represent in the context of using coins.

The problems refer to two students who are making a common purchase from a vending machine. Diane has a roll of dimes, and Quentin has a roll of quarters. They want to use their respective coins to each buy the same item. What is the cost of the cheapest item they can each buy if the machine gives no change? This context provides one way to think about the LCM of 10 and 25. The students may also think about the GCF using coins. Diane and Quentin want to exchange one of their dimes and one of their quarters for coins that are all a single denomination. What is the largest value coin for which they can do this? In this conversation, students may note that although pennies with the 1 cent value would be a solution that converts their dime and quarter, the nickels with its 5 cent value would be of largest value, and fewer coins as well. Therefore, a common denomination of 5 cents would be the GCF of 10 and 25 cents.

An area model can be used to visualize the LCM and the GCF earlier in the lesson. We placed it toward the end of the lesson, however, so that students could make the connections through numerical pattern recognition. However, the area model can be brought in earlier as well.

The teacher may then consider listing factors and multiples of two numbers and identifying the GCF and LCM directly, to increase student understanding. We also suggest that students be given several examples to strengthen their working definitions of the terms. After this, introduce the class to an alternative method to find the GCF and LCM of the same two numbers using their prime factorizations. When viewing the prime factorizations of two numbers, their GCF can be found by taking the product of the common prime factors raised to the lowest power that is appearing in each of the prime factorizations. Similarly, the LCM can be found by taking the product of the prime factors raised to the highest power appearing in the prime factorizations.

In the Practice Problems section of the **activity sheet**, which can be implemented individually or in groups, we ask students to practice their skills of finding the GCF and LCM by using prime factorization. This strategy will then be applied later in the main activity. To begin the main part of the activity, each student should receive an assigned number at the top of the chart on the second page of the activity sheet. Students are then asked to find ten other students. They are to work with each of them to figure out the GCF and LCM of their numbers and fill in the last two columns. As the students interact and work together to fill out their charts, they should be encouraged to explore the methods being used by other students. In that way, they will become aware of the advantage of using the prime factorization of numbers (especially for larger numbers).

EXPLORATION

When the students finish the chart, they can work on the last section of the activity sheet, which allows them to explore their data from the activity sheet. The students may have noticed that the numbers in the last two columns of the activity sheet are the same. The questions that follow ask students to show an example of finding the GCF and LCM of two larger numbers by only looking at their prime factorizations, which guides the students to the reasoning behind the rule they observed. Since the GCF can be calculated by multiplying each common prime factor appearing in the prime factorizations taken to the lower of the two powers in the prime factorizations, and the LCM is found by multiplying each of the primes appearing in the prime factorizations taken to the larger of the two powers in the prime factorizations, both of the numbers are being used in their entirety between the GCF and LCM. Thus, it must hold true that $\text{GCF}(a, b) \times \text{LCM}(a, b) = a \times b$. To see this more clearly, consider the following example:

$$a = 2^3 \times 3^1,\ b = 2^2 \times 3^4,$$
$$\text{GCF}(a, b) = 2^2 \times 3,$$
$$\text{LCM}(a, b) = 2^3 \times 3^4$$

We see that the maximum and minimum exponent for 2 and 3 in the factorizations have been used in the GCF and LCM, respectively; thus, all factors of a and b appear in the GCF and LCM. Therefore, the product of the two numbers is the same as the product of the GCF and LCM. The additional questions encourage students to think deeply about the relationship of two numbers in terms of their GCF and LCM. For example, question 5 asks them to consider what the LCM would be if the GCF is 1, which allows students to solidify their understanding of the definition of relatively prime and also use the rule that they justified above to conclude that the LCM must be the product of the two numbers. An interesting extension that shows the usefulness of this formula is to have the students determine the LCM of two numbers by calculating the product of the numbers and dividing this product by their GCF.

As we implemented the follow-up questions in our classes, the majority of students were able to understand why the product of the GCF and LCM equals the product of the two numbers, but they had a hard time putting their ideas into words. For this reason, students drew diagrams or wrote equations to explain their reasoning. However, encouraging your students to put their ideas into words can increase their understanding of this result by eliciting deeper thinking.

VISUALIZATIONS THROUGH AREA

Finding the LCM was more difficult for students because it required more calculations than did determining the GCF. We had some students finish all the GCF, but leave the LCM column empty. Encourage students to work with their partner to complete the entire row before moving on to a new partner. Difficulties also arose when students who had relatively prime numbers tried to find the GCF of their two numbers. The students thought that because the two numbers had “nothing in common” that the GCF was 0. This is extremely important to have the students correct as soon as possible because not only is there a question in the exploration section based on this observation but also the last two columns depend on this finding.

As stated above, some students had different answers in the last two columns of the activity sheet, which confused them when they were exploring the relationship of GCF (My Number, Friend’s Number) × LCM (My Number, Friend’s Number) and the product My Number × Friend’s Number. When this happened, and it was not because the numbers were relatively prime, we encouraged students to check their answers with those around them. A few students, even with the appropriate answers, had difficulty seeing the connection between the last two columns. For maximum understanding, teachers should plan for significant time to explore this question, and discuss any confusion that arises.

A concrete way of viewing GCF and LCM of two numbers in relation to the two numbers is by using an area model. For example, using the two numbers 6 and 8, have the students pair up and each draw a 6 × 8 rectangle on grid paper and determine that each rectangle has area 48 square units. Ask the students to partition the entire 6 × 8 rectangle into the fewest number of congruent squares and have them determine the length of its sides. The students should have twelve 2 × 2 squares tessellating the rectangle. We note that 2 is also the GCF of 6 and 8. Then ask the students to cut the rectangle into strips that are the width of their squares so that each student’s strips are cut in a different direction. One student lines up the 4 strips into one long strip. The other student lines up the 3 strips into one long strip. The two students should note their lengths are equal to 24, which is equal to the LCM. In addition, the areas as long strips with dimension GCF by LCM, or 2 × 24, equals 48 square units, the area of the original rectangle.

CONCLUSION

Students seemed to enjoy being able to move around and interact with several of their peers instead of being in a static group. This interaction allowed them to show one another multiple ways of thinking; it also gave them a sense of connection between the variety of methods used. The multiple examples also allowed them to generalize different ideas into a conjecture that they could reason through in the extension worksheet. By the end of this activity, students were able to determine and justify GCFs and LCMs for pairs of numbers as well as find GCFs and LCMs that fit given criteria while using the definitions of prime numbers and relatively prime numbers, or when given relationships between the GCF and LCM. In addition to developing greater proficiency in working with these numbers, students showed a deeper understanding of the definition of GCF and LCM and shared with others their insights into the relationship between the two numbers.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Van de Walle, John A., Karen S. Karp, and Jennifer M. Bay Williams. 2015. *Elementary and Middle School Mathematics: Teaching Developmentally* (International Edition). New York: Pearson.




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activity sheet

Name ______________________________

WARM UP: INTRODUCTIONS TO GCFS AND LCMS

A. Diane has a roll of dimes, and Quentin has a roll of quarters. They do not want to pool their money, but they each want to buy the same snack from a vending machine that requires exact change. Find prices for at least three items that Diane and Quentin can buy. What is the price of the cheapest item they can each buy if they want to purchase the same item?

The lowest price that you calculated above is called the Least Common Multiple (LCM) of 10 and 25 because out of the multiples they have in common, it is the least number. We use the notation LCM (10, 25) to represent this value.

B. Now Diane and Quentin go to a store and want to obtain change for their coins. They would both like to receive the same type of coin in exchange for their dimes and quarters. Find at least two values of coins that Diane and Quentin can get in exchange for theirs. What is the largest value coin that Diane and Quentin can exchange for one of their dimes and one of their quarters?

The largest value coin that you found above is called the Greatest Common Factor of 10 and 25 because out of the common factors, it is the greatest. We can use the notation GCF (10, 25) to represent this value.

PART 1: PRACTICE PROBLEMS

1. Determine the prime factorizations of the following numbers:

a. 18

b. 25

c. 51

d. 60

2. Determine the GCFs of the following pairs of numbers:

a. GCF (18, 51)

b. GCF (18, 60)

c. GCF (25, 51)

3. Determine the LCMs of the following pairs of numbers (you may leave your answer in its prime factorization):

a. LCM (18, 51)

b. LCM (18, 25)

c. LCM (25, 60)

Name ______________________________

PART 2: ACTIVITY CHART

Activity instructions:

1. Find the prime factorization of your assigned number (at the top of your paper).
2. Meet with another student and record his or her number and prime factorization in the first two columns of your table. Partners should check each other's work for accuracy.
3. Work together to find the GCF and LCM of the two numbers and record them in the appropriate place in the table.
4. Calculate the product GCF × LCM and My Number × Friend's Number and record these values in the last two columns of the table.
5. Repeat steps 2 through 4 until you have met with ten other students.

My number is ____________ and its prime factorization is ______________________.

Friend's Number	Friend's Prime Factorization	GCF	LCM	GCD × LCM	My Number × Friend's Number

PART 3: FOLLOW-UP QUESTIONS

1. What did you observe about the relationship between the product GCF × LCM and the product My Number × Friend's Number?

Name ______________________________

2. Determine the GCF and LCM of the following numbers and write each in their prime factorization form.

$2^3 \times 3^4 \times 5^2$ $2^5 \times 3^0 \times 5^3$

GCF:

LCM:

3. Justify your conclusion in question 1. It may be helpful to use the example in question 2.

4. Give an example of two numbers whose GCF is 1 (do not use numbers already in your chart).

5. When the GCF of your number and your friend's number is 1, what is the LCM?

6. When the GCF of your number and your friend's number is your number, what is the relationship between their number and your number?

PART 4: VISUALIZATION WITH AREA: GCF, LCM, AND THE PRODUCT OF THE TWO NUMBERS

1. To gain a visual idea of this process, work with your friend and each of you draw a rectangle on your grid paper that has dimensions 6×8 units. Record the area of the rectangle.

2. Partition one of the rectangles into congruent squares that cover the entire rectangle without overlap using the fewest number of squares. What are the dimensions of the square that satisfies the requirements?

3. Cut the first rectangle into strips, each with width equal to the square's side and length equal to 6. Use tape to connect them into one long strip. With the other rectangle, cut the rectangle into strips, each with width equal to the square's side and the length equal to 8. Again use tape to connect them into one long strip. Determine the two dimensions. What do you notice?

4. What connection, if any, do you see between the dimensions of the rectangle and the GCF and LCM of the 6 and 8?

5. What can you say about the area of the rectangle obtained using the GCF and LCM and the area of the original rectangle?

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