Math Explorations
Algebra I

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Preface and Introduction

Math Explorations follows several fundamental principles. It is important to carefully state these at the beginning, and describe how these are a perfect fit not only in educating the general student population, but also in teaching students whose native language is not English. These guiding principles will help the curriculum come alive for all students.

First, learning math is not a spectator sport. The activities that fill the text and accompanying workbooks encourage students to develop the major concepts through exploration and investigation rather than by given rules to follow. **A crucial element is to understand the importance of small-group work, and to appreciate the extent to which everyone can benefit from working together.** In fact, often the process of explaining how to work a problem helps the explainer as much or more than the person who asks the question. As every teacher knows, explaining an idea to someone else is one of the best ways to learn it for oneself. Some basic rules for discussion within a group include

1. **Encourage everyone to participate**, and value each person's opinions. Listening carefully to what someone else says can help clarify a question. The process helps the explainer often as much as the questioner.
2. If one person has a question, remember that the chances are someone else will have the same question. Be sure everyone understands new ideas completely, and **never be afraid to ask questions**.
3. **Don't be afraid to make a mistake.** In the words of Albert Einstein, "A person who never made a mistake never discovered anything new." Group discussion is a time of exploration without criticism. In fact, many times mistakes help to discover difficulties in solving a problem. So rather than considering a mistake a problem, think of a mistake as an opportunity to learn

more about the process of problem-solving.

4. **Finally, always share your ideas with one another**, and make sure that everyone is able to report the group reasoning and conclusions to the class. Everyone needs to know why things work and not just the answer. If you don't understand an idea, be sure to ask "why" it works. You need to be able to justify your answers. The best way to be sure you understand why something works is to describe your solution to the group and class. You will learn more by sharing your ideas with one another.

If an idea isn't clear, there are several things to try.

1. First, look for simpler cases. Looking deeply at simple cases can help to you see a general pattern.
2. Second, if an idea is unclear, ask your peers and teacher for help. Go beyond "Is this the right answer?"
3. Third, understand the question being asked. Understanding the question leads to to mathematical progress.
4. Focus on the process of obtaining an answer, not just the answer itself; in short become problem centered, not answer centered. One of the major goals of this book is to develop an understanding of ideas that can solve more difficult problems as well.
5. After getting help, work the problem yourself, and make sure you really understand. Make sure you can work a similar problem by yourself.

Some hints to help in responding to oral questions in group and class discussion: As you work through the Explorations in the book, working both individually and in groups can make understanding the material easier. Sometimes it is better to explore problems together, and other times try exploring first by yourself and then discuss your ideas with others. When you discuss the problems as a group, it is more productive if you try to remember these simple rules:

1. Try not to interrupt when someone else is talking.
2. In class, be recognized if you want to contribute or ask a

question.

3. Be polite and listen when others in your group or class are talking. This is one of the best ways to learn.
4. Finally, don't be shy. If you have a question, raise your hand and ask. Remember, there is almost always someone else with a similar or identical question.

Last, some general advice about reading and taking notes in math.

1. Reading math is a specific skill. Unlike other types of reading, when you read math, you need to read each word carefully. The first step is to know the mathematical meaning of all words.
2. It is often necessary to write definitions of new words and to include mathematical examples. Try to write definitions in your own words without changing the meaning or omitting any important point. When you write down a definition, look for an example that illustrates what you are learning. This will help you to relate what you are learning to real world situations.
3. Explaining new ideas and definitions that you read to your peers and teachers is very helpful. This will provide practice with any new definitions, and make sure that you are using the words correctly. Explaining a concept can help to correct any misconceptions and also reinforces learning.
4. If possible, try to draw a visual representation to make a difficult or new concept clear. It is really true that "a picture is worth a thousand words." Visual cues help you understand and remember definitions of new terms.

Throughout this book, students learn algebraic thinking and the precise use of mathematical language to model problems and communicate ideas. The communication that makes this possible can be in small groups, in class discussion, and in student notes. It is important to note that the use of variables and algebra is not an afterthought, but is woven throughout all of our books. By using language purposefully in small groups, class discussions, and in written work, students develop the ability to solve progressively more challenging problems.

The authors are aware that one important member of their audience is the parent. To this end, they have made every effort to create explanations that are as transparent as possible. Parents are encouraged to read both the book and the accompanying materials.

The authors have written a 3 volume set of books that is designed to take all students from pre-algebra through Algebra I. This includes students who may not have understood the previous years math. Students from 4th through 8th grade should enjoy the ingenuity and investigation problems at the end of each set of exercises. Math Explorations is intended to prepare all students for algebra, with algebraic concepts woven in throughout. In addition, ME Part II, and ME Algebra I cover all of the 7th and 8th grade Texas Essential Knowledge and Skills (TEKS). The Teacher Guide has been written to make the textbook and its mathematical content as easy and intuitive for teachers as possible. Answers to the exercises are readily available and readable in the teacher edition. The teacher edition and accompanying CD contain supplementary activities that the students might enjoy.

This text has its origins in the Texas State Honors Summer Math Camp (HSMC), a six-week residential program in mathematics for talented high school students. The HSMC began in 1990 modeled after the Ross program at Ohio State, teaching students to think deeply of simple things (A. E. Ross). Students learned mathematics by exploring problems, computing examples, making conjectures, and then justifying or proving why things worked. The HSMC has had remarkable success over the years, with numerous students being named Siemens- Westinghouse semi-finalists, regional finalists, and national finalists. Initially supported by grants from the National Science Foundation and RGK Foundation, the HSMC has also had significant contributions from Siemens Foundation, Intel, SBC Foundation, Coca-Cola, the American Math Society Epsilon Fund, and an active, supportive Advisory Board.

In 1996, two San Marcos teachers, Judy Brown and Ann Perkins, suggested that we develop a pipeline to the HSMC that would introduce all young students to algebra and higher-level mathematics. Following their suggestion, we began the Junior Summer Math Camp (JSMC) as a two-week program for students in grades 48. We carefully developed the JSMC curriculum by meeting regularly with Judy and Ann, who gave us invaluable feedback and suggestions.

With support from the Fund for the Improvement of Postsecondary Education (FIPSE), Eisenhower Grants Program, Teacher Quality Grants, and the Texas Education Agency, we developed the JSMC into a replicable model that school districts throughout the state could implement. The JSMC curriculum was designed to prepare all students for higher-level mathematics. In some districts the JSMC targeted gifted students; in other districts the program was delivered to a mixed group of students; and in other districts the program was used especially for ELL. In every setting, the program had remarkable results in preparing students for algebra as measured by the Orleans-Hanna algebra prognosis pre- and post-tests.

Over the years, we trained hundreds of teachers and thousands of students. Although we cannot thank each personally, we should mention that it has been through their suggestions and input that we have been able to continually modify, refine, and improve the curriculum.

The problem with the JSMC curriculum was that it was only supplementary material for teachers, and many of the state-required mathematics topics were not included. The 3 volume Math Explorations texts that we have written has taken the JSMC curriculum and extended it to cover all of the TEKS (Texas Essential Knowledge and Skills) for grades 6-8 while weaving in algebra throughout. In particular, this volume, Math Explorations, Algebra I, was developed especially for younger students. By learning the language of mathematics and algebra, young students can develop careful, precise mathematical models that will enable them to work multi-step problems that have been a difficult area for U. S. students on international tests.

This project had wonderful supporters in the Meadows Foundation, RGK Foundation Foundation, and Kodosky Foundation. A special thanks to our Advisory Board, especially Bob Rutishauser and Jeff Kodosky, who have provided constant encouragement and support for our curriculum project. The person who motivated this project more than any other was Jeff Kodosky, who immediately realized

the potential it had to dramatically change mathematics teaching. Jeff is truly a visionary with a sense for the important problems that we face and ideas about how to solve them. His kind words, encouragement, and support for our JSMC and this project have kept me going whenever I got discouraged.

Our writing team has been exceptional. The basis for the book was our junior summer math camp curriculum, coauthored with my wife Hiroko, and friend, colleague and coauthor Terry McCabe. We were very fortunate when we decided to extend that curriculum to cover all of algebra I to find an extraordinarily talented co-author, Alex White. Alex has taken the lead on the algebra book in working with our team of students, faculty, and teachers, while also doing the amazing job of both making edits and doing typesetting. His specialty is math education and statistics, which are important and often neglected parts of an algebra book. Our team of authors has many lively discussions where we debate different approaches to introducing a new topic, talk about different ways to engage students to explore new ideas, and carefully go through each new idea and how it should be sequenced to best guide student learning.

Over the summers of 2005-2012, we have been assisted by an outstanding group of former Honors Summer Math Camp students, undergraduate and graduate students from Texas State, as well as an absolutely incredible group of pilot teachers. While it would take a volume to list everyone, we would be remiss not to acknowledge the help and support from this past summer, as well as describe the evolution of this three-volume series.

Briefly, in 2008, the Math Explorations Book was only one volume. This was piloted by a group of 6th and 7th grade teachers in McAllen, San Marcos, and New Braunfels. The results of these pilots have been extremely encouraging. We are seeing young 6th and 7th grade students reach (on average) 8th grade level and above as measured by the Orleans-Hanna test by the end of 7th grade. However, there was a consensus that it would be beneficial to split the Math Explorations book into a separate 6th grade and 7th grade book.

After meeting with the McAllen teachers in the summer, 2009, we carefully divided the Math Exploration text into two volumes. Hiroko Warshauer led the team in developing this new book, Math Explorations Part I, assisted by Terry McCabe and Max Warshauer. Alex White from Texas State provided valuable suggestions, and took over the leadership of the effort for the third volume, Algebra 1. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8, while also covering Algebra I.

Although there is naturally some overlap, the new books, Math Explorations Part 1 and Part 2, much more closely align with what the teachers felt would work best with their students. Math Explorations Part 1 should work for any 6th grade student, while Math Explorations Part 2 is suitable for either an advanced 6th grade student or any 7th or 8th grade student. The complete set of 3 books covers all of the Texas Essential Knowledge and Skills (TEKS) for grades 6-8, while also covering all of the TEKS for Algebra I.

In this project, we have been incredibly fortunate to have the help of several talented teachers. Major contributors this past summer include Amanda Voigt and Ashley Beach from San Marcos, Patricia Serviere from McAllen, and Amy Warshauer from Austin. These teachers provided wonderful help in the development of a an accompanying workbook, that provides a template for how to teach the book, with new explorations and supplemental problems. They also made numerous suggestions and edits, while checking that we covered all of the state mandated topics for Algebra I.

Sam Baethge and Michael Kellerman gave the entire book a careful reading, which provided amazing support for editing and revising the book. Michael focused primarily on readability edits. Sam continued to develop new challenge and ingenuity problems which should engage and excite young students in mathematics. Cody Patterson made key contributions to the original design, problems and content of the text. Robert Perez developed additional resources for English Language Learners, including a translation of the glossary and key mathematical terms into Spanish.

As we prepared our books for state adoption, Bonnie Leitch came on board to help guide and support the entire project. Bonnie worked tirelessly to find where each of the Texas Essential Knowledge and Skills (TEKS) and English Language Proficiency Skills (ELPS) was covered in both the text and exercises. We added additional exercises and text to cover any TEKS that were not sufficiently addressed. Bonnie also edited these revisions and gave a final proofreading for each of the books, working with the authors to proofread every edit. However, in the end the authors take total responsibility for any errors or omissions. We do, however, welcome any suggestions that the reader might have to help make future editions better. In short, we had an incredible, hard-working team that did the work of an entire textbook company in a few short weeks! Without their help, the project could not have reached its present state.

Any curriculum will only be as effective as the teachers who use it, and without the support and encouragement of the administration and parents, this can never happen. In this, we have been very fortunate to be able to work with fabulous teachers and administrators from San Marcos, McAllen, New Braunfels, Midland, and Austin. The Mathworks staff gave invaluable help. Patty Amdende and Andrew Hsiau have provided support whenever needed. I hope you will join our team by giving us feedback about what works, what doesnt and how we can improve the book. By working together, I believe that we can develop a mathematics curriculum that will reach out to all students and that will engage students at a higher level than we have previously been able to achieve.

Max Warshauer

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Guiding Principles

Based on available research, our experience in mathematics education and after long hours of discussion and trial and error, the Mathworks team have developed a set of guiding principles that we feel will help foster a learning environment in which all children are challenged, engaged and have an opportunity to learn.

1. Math is about making sense of things, thinking deeply of fundamental concepts. Students need to:
   - Make connections using mathematics.
   - Investigate challenging, well-sequenced problems.
   - Build on prior knowledge.
   - Explore problems to make sense of mathematical ideas for themselves.
   - Focus on the math problems, not the answers.
   - Reflect on what they have learned.
2. Teachers need to establish a classroom culture where students are not afraid of failure. The keys to establishing this culture are to:
   - Encourage students to take risks without fear of failure.
   - Value curiosity. Make the problems fun, interesting and relevant.
   - Promote productive struggle.
   - Allow sufficient time for learning ideas deeply.
   - Consider all attempts seriously.
3. Persistence is critical to success. To develop students with the confidence to not give up easily, students need to:
   - Develop a "growth mindset."
   - Take ownership of their own learning.
   - Understand and believe that ability can be developed with hard work.
   - Be challenged by problems with high expectations.

4. Communication between students and teachers is critical for learning. To facilitate this, teachers should:
    - Expect students to present work and defend reasoning using precise mathematical language.
    - Take student ideas seriously, and examine both right and wrong approaches.
    - Expect students to articulate the big ideas (justify reasoning).
    - Balance individual and group work; both can be appropriate depending on the problem. Group work needs careful management.
5. Dispositions and external factors are powerful and need to be taken into account and dealt with. To do this, the teacher must:
    - Identify and deal with extraneous issues that can affect learning.
    - Inspire desire to build a nurturing classroom environment where students take charge of their own learning.

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How to use the book

Different Types of Problems

With the guiding principles in mind, the Mathworks team has developed a set of well-sequenced rich problems. The problems in this book are divided into six categories described below. Note that **Explorations**, **Problems**, and **Examples** appear in the body of text in each section. Copies of these problems with space to work are provided in the Student Workbook. This workbook is intended for students to use during class as they explore the mathematical ideas and take notes. **Exercises** appear at the end of each section in the textbook. **Warm Up** problems are provided at the beginning of each section in this Teacher Edition of the textbook. Many teachers choose to have the students keep their textbooks at home. This way they can read the explanations given there and work on the exercises using both the workbook and the textbook as a guide. For added flexibility, electronic copies of all the exercises and warm up problems are available in MS word format on the accompanying CD.

Warm Up: These problems appear in the teacher's edition at the beginning of each section. They are intended to be used at beginning of each class to get students "in the mood" and to provide a mathematical activity while the teacher is taking attendance,or doing other required paperwork. Each of the problems satisfies at least one of three purposes: (1) to review material from prior sections or grades which is related to the current section, (2) to practice concepts from the previous section, and (3) to practice multiple-choice type questions that might be covered in a standardized testing environment. In most sections, multiple warm-up problems are provided in case multiple days are needed to cover the section. We suggest that each day the teacher give one of the problems as students enter the classroom, allow the students 5 minutes to work on it and then lead a brief discussion where a few students share their solutions. It is important to spend time discussing each warm-up problem carefully to establish a baseline of knowledge for the upcoming section, as well as to address possible misconceptions that students might have from previous sections.

Exploration: These problems form the heart of the Mathworks curriculum. Typically there is one exploration for each big idea presented in a section. These problems are particularly well-suited to group work and/or whole class discussion. The explorations allow students to work with the mathematical concepts before they are formally defined. We recommend that students be given time to work on the problems, share their solution strategies and reflect on patterns they discover. Ideally, students will recognize the underlying concept and will only need to the teacher's help to verbalize or name what is going on. However, experience informs us that this may not happen each time. The teacher must be prepared with leading questions. It is important that by the end of the exploration, the class has reached some consensus or closure on the big idea. It is not necessary in each case, however, that the big idea be expressed in formal mathematical language. We have tried to give guidance in the teacher notes about what to expect during each exploration.

Problem: Most explorations are followed by a "problem". These are intended to give students practice at the concept, skill or big idea developed in the exploration. These problems are well-suited for individual seat work. The teacher can use this time to informally assess student understanding.

Example: Each section contains at least one example. In the examples, the complete solutions are presented including explanations for each step. Based on the class discussion in the exploration and student work on the problems, the examples can be used in variety of ways. If the students have demonstrated mastery, the teacher may choose to skip the example. Or the teacher may present the problem using only the workbook and use the example for guided practice as the students find the solution themselves. Finally, the teacher may feel it necessary to present the problem and its complete solution. Additionally, we feel the students or their parents may find the examples useful when they work on the homework exercises.

Exercise: At the end of each section, a large number of practice exercises are given. Where possible, we have tried to sequence the problems so that they increase in complexity and so that patterns emerge from which the students can learn. Solutions to all the exercises are included in this teacher's edition.

Investigation: Exercises that are labeled as investigations involve multiple steps and are similar in nature to the explorations. Investigations extend the concepts discussed in the section and foreshadow concepts that appear in the subsequent sections. Unlike regular exercises, we do not expect all students to successfully complete these problems. However, all students should be able to start the investigation and explore related concepts. In many cases, the concept of the investigation will be revisited in the next section. Teachers should expect to have class discussion about the investigations when the students hand in their homework.

Ingenuity: Exercises that are labeled as ingenuities are challenging problems which often involve a leap in thinking. These exercises are intended to encourage students to think creatively, and to develop the mathematical abilities of even the most talented students. However, all students should be encouraged to explore these problems, and to develop a mindset that they will learn by exploring hard problems even when they cannot immediately see how to do them.

Practice Problems: Each chapter ends with a review section which summarizes the important terms and formulas from the chapter. This review also contains a set of practice problems covering the most important skills and concepts from the chapter.

Structure of the Teacher's Edition

Chapter Preview: Each chapter in the teacher's edition begins with a brief summary of the content and pedagogical intent of the chapter. In this preview we have highlighted the more innovative aspects and approaches of the Math Explorations: Algebra I.

Section Summary: Each section in the teacher's edition begins with a page detailing that section's *Big Idea*, *Key Objectives*, and *TEKS*. Any materials needed for the section's explorations are listed and in some cases a *Launch* is given. These launches suggest ways to motivate the sections' lessons. In most cases, we predict that the first exploration in the section will be a great launch to the lesson.

Warm Ups Following the section summary in the teacher's edition, there is a page with the warm up problems and a page with the key to these problems. These are also available in Word format in the accompanying CD.

Augmented Student Edition The teacher's edition includes a complete version of the student textbook. Where possible, solutions to problems are included and highlighted within the text. Due to space considerations, some solutions are included in the margins. Also in the margins are teacher tips pointing out the authors pedagogical intent and advising teacher's what to look out for as students work through explorations and problems.

Corrections

With help of many teachers who have read and piloted this textbook, errors from a previous version have been detected and corrected. Since the publication of this textbook, some additional errors have been caught by the authors. These errors have been corrected in this teacher's edition, but are still present in the published student edition. A complete list of errors is available at the Mathworks web site, `www.txstate.edu/mathworks`. If and when you or your students find more errors, please let us at Mathworks know via email at max@txstate.edu. Every textbook is a work in progress and we appreciate your help in improving the textbook and creating better resources for our children.

- TODO Add any errors.

TEKS

This textbook is designed to cover the Texas Essential Knowledge and Skills for Algebra I. Additionally, since the book is intended as the third book in a series for Middle School, the content includes many eighth grade TEKS as well. When considered as a series, all of the TEKS from 6th-8th as well as Algebra I are covered. In April 20 2012, the State Board of Education approved a significant revision of the TEKS. These revised TEKS added a considerable amount of Algebra content to the middle school curriculum. Making this textbook series a particularly good fit for the new standards. At the time of publication of this text, the date at which the new TEKS would go into effect was not available. For this reason, the following tables refer to the Original TEKS where were amended in 2006 and the Revised TEKS which were approved in 2012. The first set of tables is for the Original TEKS and the second set for the Revised TEKS. For convenience, the alignment is organized in two ways: by section and by TEK.

Table 1 Orginal TEKS (2006) by Section

Section	8th	Alg 1	Alg2
1.1	8.1a		
1.2	8.1a, 8.2a	A3b	
1.3	8.15a	A3a	
1.4	8.2b, 8.4b, 8.5a, 8.15a, 8.15b	A1c	
1.5		A3a, A3b, A4b	
1.6			
1.7	8.10a, 8.10b		
2.1		A1a, A1b, A2b, A4c, A5b	
2.2	8.5a	A1d, A1e, A2b, A5a, A5c, A6e	
2.3	8.5b	A1b, A1c	
2.4	8.2d, 8.4, 8.5a	A1a, A1b, A1c, A1d, A2b, A6e, A7a	
3.1		A1d	
3.2		A6a, A6b, A6c, A6d, A6f	
3.3		A6c, A6d, A6e, A6f	
3.4		A6d	
3.5		A6d, A6e	
3.6		A6e, A6f	
4.1		A8a, A8b, A8c	
4.2		A8a, A8b, A8c	
4.3		A8a, A8b, A8c	
4.4		A8a, A8b, A8c	
4.5		A8a, A8b, A8c	
5.1		A7a, A7b	
5.2		A7a, A7b, A7c	
5.3		A7a, A7b, A7c	

Table 2 Original TEKS (2006) by section CONT'd

Section	8th	Alg 1	Alg2
6.1			
6.2			
6.3			
6.4			
6.5			
6.6			
6.7			
7.1		A3a	
7.2		A4a, A4b	
7.3		A4a, A4b	
7.4		A4a, A4b	
7.5		A4a, A10a	
7.6		A4a, A10a	
7.7		A4a, A10a	
8.1		A2a, A9c, A9d	
8.2		A2a, A9b	
8.3		A10b	
8.4		A9b, A9c, A9d, A10b	
8.5		A10a, A10b	
9.1		A1b, A2d	
9.2		A11b, A11c	
10.1	8.7c, 8.9a		
10.2	8.1c, 8.7c		
10.3	8.7d		
10.4		A6g, A11b	
10.5		A6g, A11b	
10.6		A6g, A11b	
11.1			A4, 4C, 9A, 9B, 9G
11.2			2A
11.3			9D

Table 3 Revised TEKS (2012) by Section

1.1	8.2A		
1.2	8.8C		
1.3			
1.4			
1.5		A10D	
1.6	8.8A,8.8B,8.8C	A5A	
1.7	8.7A, 8.10D	A12E	
2.1		A2A,A12A,A12B	
2.2	8.5G	A2A	
2.3		A12C, A12D	
2.4	8.5H	A2C	
3.1		A2G	
3.2	8.4A,8.5A	A2D	
3.3	8.5B, 8.5F, 8,5I	A2B, A2E, A3A	
3.4	8.2E, 8.4B, 8.5H	A2C, A3B, A3C, A12C,A12D	
3.5		A2B, A2C, A3A	
3.6		A2F	
4.1	8.9	A2H, A3F, A3G	
4.2		A5C	
4.3		A5C	
4.4		A3G	
4.5			
5.1			
5.2		A5B	
5.3		A2H, A3D, A3H	

Table 4 Revised TEKS (2012) by Section, CONT'D

6.1		A11B	
6.2		A11B	
6.3		A9B, A9C, A9D	
6.4		A9B, A9C, A9D	
6.5	8.12A, 812B, 812.C	A9B, A9C	
6.6		A12C, A12D	
6.7	8.2C		
7.1			
7.2		A10A	
7.3		A10B	
7.4		A10D	
7.5		A10D, A10F	
7.6		A8A, 10D	
7.7		A8A, A10E	
8.1		A7A, A7C	
8.2		A7A, A7C	
8.3		A7B	
8.4		A7A	
8.5		A8A	
9.1	8.5D, 8.11A	A4A, A4B, A4C	
9.2	8.5C, 8.5H	A8, A9E	
10.1	8.6c,8.7c		
10.2	8.2b,8.7c		
10.3	8.7c, 8.7d		
10.4			4C
10.5		A11A, A11B	
10.6			4F
11.1		A10C	
11.2		A2D	
11.3			

Table 5 Sections by Original TEKS (2006)

8th TEK	Sections	Alg 1 TEK	Sections
8.1A		A1A	2.1
8.1B		A1B	9.1
8.1C	10.2	A1C	2.4, 3.4, 3.5, 4.4, 5.2, 5.3, 9.1, 9.2
8.1D	6.7	A1D	all
8.1E	10.2	A1E	2.4,3.4,6.5,9.1,9.2
8.2A		A2A	3.2, 8.1
8.2B		A2B	2.1, 2.2
8.2C		A2C	
8.2D		A2D	9.1, 9.2
8.3A	3.3	A3A	1.3,
8.3B		A3B	2.3, many others
8.4	all	A4A	2.1-2.4, 6.3, 6.4, 8.1-8.4
8.5A	2.1-2.4	A4B	1.5
8.5B	2.3, 3.4	A4C	3.4
8.6A		A5A	2.3, 2.4, 6.3, 9.2
8.6B		A5B	2.1, 2.2
8.7A		A5C	2.2, 3.1-3.5
8.7B		A6A	3.2
8.7C		A6B	3.2
8.7D		A6C	3.3
8.8A	1.7	A6D	3.1-3.6
8.8B		A6E	3.3, 3.5
8.8C		A6F	3.4
8.9A	10.1 - 10.3	A6G	3.2, 11.2
8.9B		A7A	2.4, 3.4
8.10A	1.7	A7B	1.2, 1.4, 1.6, 5.2
8.10B	1.7	A7C	1.2, 1.4, 1.6, 5.2
8.11A		A8A	4.1- 4.4
8.11B		A8B	4.1-4.4
8.11C		A8C	4.1-4.4

Table 6 Sections by Original TEKS (2006) CONT'd

8th TEK	Sections	Alg 1 TEK	Sections
8.12A		A9A	8.1-8.4
8.12B	9.1	A9B	8.2
8.12C		A9C	8.1
8.13A		A9D	8.1-8.4
8.13B		A10A	7.6, 7.7
8.14	all	A10B	8.3
8.15	all	A11A	6.1, 6.2
8.16	all	A11B	11.2
		A11C	6.3, 6.4

Table 7 Sections by Revised TEKS (2012)

8th TEK	Sections	Alg 1 TEK	Sections
8.2A	1.1, 10.2	A1	all
8.2B	10.2	A2A	2.1 - 3.4
8.2C	6.7	A2B	3.1-3.5
8.2D		A2C	2.4-3.5
8.3A		A2D	3.2, 11.2
8.3B		A2E	3.1, 3.3, 3.6
8.4A	3.2	A2F	3.6
8.4B	2.2,2.3,2.4,3.2, 3.3,3.4	A2G	3.1
8.4C	3.2, 11.2	A2H	5.3
8.5A	3.2	A2I	4.1-4.4
8.5B	3.3	A3A	3.2, 3.3, 3.5
8.5C	2.3, 6.6, 9.2	A3B	2.4, 3.4
8.5D	9.1	A3C	3.2-3.5
8.5E	3.2, 11.2	A3D	5.3
8.5F	3.3	A3E	
8.5G	2.1, 2.2	A3F	4.1, 4.5
8.5H		A3G	4.1
8.5I	3.3	A3H	5.3

Table 8 Sections by Revised TEKS (2012) CONT'd

8th TEK	Sections	Alg 1 TEK	Sections
8.6A		A4A	9.1
8.6B		A4B	9.1
8.6C	10.1, 10.2	A4C	9.1
8.7A	1.7	A5A	1.2, 1.4, 1.6
8.7B	1.7	A5C	4.1-4.4
8.7C	10.1, 10.2	A6A	8.1, 8.2, 8.4
8.7D	10.3	A6B	8.4
8.8A	1.6	A6C	8.3, 8.5
8.8B		A7A	8.1-8.4
8.8C	1.2, 1.4, 1.6	A7B	8.3, 8.5
8.8D		A7C	8.1, 8.2
8.9	4.1-4.4	A8A	7.6, 7.7, 8.5
8.10A		A8B	9.2
8.10B		A9A	
8.10C		A9B	6.3, 6.4
8.10D		A9C	6.3, 6.4
8.11A	9.1	A9D	6.3, 6.4
8.11B		A9E	9.2
8.11C		A10A	7.2
8.12A	6.5	A10B	7.3
8.12B	6.5	A10C	11.1
8.12C	6.5	A10D	1.5, 7.3, 7.5
8.12D	6.5	A10E	7.5, 7.7
8.12E		A10F	7.5
8.12F		A11A	10.2, 10.5
8.12G		A11B	6.1, 6.2, 10.5
		A12A	2.1
		A12B	2.1-2.4
		A12C	2.3, 3.4, 6.6
		A12D	2.3, 3.4, 6.6
		A12E	1.7

MATH EXPLORATIONS

VARIABLES

Chapter Preview In this chapter, students explore the number line, variables and expressions. In section 1.1, we construct the number line and discuss important subsets of the real numbers: natural numbers, whole numbers, integers and rational numbers. This is an opportunity for students to review arithmetic with integers and rational numbers and to discuss distance on the number line. We feel the number line is a very useful model that can help students visualize the properties of arithmetic.

In Section 1.2, the students explore the idea of labeling a position on the number line with a variable. This allows the students to ease their way into algebra using a visual model. Many of the common algebraic procedures arise very naturally in this model. One of the most important concepts in this section is the effect of taking the opposite of a number. In particular, $-(-a) = a$. Expect the students to struggle with the idea that $-b$ is a positive number when $b < 0$.

In Section 1.3, students explore the idea of a variable more deeply. Students practice converting from word expressions to algebraic expressions. Variables are also used in set notation. In Section 1.4, students use the balance model as well as the linear model to solve linear equations. The first equations solved are simple, and emphasis should be placed on the properties of equality and the models. The more ways students have to think about what the algebraic manipulations are doing, the better. In Section 1.5, the concept of equivalence is discussed in detail. This is a key idea in algebra that causes students many problems. First of all, the word "equivalent" is abstract and hard for students to understand. Research has shown that students score significantly worse on questions on standardized tests that use the word equivalent. The use of the equal sign in mathematics can be subtle. For many students, equals means find the answer, much like the button on the calculator. In some cases, the equal sign is used as assignment for a definition (e.g. $d =$ the distance traveled by the car). But in algebra, most often the equals represents equivalence. This can refer to a sense of balance as in equations, or to the fact that two different expressions represent the same number (as in $2(x + 3)$ is equivalent to $2x + 6$).

Most of algebra can be thought of as finding an equivalent form that is most useful. For example, we convert $3x + 6 = 2x + 8$ into $x = 2$. Emphasize to the students that the most useful equivalent form depends on the context and the question being asked. In our development of equivalence we begin with the famous pool border problem. The point here is that students naturally have different but correct strategies for computing the area of the border. After we list the different strategies, we use algebra and the idea of equivalence to show that all the strategies produce the same answer. We discuss the important properties of arithmetic (associative, commutative and distributive) that are used to establish equivalence. We conclude with the definition of the multiplicative inverse and the meaning of division. In particular, we discuss the equivalence of $\frac{4y}{5}$ and $\frac{4}{5}y$. In Section 1.6, students solve equations in a single variable with the variable appearing on both sides of the equation. Students also explore examples of equations with no solution or where the solution set includes all numbers.

In Section 1.7, students work with formulas and literal equations. In addition to substitution, students will explore the idea of solving a given formula for another variable. An example is transforming from the formula for circumference of a circle $C = 2\pi r$ to $r = \frac{C}{2\pi}$.

CONSTRUCTING A NUMBER LINE

Big Idea Develop rational numbers in stages using the visual model of the number line

Key Objectives

- Build numbers up from the natural numbers to the rational numbers
- Place numbers on the number line relative to each other
- Model arithmetic on the number line
- Relate absolute value and distance on the number line.

TEKS 8.1a

Revised TEKS 8.2A

Materials Needed Graph paper to make number lines.

Launch Have students divide into small groups (3–5 students per group) and discuss properties of the number line. Listen for groups to mention properties such as "zero goes in the middle of the number line," "numbers increase as you go from left to right across the number line," "negative numbers are to the left of the zero and positive numbers are to the right of the zero on the number line," and "numbers (integers) are evenly spaced on the number line." Question to spur thinking: What different kinds of numbers are there on the number line? Students may not know the exact names for different types of numbers; however they should be able to identify that there are positive numbers and negative numbers, zero, whole numbers, fractions and decimals. Once all of the groups have listed off the number line properties they know, tell the class that today they will be reviewing number lines and using number lines to look at different categories of numbers.

VARIABLES, EXPRESSIONS AND EQUATIONS

1

SECTION 1.1 CONSTRUCTING A NUMBER LINE

"What is Algebra?" Rather than give you an incomplete answer now, we hope that through learning, you will soon be able to answer this question yourself. As a preview, let's look at some questions that algebra can help us answer that we could not have answered before:

- If I drop a marble off a two-story building, how long will it take the marble to hit the ground?
- If I have $10 and go into a candy shop, where chocolate costs $.50 and licorice costs $.65, how many of each could I buy?

We begin by reviewing numbers and the ways you manipulate and represent them. We will develop collections of numbers in stages, building up smaller groups of numbers until we get all numbers.

We first encounter numbers as children by counting, starting with one, two, and three. We call the numbers that we use in counting the *natural numbers*, or sometimes the counting numbers. They include the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots,$ where the "..." means that they go on forever in the same way. These numbers describe how many of something, for example, how many brothers or sisters you have, how many days in a week, the number of people in your town and even the number of grains of sand on a beach.

EXPLORATION 1

Make a number line on a large piece of paper. Put the number 1 in the middle of the line. Locate and label the first 20 natural numbers.

Including the number 0 in this set of numbers gives us the *whole numbers*. The whole numbers take care of many situations, but if we want to talk about the temperature, there are places on Earth that routinely have temperatures below zero, like -5 °C or -20 °C. We must expand our idea of number to include the negatives of the natural numbers. This larger collection of numbers is called the *integers*, it is denoted by the symbol $\mathbb{Z}$ and includes the whole numbers. Notice that every integer is either positive, zero or negative. The natural numbers are positive integers and denoted sometimes by $\mathbb{Z}^+$.

Use a different colored writing pen if possible, locate and label the point that represents the number 0 on your number line.

In this book we will try to be very precise in our wording, because we want our mathematics and words to be clear. Just as you learn new words in English class to express complicated concepts, we must learn new words and symbols in mathematics. We have discussed 3 collections of numbers so far: the integers, the whole numbers, and the natural numbers. In mathematics, we call collections of numbers (or other objects) *sets*. A set is defined by its members. We call these members *elements*. In order to write out what a set is, we want to describe its elements in *set notation*. This is done for example by listing the elements of a set inside braces. For example, the natural numbers are $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$ and the integers could be written $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Not all sets go on forever, for example, the set of even natural numbers between 4 and 14 is $\{6, 8, 10, 12\}$.

Point out that since the integers do not have a beginning or end, we need the "..." on both sides of the list.

Every natural number is also an integer. In our more precise language, this could be described as "every number in the set of natural numbers is also in the set of integers". For this reason, we call the natural numbers a *subset* of the integers. We will call a set a subset of another set when any element in the first set is also in the second. We use such precise wording in order to be clear in our discussion of mathematics. The use of precise language is more important in mathematics than it is in everyday life.

EXPLORATION 2

Continue to work on the number line from Exploration 1. Using a red marker, plot and label the negative integers from -1 to -20. What properties does the set of integers have that the set of whole numbers did not?

Includes numbers less than 0. Can subtract larger number from smaller number. This is because the integers are closed under subtraction: the difference of two integers is also an integer. For example, $12 - 25 = -13$, which is in $\mathbb{Z}$ but is not a natural number.

It does not take long to see the need for numbers that are not integers. You might hear in a weather report that it rained $2\frac{1}{2}$ inches or know that a person's normal body temperature is 98.6°Fahrenheit. So sometimes we need to talk about parts of whole numbers called fractions. This expanded set of numbers that includes fractions is called the set of *rational numbers*.

Some decimals are included in this subset of numbers, the ones that terminate or are repeating decimals. Students might ask if all decimal numbers are in the rational numbers. The answer is no but it is not so easy to describe a non-repeating decimal number, such as the $\sqrt{2}$.

EXPLORATION 3

Using a different colored marker, plot and label 3 fractions between each of the following pairs of integers:

2 and 3   4 and 5 -1 and 0 -3 and -2

Make sure the students have the fractions in the right order.

A rational number is the quotient of 2 integers and the denominator can not be zero. For example, both $\frac{3}{7}$ and $\frac{9}{4}$ are rational numbers. They are called rational numbers because they are the ratio of 2 integers. A rational number can be represented as quotient in more than one way. Also, every rational number can be written in decimal form. For example, $\frac{1}{2}$ is equivalent to $\frac{2}{4}$ and to 0.5, $\frac{3}{4}$ is the same as 0.75 and $2\frac{2}{5}$ is equal to 2.4 and $\frac{12}{5}$.

We asked you to find two fractions between 2 and 3. Could you find two fractions between the fractions you just found? How about two fractions between these two?

PROBLEM 1

How many fractions are there between 0 and 1? How many fractions are there between 2 and 3?

The concept of infinitely many fractions between 0 and 1 might be difficult to understand for students. Even though there is only finite space between the two intervals, there can still be an infinite number of fractions. Ask them how many fractions there should be? Also, ask the students which numbers we should draw on the number line when we draw it. Why not all? The number line gives students a feel for how dense the rational numbers are and how scattered the integers are. Some students make the mistake that a decimal number such as .1385 is not a rational number, but it is equal to $\frac{1385}{10000}$.

Notice that every integer is a rational number. There are, however, rational numbers that are not integers. This means that the set of integers is a subset of the set of rational numbers, but the set of rational numbers is not a subset of the set of integers.

PROBLEM 2

List 3 examples of rational numbers that are not integers and list 3 examples of integers that are not whole numbers. Locate these numbers on your number line.

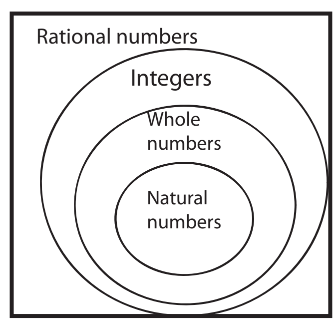
Examples are $-\frac{1}{2}$, $\frac{3}{4}$, $6\frac{1}{3}$ for rational numbers that are not integers, and -1, -29, -3 are examples of integers that are not whole numbers. Have students associate the names of the sets of numbers with where they are on the number line.

EXAMPLE 1

Create a Venn Diagram to show the relationship between the following sets of numbers:

- rational numbers
- whole numbers
- integers
- natural numbers

Solution The sets of numbers are nested. For example, every integer is a rational number, but not every rational number is an integer.



Operations on the Number Line

One advantage of representing numbers on the number line is that it shows a natural order among its members. The location of the points representing 2 numbers reflects a relationship between those

2 numbers that we define as greater than, equal to or less than. You can also examine distances between numbers and model the operations of addition, subtraction, multiplication and division on the number line. In fact, we will frequently use examples on the number line to illustrate algebraic ideas.

Let's review how to do arithmetic with integers using a *linear model*, that is, by representing numbers on a number line. We will use the number line in discussing algebraic concepts. Let's get familiar with using this line.

This might take some time, depending on the background of your students. Use *Math Explorations, Part 2*: Adding and Subtracting on the Number Line (Chapter 2) and Multiplication and Division (Chapter 4).

EXPLORATION 4

1. Use the number line to illustrate the sum $3 + (-4)$ and the difference $3 - 4$. Explain how you arrived at your answer and location for each problem. Then, using the same pattern, explain how you compute the sum $38+(-63)$ and the difference $38 - 63$ without a detailed number line.
2. Use the number line to illustrate the difference $3 - (-5)$ and sum $3 + 5$. Then explain how you compute the difference $38 - (-63)$ without a detailed number line. 101
3. Summarize the rules for addition and subtraction of integers.
4. Use the number line to illustrate the product $3(-4)$ and $-3(4)$. Explain how you arrived at your answer and location for each problem. Then using the same pattern, explain how you compute the products $18(-6)$ and $-5(12)$ without a detailed number line.
5. Use the number line to illustrate the product $-3(-4)$. Explain how you arrived at your answer and location for each problem. Then using the same pattern, explain how you compute the product $-28(-3)$.
6. Summarize the rules for multiplication of integers.

Look for student understanding that you subtract 38 from 63 and then make the answer be negative, -25.

Review modeling subtraction on the number line from *Math Explorations, Part 2*, Section 2.2. Emphasize patterns constantly and ask students to give possible explanations for the patterns. Encourage them to seek out reasons for the patterns as well.

We want the students to have mental image of numbers on the number line to go along with any formal rules of arithmetic. So encourage the students to write the rules in their own words and match them with a number line picture.

Review multiplication using the number line and discuss the meaning associated to each of the 2 factors in the linear model.

The number line is also useful for thinking about operations with rational numbers and exploring the relationship between numbers.

Use this problem to illustrate how you can use the "count on" method using fractions. Start at $1\frac{3}{4}$ and then add 2 to arrive at $3\frac{3}{4}$ and then add $\frac{3}{4}$ to get the final answer. This perspective may help some students get a good estimate of the answer before actually computing it.

Make sure students understand how the words translate into moves on the number line. E.g. five more than a number, means hop to the right 5 units. In the next section, students will explore algebra on the number line using these same moves.

EXPLORATION 5

1. Use the number line to illustrate the sums $1\frac{3}{4} + 2\frac{3}{4}$ and $\frac{4}{5} + \frac{3}{5}$. $4\frac{1}{2}$, $1\frac{2}{5}$
2. Starting at the point representing 3, determine and locate on the number line the following numbers. Explain how you arrived at your answer.
   a. The number that is 5 more than this number.
   b. The number that is 5 less than this number.
   c. The number that is 3 times this number.
   d. The number that is half as big as this number.
3. Locate and label three numbers that are greater than -5. Locate and label three numbers that are less than -6.

Distance on the Number Line

Another important concept to study on the number line is the *distance* between points.

EXPLORATION 6

Make a new number line from -15 to 15, labeling all of the integers between them. Locate the points 6 and 13 on the new number line. Determine the distance between 6 and 13. 7 by counting from 6 to 13

1. What is the distance from 12 to 4? Explain how did you got your answer. 8 by counting from 4 to 12
2. What is the distance from -3 to -11? From -9 to -2? How did you get your answers? 8, 7
3. What is the distance from -7 to 4? From 5 to -7? Explain.
4. Find the distance between $\frac{1}{2}$ and $3\frac{1}{2}$. 3
5. Find the distance between $\frac{1}{2}$ and $\frac{3}{4}$. $\frac{1}{4}$
6. Find the distance between $\frac{3}{4}$ and $3\frac{1}{2}$. $2\frac{3}{4} = \frac{11}{4}$
7. What is the distance from $-\frac{1}{2}$ to $\frac{7}{8}$? $1\frac{3}{8} = \frac{11}{8}$
8. What is the distance between $4\frac{2}{3}$ and $1\frac{1}{2}$? $3\frac{1}{6} = \frac{19}{6}$

11, 12

1–3. Watch for the idea of subtraction but do not tell them to use subtraction. We will cover this idea in the next paragraph. They will probably find this distance by counting from one to the next.

5–6. Students may count by halves and fourths to compute the distance. Of course, they could use subtraction.

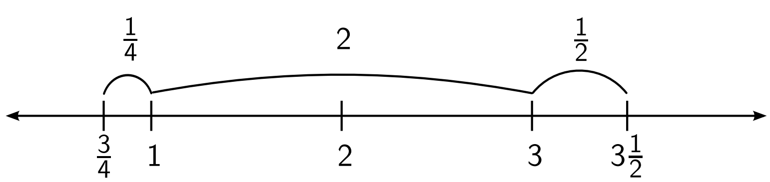
One way you might have found the distance between two points representing integers on a number line is to "count up" from the left most number until you reach the one on the right or to "count down" from the right most number until you reach the one on the left. For example from 6 you might have counted up and noted that it took 7 units to arrive at 13 and so concluded that the distance between 6 and 13 is 7. Or in the second question asking for the distance between 12 and 4, you might have counted down from 12 until you reached 4 and noted that it took 8 units, to conclude that the distance between 12 and 4 is 8. However, you might also have noticed that $12 - 4 = 8$ and $13 - 6 = 7$. The distance between two numbers is the difference of the lesser from the greater.

Discuss why you subtract the number that is on the left on the number line from the number on the right. Distance should always be positive. If you check in the third question above, look at the difference $-3 - -7$. Notice that the difference is 4, which is also the distance between -3 and -7. Note that even though 3 is less than 7, -3 is greater than -7.

In part 6, you might want to break the distance from $\frac{3}{4}$ to $3\frac{1}{2}$ into three parts:

- the distance from $\frac{3}{4}$ to 1 is $\frac{1}{4}$,
- the distance from 1 to 3 is 2,
- the distance from 3 to $3\frac{1}{2}$ is $\frac{1}{2}$.

These parts add up to $\frac{1}{4} + 2 + \frac{1}{2} = 2\frac{3}{4}$.



Note we introduce absolute value below. It is used sparingly throughout the book, mostly in Chapter 11. Since it is not a major topic in Algebra 1, we do not develop the concept fully but we use it to describe distances on the number line and coordinate plane. You may choose to forego discussing absolute value and simply talk about distance as the difference of the lesser from the greater.

The absolute value of a number is the distance from 0. We have a special symbol to represent absolute value. For example, we write $|6|$ and read it as absolute value of 6. We write $|-6|$ and read it as absolute value of -6. Since 6 and -6 are both 6 units from 0, we see that $|6| = |-6| = 6$. Since the absolute value is a distance, it is never negative. We often use absolute value when computing or representing distances between numbers. For example, if we want to compute the distance between -5 and 3, we can either subtract the lesser number from the greater number $3 - (-5) = 8$. Or we can take the absolute value of the difference, $|-5-3| = |-8| = 8$. The advantage of using the absolute value is that we can compute the difference in either order. Why is this true?

Conceptually, we can compute the difference in either order, because the distance from a to b is the same as the distance from b to a. Algebraically, we can see that $|a-b| = |-(b-a)|$ because of the absolute value.

PROBLEM 3

Compute the distance between the following pairs of numbers.

1. -12 and 6 18
2. -52 and 27 79
3. -23 and -35 12
4. 1.75 and -1.25 3
5. $\frac{3}{4}$ and $-\frac{1}{3}$ $1\frac{1}{2}$

EXERCISES

1. Compute the following sums or differences.
   a. $45 - 64$ $= -19$
   b. $42 + (-36)$ $= 6$
   c. $19 - (-33)$ $= 52$
   d. $17 - (-25)$ $= 42$
   e. $-13 + 26$ $= 13$
   f. $\frac{2}{3} + \frac{1}{5}$ $= \frac{13}{15}$
   g. $\frac{3}{5} + \frac{2}{3}$ $= \frac{19}{15} = 1\frac{4}{15}$
   h. $\frac{4}{5} - \frac{2}{3}$ $= \frac{2}{15}$
   i. $\frac{5}{7} + \frac{1}{3}$ $= \frac{22}{21} = 1\frac{1}{21}$
   j. $2\frac{3}{4} + 3\frac{1}{5}$ $= \frac{119}{20} = 5\frac{19}{20}$
   k. $5\frac{3}{4} - 2\frac{2}{3}$ $= \frac{37}{12} = 3\frac{1}{12}$
   l. $5\frac{1}{4} - 2\frac{2}{3}$ $= \frac{31}{12} = 2\frac{7}{12}$

2. Compute the following products and quotients.
   a. $-2 \cdot 7$ $= -14$
   b. $5 \cdot (-5)$ $= -25$
   c. $-11 \cdot (-6)$ $= 66$
   d. $-24 \div 6$ $= -4$
   e. $-33 \div (-5.5)$ $= 6$
   f. $\frac{2}{3} \cdot \left(-\frac{4}{5}\right)$ $= -\frac{8}{15}$
   g. $-\frac{5}{7} \div \left(-\frac{15}{16}\right)$ $= \frac{16}{21}$
   h. $-6 \div \frac{3}{5}$ $= -10$
   i. $3\frac{1}{2} \cdot 2\frac{2}{5}$ $= \frac{42}{5} = 8\frac{2}{5}$
3. Evaluate the following expressions.
   a. $5 + 6 \cdot (-3)$ $= -13$
   b. $6 \cdot 7 - (-3) \cdot 7$ $= 63$
   c. $9 \cdot (-14 + 5)$ $= -81$
   d. $-13 - (-6 - 29)$ $= 22$
   e. $\frac{-2+20}{-3}$ $= -6$
   f. $\frac{2 \cdot 8 - 21}{-3 \cdot 10}$ $= \frac{1}{6}$
4. Compute the distance between each of the following pairs of numbers.
   a. 8 and -3 11
   b. 4 and -5 9
   c. 1.1 and .9 .2
   d. 3.4 and 2.95 .45
   e. .26 and .3 .04
   f. $\frac{2}{3}$ and 2 $1\frac{1}{3}$
   g. $\frac{2}{3}$ and $\frac{1}{2}$ $\frac{1}{6}$
   h. $\frac{3}{5}$ and $\frac{3}{10}$ $\frac{3}{10}$
   i. 3.01 and 2.9 .11
   j. 3.01 and 2.99 .02
   k. 3.1 and 2.9 .2
   l. 3.1 and 2.99 .11
5. Using words, describe 3 subsets of whole numbers that are each infinite. Describe another infinite subset of whole numbers that is a subset of one of your first 3 subsets.

5. There are many possibilities, examples are the set of even whole numbers, the set of odd whole numbers, the multiples of 5. The multiples of 4 is a subset of the even numbers, whole numbers ending in 1 form a subset of odd whole numbers etc. We will use set notation for these sets in Exercise 19 in Section 1.3.

6. Copy the Venn Diagram from Example 1. For each condition given below, find a number that satisfies the condition and then place it on the Venn diagram.
   a. A whole number that is not a natural number.
   b. An integer that this not a whole number.
   c. A rational number that is not an integer.
   d. A rational number that is an integer, but not a whole number.

7. The first three examples given here are easy to find, the rest are more challenging. Ask the students to prove that their examples are correct, point out that there are other ways of comparing two fractions than finding the common denominator.

Answers will vary. $\frac{11}{15}, \frac{12}{15}, \frac{14}{15}$ will work; so will $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}$. Note that $1 - \frac{1}{3} < 1 - \frac{1}{4} < \ldots$.

$\frac{13}{48}, \frac{14}{48}, \frac{15}{48}, \frac{2}{7}$ are possibilities. Note that $\frac{2}{8} < \frac{2}{7} < \frac{2}{6}$.

$2\frac{7}{8}, 3, 3\frac{1}{10}$ $\quad 2\frac{4}{5}, 3\frac{1}{6}$

$\frac{31}{14}, \frac{32}{14}, \frac{34}{14}$ will work; so will $2\frac{1}{6}, 2\frac{1}{5}, 2\frac{1}{3}$.

8.

$$1.01; \quad 1.69; \quad \frac{3}{2} = 1\frac{1}{2}; \quad \frac{23}{12} = 1\frac{11}{12}$$

First is closest, last is greatest distance apart.

7. Find 3 numbers between each of the following pairs of numbers. Sketch a number line and plot the numbers on it.
   a. $\frac{2}{3}$ and 1
   b. $\frac{1}{4}$ and $\frac{1}{3}$
   c. $2\frac{3}{4}$ and $3\frac{1}{5}$
   d. $\frac{15}{7}$ and $\frac{5}{2}$

8. Compute the distance between each pair of the following list of numbers. Explain which pair is closest and which pair is the greatest distance apart.

$$1.39 \text{ and } 2.4 \qquad 1.41 \text{ and } 3.1 \qquad 1\frac{5}{6} \text{ and } 3\frac{1}{3} \qquad \frac{7}{4} \text{ and } \frac{11}{3}$$

9. In each of the following problems, 3 numbers are given. Draw a number line and mark and label the 3 numbers. Pay attention to the distances between the numbers. Your picture should give an approximate sense of where the 3 numbers lie in relation to each other.
   a. 1, 4, and 7
   b. 5, 19, and 23
   c. 2, 4, and −5
   d. −2, −7, and −12
   e. −10, 20, and 30
   f. 6, 8, and −97

VARIABLES ON THE NUMBER LINE

Big Idea Visualize variables and expressions on a number line.

Key Objectives

- Represent arithmetic operations visually on the number line
- Use variables to state the properties of the integers, e.g. the double opposite theorem
- Given an expression in a variable, perform operations that "undo" the operations in the given expression

TEKS 8.1a, 8.2a, A3b

Revised TEKS 8.8C

Materials Needed String or cardstock for the students. Rulers are to be substituted only when necessary, otherwise the students would be tempted to use the scale on it instead of comparing distances. It is more convenient to use cardstock for the number lines in the book, however, it is easier to half a distance with a string. A number line should also be drawn on the board or on a long strip of paper and a string is to be used for comparing and copying distances.

Launch Students will be using a string of variable length to explore variable expressions on a number line throughout this section. Using a string of specific length to represent the value of a may be a new concept for many students. Have several pieces of string of the same length cut to represent a. The teacher should start by holding up one piece of string and telling the class that we will call the length of the string a. The teacher can then have a student volunteer to hold up the second piece of string so that the two pieces of string line up end to end. Then the teacher should ask the class what 2 pieces of string would be called. (Answer: $2a$). The teacher can then have a third student volunteer to hold up a third piece of string so that all 3 pieces of string line up end to end. Then the teacher should ask the class what 3 pieces of string would be called. (Answer: $3a$). The 2 student volunteers may return to their seats and one piece of string is again held up. This time the teacher should fold the piece of string in half and ask the students what this length should be called. The students should come up with answers such as "less than $1a$," "one half of a" or "a over 2." At this point the teacher should tell the students that in mathematics we often use letters to represent unknown quantities, such as the length of the string. From this point have students divide into small groups (3–4 students) and proceed to Exploration 1 for which they will use string to measure multiples of a variable on a number line.